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Survey

Hopf-Lax-Oleinik-Type Formulas for Viscosity Solutions to Some Hamilton-Jacobi Equations*

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Abstract. In this paper, we investigate explicit viscosity solutions of the Cauchy problem to Hamilton-Jacobi equations in connection with their Hopf-Lax-Oleinik-type formulas.

1. Introduction

In this paper we consider the Cauchy problem for the Hamilton-Jacobi equation

$$u_t + H(t, u, Du) = 0 \quad \text{in} \quad \{t > 0,\ x \in \mathbb{R}^n\}, \tag{0.1}$$
$$u(0, x) = \phi(x) \quad \text{on} \quad \{t = 0,\ x \in \mathbb{R}^n\}. \tag{0.2}$$

If the Hamiltonian $H(t, \gamma, p)$ denoted by $H(p)$ depending only on $p := Du$ is strictly convex with its Fenchel conjugate $H^* = H^*(p)$ and

$$\lim_{|p|\to+\infty} H(p)/|p| = +\infty$$

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and if the initial function $\phi = \phi(x)$ is globally Lipschitz continuous, Hopf in 1965 [21] established the following formula for a *global Lipschitz solution* of (0.1)-(0.2)

$$u(t,x) = \min_{y\in\mathbb{R}^n}\big\{\phi(y) + t\cdot H^*\big((x-y)/t\big)\big\}. \tag{0.3}$$

If the Hamiltonian $H = H(p)$ is continuous and if the initial function $\phi = \phi(x)$ is globally Lipschitz continuous and convex with the Fenchel conjugate $\phi^* = \phi^*(p)$, Hopf [21] also proved that the second formula

$$u(t,x) = \max_{p\in\mathbb{R}^n}\{\langle p,x\rangle - \phi^*(p) - tH(p)\} \tag{0.4}$$

determines a global Lipschitz solution of the Cauchy problem (0.1)-(0.2).

In the one-dimensional case for the conservation law

$$u_t + H_x(u) = 0 \quad \text{in} \quad \{t > 0,\ x\in\mathbb{R}^n\}, \tag{0.5}$$

$$u(0,x) = \phi(x) \quad \text{on} \quad \{t = 0,\ x\in\mathbb{R}^n\}, \tag{0.6}$$

Lax [36] and Oleinik [49] had found analogous solutions $u(t,x)$ by the method of vanishing viscosity, and under essentially analogous hypotheses: a) $H''(p) \geq 0$ for all p, $\phi(x)$ largely arbitrary, and b) $\phi(x)$ monotone, $H(p)$ arbitrary.

However, as Hopf mentioned in [21], it is unlikely that such restrictions, either on $H(p)$ or on the initial functions $\phi(x)$, are really vital. A relevant solution is expected to exist under much wider assumptions. Since that time many mathematicians have obtained their generalizations of formulas (0.3) and (0.4) for wider classes of first order nonlinear PDEs. This paper is devoted to some of these generalizations and we call them *Hopf-Lax-Oleinik type formulas*.

Since the early 1980s, the concept of *viscosity solutions* introduced by Crandall and Lions [15] has been used in a large portion of research in a nonclassical theory of first-order nonlinear PDEs as well as in other types of PDEs. For convex Hamilton-Jacobi equations, the viscosity solution-characterized by a semiconcave stability condition, was first introduced by Kruzkov [35]. There is an enormous activity which is based on these studies. The primary virtues of this theory are that it allows merely nonsmooth functions to be solutions of nonlinear PDEs, it provides very general existence and uniqueness theorems, and it yields precise formulations of general boundary conditions. Let us mention here the names: Crandall, Lions, Evans, Ishii, Jensen, Barbu, Bardi, Barles, Barron, Cappuzzo-Dolcetta, Dupuis, Lenhart, Osher, Perthame, Soravia, Souganidis, Tataru, Tomita, Yamada, and many others, whose contributions make great progress in nonlinear PDEs. The concept of viscosity solutions is motivated by the classical maximum principle which distinguishes it from other definitions of generalized solutions. For the results in this field the redear is referred to [14, 16 - 19, 28,..., 35, 38],...

Bardi and Evans [5, 19] and Lions [38] showed that the formulas (0.3) and (0.4) still give the unique Lipschitz viscosity solution of (0.1) - (0.2) under the assumptions that H is convex and ϕ is uniformly Lipschitz continuous for (0.3) and H is continuous and ϕ is convex and Lipschitz continuous for (0.4). Furthermore, Bardi and Faggian [6] proved that the formula (0.3) is still valid for unique viscosity solution whenever H is convex and ϕ is uniformly continuous.

Lions and Rochet, [40] studied the multi-time Hamilton-Jacobi equations and obtained a Hopf-Lax-Oleinik type formula for these equations.

The Hopf-Lax-Oleinik type formulas for the Hamilton-Jacobi equations

$$u_t(t,x) + H(u, D_x u) = 0 \tag{0.7}$$

were found in the papers by Barron, Jensen, and Liu [10 - 12], where the first and second conjugates for quasiconvex functions - functions whose level set are convex - were successfully used.

The paper by Alvarez, Barron, and Ishii [4] is concerned with finding Hopf-Lax-Oleinik type formulas of the Hamilton-Jacobi equation (0.7), $(t,x) \in (0,\infty)\times \mathbb{R}^n$ with

$$u(0,x) = g(x),\ x \in \mathbb{R}^n, \tag{0.8}$$

when the initial function g is only lower semicontinuous (l.s.c.), and possibly infinite. If $H(\gamma, p)$ is convex in p and the initial data g is quasiconvex and l.s.c., the Hopf-Lax-Oleinik type formula gives the l.s.c. solution of the problem (0.7) - (0.8). If the assumption of convexity of $p \mapsto H(\gamma,p)$ is dropped, it is proved that $u = (g^{\#} + tH)^{\#}$ is still characterized as the minimal l.s.c. supersolution (here, $\#$ means the second quasiconvex conjugate, see [9 - 10]).

If H is a concave-convex function given by a D. C. representation

$$H(p', p'') := H_1(p') - H_2(p'')$$

and ϕ is uniformly continuous, Bardi and Faggian [6] have found explicit pointwise upper and lower bounds of Hopf-Lax-Oleinik type for the viscosity solutions. If the Hamiltonian $H(\gamma, p), (\gamma, p) \in \mathbb{R}\times\mathbb{R}^n$, is a D. C. function in p, i.e.,

$$H(\gamma,p) = H_1(\gamma,p) - H_2(\gamma,p), \quad (\gamma,p)\in\mathbb{R}\times\mathbb{R}^n,$$

Barron, Jensen, and Liu [12] have given their Hopf-Lax-Oleinik type estimates for viscosity solutions to the corresponding Cauchy problem.

Penot and Volle [50] recently show that the Hopf-Lax formulas for solving (0.1) - (0.2) are valid under very weak continuous assumptions and mild convexity assumptions on H.

We also want to mention the investigations on explicit solutions of PDEs based on the Cole-Hopf transformation, which have been discovered by Maslov and his coworkers [22], Gesztesy and Holden [22], Sachdev [55], Joseph K.T. and his coworkers [44, 41, 45], Truman and Zhao [62], and many others. The Hopf-Lax-Oleinik type and explicit formulas have obtained in the recent papers by Adimurthi and Gowda [1 - 3], Barles and Tourin [8], Barles [7], Rockafellar and Wolenski [53], LeFloch [37], Manfredi and Stroffolini [43], Ngoan [47], Plaskacz and Quincampoix [51], Thai Son [61], Subbotin [59], Melikyan [46], Rublev [54], Silin [56], Stromberg [57],...

This paper is a survey of results on Hopf-Lax-Oleinik type and explicit formulas for the solutions of (0.1) - (0.2) obtained by the author, Mai Duc Thanh and Rudolf Gorenflo in [70, 72, and 69].

The paper consists of 4 sections. Sec. 1 being introduction gives the contents of the present paper and notations that we use in the following sections..

Sec. 2 is to deal with Hopf-Lax-Oleinik type formulas of viscosity solutions to Hamilton-Jacobi equations when the Hamiltonian does not depend on the unknown function, the initial function need not be uniformly continuous. Sec. 3 is to do the same work with the convex Hamiltonian depending also on the unknown function and its spatial gradient. The Hamiltonian H will be supposed to satisfy the conditions: $H = H(\gamma, p)$ is nondecreasing in $\gamma \in \mathbb{R}$, convex and positively homogeneous of degree one in $p \in \mathbb{R}^n$. The initial function will be assumed to be only continuous. In Sec. 4, the Hamiltonian also contains the time variable, the unknown function and its spatial gradient. Our assumptions are: the Hamiltonian $H = H(t, \gamma, p)$ is nondecreasing in $\gamma \in \mathbb{R}$, positively homogeneous of degree one in $p \in \mathbb{R}^n$ and some additional conditions on the time variable; the initial data are quasiconvex. In this paper, the following notations will be used:

$$U := (0, T) \times \mathbb{R}^n,$$

thus,

$$\bar{U} = [0, T] \times \mathbb{R}^n,$$

for some given numbers $a, b \in \mathbb{R},\ r > 0,\ N > 0$, for some given $x \in \mathbb{R}^n$,

$$\begin{aligned} a \vee b &:= \max\{a, b\}, \\ B(x; r) &:= \{y \in \mathbb{R}^n : |y - x| < r\}, \\ S(x; N) &:= \{y \in \mathbb{R}^n : |y - x| = N\}; \end{aligned}$$

for some given open subset $\mathcal{D} \subset \mathbb{R}^n,\ f : \mathcal{D} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$,

$$\begin{aligned} Q(f) &:= \{x \in \mathcal{D} : f(x) = -\infty\}, \\ C^{0,1}(\bar{\mathcal{D}}) &:= \{v \in C(\bar{\mathcal{D}}) : \exists k > 0,\ |v(x) - v(y)| \le k|x - y|, \quad \forall x, y \in \bar{\mathcal{D}}\}, \\ C^{0,1}(\mathcal{D}) &:= \{v \in C^{0,1}(\bar{W}),\ \forall\, W \quad \text{open bounded set such that} \quad \bar{W} \subset \mathcal{D}\}, \\ W^{1,\infty}_{\mathrm{loc}}(\mathcal{D}) &:= \{v \in L^\infty_{\mathrm{loc}}(\mathcal{D}) : D_j v \in L^\infty_{\mathrm{loc}}(\mathcal{D}), \quad \forall j = 1, 2, ..., m\}, \end{aligned}$$

where $D_j v$ are understood in the sense of distributions;

$$\begin{aligned} UC(\mathcal{D}) &:= \{v \in C(\mathcal{D}) : v \quad \text{is uniformly continuous in} \quad \mathcal{D}\}, \\ UC(\bar{\mathcal{D}}) &:= \{v \in C(\bar{\mathcal{D}}) : v \quad \text{is uniformly continuous in} \quad \bar{\mathcal{D}}\}, \\ BUC(\mathcal{D}) &:= \{v \in UC(\mathcal{D}) : v \quad \text{is bounded on} \quad \mathcal{D}\}, \\ BUC(\bar{\mathcal{D}}) &:= \{v \in UC(\bar{\mathcal{D}}) : v \quad \text{is bounded on} \quad \bar{\mathcal{D}}\}, \end{aligned}$$

$UC_x(\bar{U}) := \{v \in C(\bar{U}) : v$ is uniformly continuous on $\mathbb{R}^n$ uniformly in $t \in [0, T]\}$.

Remark 1. It is indicated (see [38] for example) that

$$C^{0,1}(\mathcal{D}) = W^{1,\infty}_{\mathrm{loc}}(\mathcal{D}).$$

Now we give a definition of viscosity solutions for the Cauchy problem of Hamilton-Jacobi equations (0.1) - (0.2). Namely, consider the Cauchy problem

$$u_t + H(t, u, Du) = 0 \text{ in } U := (0, \infty) \times \mathbb{R}^n \tag{0.1}$$
$$u(0, x) = g(x) \text{ on } \{t = 0\} \times \mathbb{R}^n. \tag{0.2}$$

Here the Hamiltonian $H(t, \gamma, p)$ is continuous in the variables t, γ and p.

Definition 1.1. *A bounded, uniformly continuous function u is called a viscosity solution of* (0.1) - (0.2) *provided:*
(i) $u = g$ *on* $\{t = 0\} \times \mathbb{R}^n$ *and*
(ii) *for each* $v \in C^\infty\big((0, \infty) \times \mathbb{R}^n\big)$,

$$\begin{cases} \text{if } u - v \text{ has a local maximum at a point } (t_0, x_0) \in (0, \infty) \times \mathbb{R}^n \\ \text{then } \ v_t(t_0, x_0) + H(t_0, v(t_0, x_0), Dv(t_0, x_0)) \le 0, \end{cases}$$

(*in this case u is called a subsolution*) *and*

$$\begin{cases} \text{if } \ u - v \text{ has a local minimum at a point } (t_0, x_0) \in (0, \infty) \times \mathbb{R}^n \\ \text{then } \ v_t(t_0, x_0) + H(t_0, v(t_0, x_0), Dv(t_0, x_0)) \ge 0, \end{cases}$$

(*in this case u is called a supersolution*).

If the solution u is locally Lipschitz continuous in $(0, \infty) \times \mathbb{R}^n$, *then u is called* a Lipschitz viscosity solution.

2. A Hopf-Lax-Oleinik-type Formula for Viscosity Solutions

Let us consider the Cauchy problem for the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + H(D_x u) = 0 \quad \text{in} \quad U, \tag{1}$$
$$u(0, x) = \lim_{\substack{(t,y)\to(0,x) \\ t>0}} u(t, y) = g(x) \quad \text{in} \quad \mathbb{R}^n. \tag{2}$$

When g is in $BUC(\mathbb{R}^n)$ and H is convex and satisfies

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty, \tag{3}$$

the unique viscosity solution of (1)-(2) is given by the first Hopf-Lax formula

$$\begin{aligned} u(t, x) &= \inf_{y\in\mathbb{R}^n} \sup_{z\in\mathbb{R}^n} \big\{g(y) + \langle z, x - y\rangle - tH(z)\big\} \\ &= \inf_{y\in\mathbb{R}^n} \Big\{g(y) + tH^*\Big(\frac{x - y}{t}\Big)\Big\}, \end{aligned} \tag{4}$$

where, H^* is the Fenchel conjugate of H. This fact was established in [38].

The formula (4) was also proved to be the unique viscosity solution, uniformly continuous on $\mathbb{R}^n$ uniformly in $t \in [0, T]$, of (1)-(2) under the assumptions that

g is uniformly Lipschitz continuous in $\mathbb{R}^n$, H is convex, (see [5]). Furthermore, Bardi and Faggian [6] recently showed that this result is still valid whenever $g \in UC(\mathbb{R}^n)$ and H is convex.

In this section we prove

Theorem 2.1. *Assume that H is convex and satisfies* (3)*, g is continuous. Suppose in addition that for every $0 < r < +\infty$, there exists a number N such that*

$$\inf_{|z|\le N} \Big\{ g(z) + tH^*\Big(\frac{x-z}{t}\Big)\Big\} < g(y) + tH^*\Big(\frac{x-y}{t}\Big), \quad \forall (t,x) \in (0,T)\times B(0;r),\ \forall |y| > N. \tag{5}$$

Then (4) *determines a Lipschitz viscosity solution of* (1) - (2)*.*

Remark 2. The condition (5) may be considered as a compatible one between the Hamiltonian and the initial function for the existence of generalized solutions of Problem (1) - (2).

Proof. Fix an arbitrary $r > 0$ and let N be the corresponding constant as in (5). First, we will show that u is in $C^{0,1}(U)$. Define

$$\begin{aligned} h(t,x) &:= \operatorname*{Argmin}_{y\in\mathbb{R}^n} \Big\{ g(y) + tH^*\Big(\frac{x-y}{t}\Big)\Big\} \\ &:= \Big\{ y_0 \in \mathbb{R}^n : g(y_0) + tH^*\Big(\frac{x-y_0}{t}\Big) = \min_{y\in\mathbb{R}^n} \Big\{ g(y) + tH^*\Big(\frac{x-y}{t}\Big)\Big\}\Big\}, \\ &\qquad (t,x) \in (0,T]\times\mathbb{R}^n. \end{aligned} \tag{6}$$

Since H is convex and satisfies (3), H^* is finite and is therefore locally Lipschitz. By continuity, from (5) we can see that $h(t,x) \neq \emptyset$ for any $(t,x) \in U$ and

$$\|h(t,x)\| = \sup_{y\in h(t,x)} |y| \le N, \quad \forall (t,x) \in (0,T)\times B(0,r).$$

Choosing a constant $\delta > 0$, it follows that the function

$$(t,x,y) \mapsto tH^*\Big(\frac{x-y}{t}\Big),\ (t,x,y) \in [\delta,T]\times \bar{B}(0;r)\times\bar{B}(0;N)$$

is Lipschitz continuous. Let M be the Lipschitz constant for this function. Then,

$$u(t,x) = g(y) + tH^*\Big(\frac{x-y}{t}\Big), \quad u(t',x') \le g(y) + t'H^*\Big(\frac{x'-y}{t'}\Big), \\ \forall (t,x),(t',x') \in [\delta,T]\times\bar{B}(0;r),\ \forall y \in h(t,x).$$

Hence,

$$u(t',x') - u(t,x) \le t'H^*\Big(\frac{x'-y}{t'}\Big) - tH^*\Big(\frac{x-y}{t}\Big) \le M(|t-t'| + |x-x'|).$$

Interchanging (t, x) and (t', x'), we find $u \in C^{0,1}(U)$.

Second, let us check the initial condition (2). Clearly,

$$u(t, x') \leq g(x') + tH^*(0), \quad \forall (t, x') \in (0, T] \times \mathbb{R}^n.$$

This yields

$$\lim_{(t,x')\to(0,x)} \sup\, u(t, x') \leq g(x). \tag{7}$$

Besides,

$$u(t, x') = g(y) + tH^*\Big(\frac{x'-y}{t}\Big) \geq g(y) - tH(0), \\ \forall (t, x') \in (0, T) \times B(0; r), \forall y \in h(t, x'). \tag{8}$$

We make a claim that

$$\lim_{\substack{y\in h(t,x') \\ (t,x')\to(0,x)}} y = x. \tag{9}$$

Otherwise, by passing to a subsequence, we may assume that there are

$$(t_i, x_i) \in (0, T) \times B(0; r),\ y_i \in h(t_i, x_i),\ i \in \mathbb{N},\ |x_i - y_i| \geq \epsilon > 0.$$

Since

$$\|h(t, x)\| \leq N,\ \forall (t, x) \in (0, T) \times B(0; r),$$

it holds true for i large enough and $y_i \in h(t_i, x_i)$ that

$$+\infty > C = \max_{|x|\leq r} g(x) + T \max\{0, H^*(0)\} \geq g(x_i) + t_i H^*(0) \geq u(t_i, x_i),$$

$$u(t_i, x_i) = g(y_i) + \frac{H^*\Big(\frac{x_i-y_i}{t_i}\Big)}{\big|\frac{x_i-y_i}{t_i}\big|}|x_i - y_i| \geq \min_{|y|\leq N} g(y) + \epsilon \frac{H^*\Big(\frac{x_i-y_i}{t_i}\Big)}{\big|\frac{x_i-y_i}{t_i}\big|} \to +\infty$$

as $i \to \infty$, this is a contradiction. Thus, (9) is verified. From (8) and (9), we have

$$\lim_{(t,x')\to(0,x)} \inf\, u(t, x') \geq g(x). \tag{10}$$

The condition (2) immediately follows from (7) and (10).

Third, we suppose in addition that H is strictly convex. By an argument similar to the one in Theorem 4.1 of Chapter 4 in [76], one can see that u is a Lipschitz solution of Problem (1) - (2). Below, u will be shown to be a viscosity solution of the equation (1) in every open bounded subset of U. To this end, we intend to use Theorem 11.1 [38]. Let C be a set in $\mathbb{R}^n$. As in [38, p. 217], define

$$L_C(t, x; s, y) = \inf \Big\{ \int_s^t H^*\Big(\frac{d\xi}{d\Lambda}\Big) d\Lambda \ /\ \xi \text{ such that } \xi(s) = y,\ \xi(t) = x,\ \xi \in \bar{C}, \\ \frac{d\xi}{d\Lambda} \in L^\infty(s, t),\ \forall \Lambda \in (s, t)\Big\}, \quad t, s \in [0, T],\ x, y \in C.$$

If C is convex, arguing similarly as in [38], we have

$$L_C(t,x;s,y) = L_{\mathbb{R}^n}(t,x;s,y) = (t-s)H^*(\frac{x-y}{t-s}), \quad \forall (t,x),(s,y) \in [0,T]\times C,\ s<t. \tag{11}$$

On the other hand, it is seen that for any $(t,x) \in (0,T)\times B(0,r)$,

$$\begin{aligned} u(t,x) &= \min_{|y|\le N} \Big\{ g(y) + tH^*\Big(\frac{x-y}{t}\Big)\Big\} \\ &= \inf\Big\{u(s,y) + (t-s)H^*\Big(\frac{x-y}{t-s}\Big),\ (s,y)\in\{0\}\times \bar{B}(0,r)\Big\}. \end{aligned} \tag{12}$$

Without loss of generality, we may suppose that $N \ge r$. Put

$$Q := (0,T)\times B(0;N).$$

We have left to check the compatibility condition,

$$\begin{aligned} u(t,x) - u(s,y) \le L_{B(0;N)}(t,x;s,y) \quad &\forall (t,x)\in (0,T]\times S(0;N), \\ \forall (s,y)\in \partial_0 Q := (\{0\}\times \bar{B}(0;N)) &\cup ([0,T]\times S(0;N)),\ s<t. \end{aligned} \tag{13}$$

Let $z \in h(t,x)$ defined by (6). By convexity,

$$\frac{s}{t}H^*\Big(\frac{y-z}{s}\Big) + \frac{t-s}{t}H^*\Big(\frac{x-y}{t-s}\Big) \ge H^*\Big(\frac{x-z}{t}\Big).$$

Then,

$$\begin{aligned} u(t,x) - u(s,y) &\le tH^*\Big(\frac{x-z}{t}\Big) - sH^*\Big(\frac{y-z}{s}\Big) \\ &\le (t-s)H^*\Big(\frac{x-y}{t-s}\Big) = L_{B(0;N)}(t,x;s,y), \end{aligned}$$

i.e., (13) is verified. Set

$$v(t,x) = \inf\Big\{u(s,y) + (t-s)H^*\Big(\frac{x-y}{t-s}\Big),\ (s,y)\in \partial_0 Q,\ s<t\Big\},\ (t,x)\in Q.$$

From (11), (13) and that u is continuous on $\bar{Q}$, in view of Theorem 11.1 [38], we see that v is a viscosity solution of the equation (1) in Q. Moreover, v is accordingly the maximum element of the set $\mathcal{S}$ defined by:

$$\mathcal{S} = \Big\{w \in C(\bar{Q})\cap W^{1,\infty}_{\text{loc}},\ \frac{\partial w}{\partial t} + H(D_x w) \le 0 \text{ a.e. in } Q,\ w \le u \text{ on } \partial_0 Q\Big\}.$$

Evidently, $u \in \mathcal{S}$. Thus,

$$v(t,x) \ge u(t,x) \quad \forall (t,x)\in Q.$$

On the other hand, from (12), we have

$$u(t,x) \ge v(t,x), \quad \forall (t,x) \in (0,T)\times B(0;r).$$

Consequently,

$$u(t,x) = v(t,x), \quad \forall (t,x) \in (0,T) \times B(0;r).$$

That means u is a viscosity solution of the equation (1) in $(0,T) \times B(0;r)$. Since r is arbitrary, u is a viscosity solution of the equation (1) in U and therefore of Problem (1) - (2).

Finally, we turn to the case where H is not necessarily strictly convex. In this case, one can treat as follows. For each $k \in \mathbb{N}$, put

$$H_k(x) = H(x) + |x|^2/k.$$

Then H_k is strictly convex. It is easy to check that the condition (5) is then fulfilled with respect to H_k as well. Arguing similarly as above, we see that for every $k \in \mathbb{N}$, the function

$$u_k(t,x) = \inf_{y \in \mathbb{R}^n} \Big\{ g(y) + tH_k^*(\frac{x-y}{t}) \Big\}, \quad (t,x) \in U,$$

is a viscosity solution of the equation

$$\frac{\partial u}{\partial t} + H_k(D_x u) = 0 \quad \text{in} \quad U.$$

Besides, let $M > 0$ be arbitrary selected, by the condition (3), for any $|z| < M$, we have $\langle z, x \rangle - H(x) \to -\infty$ as $|x| \to +\infty$. Hence,

$$\langle z, x \rangle - H_k(x) \le \langle z, x \rangle - H(x) < -H(0) - 1, \ \forall |z| < M, \ \forall |x| > m,$$

for some $m > 0$. This implies that the sup in the expressions of $H_k^*(z)$ and $H^*(z)$ must be taken over the common ball $\bar{B}(0;m)$, $\forall |z| < M$. Thus, a simple calculation leads to

$$H^*(z) \ge H_k^*(z) \ge H^*(z) - m^2/k, \quad \forall |z| < M, \ \forall k \in \mathbb{N}. \tag{14}$$

For any fixed $r > 0$, $\delta > 0$, from (12) and (14), it follows that

$$u(t,x) \ge u_k(t,x) = \inf_{|y| \le N} \Big\{ g(y) + tH_k^*\Big(\frac{x-y}{t}\Big) \Big\} \ge u(t,x) - Tm^2/k, \tag{15}$$

for all $(t,x) \in [\delta, T] \times \bar{B}(0;r)$, provided M chosen is large enough such that

$$\Big| \frac{x-y}{t} \Big| \le \frac{|x| + |y|}{t} \le \frac{r+N}{\delta} \le M.$$

The inequalities (15) guarantee that the sequence u_k converges uniformly on compact subsets of $(0,T) \times \mathbb{R}^n$ to u. In view of Theorem 1.4 [16], u is a viscosity solution of (1).

In fact, we have already proved that u is a viscosity solution of Problem (1) - (2), and u is locally Lipschitz continuous in U. ■

Corollary 2.2. *Assume that H is convex and satisfies* (3)*, g is continuous and satisfies*

$$\liminf_{|x|\to\infty} \frac{g(x)}{|x|} > -\infty. \tag{16}$$

Then (4) *defines a Lipschitz viscosity solution of Problem* (1) - (2).

Proof. We need only to prove that the condition (5) is fulfilled. Actually, as seen in the proof of Theorem 2.1, H^* is finite and satisfies (3). Hence,

$$g(y) + tH^*\Big(\frac{x-y}{t}\Big) = |x-y|\Big(\frac{|y|}{|x-y|}\frac{g(y)}{|y|} + \frac{H^*(\frac{x-y}{t})}{|\frac{x-y}{t}|}\Big) \to +\infty \text{ as } |y| \to +\infty,$$

uniformly in $(t,x) \in (0,T) \times B(0;r)$. Therefore, for any fixed $r > 0$, there exists a number $N > 0$ such that for all $|y| > N$:

$$\begin{aligned} g(y) + tH^*\Big(\frac{x-y}{t}\Big) &> \max_{|z|\le N} g(z) + tH^*(0) \\ &\ge \min_{|z|\le N}\{g(z) + tH^*(\frac{x-z}{t})\} \quad \forall (t,x) \in (0,T)\times B(0;r), \end{aligned}$$

i.e., the condition (5) is verified. ■

Let us illustrate the above results by the two following examples.

Example 1.

$$\frac{\partial u}{\partial t} + \frac{1}{2}(\frac{\partial u}{\partial x})^2 = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{17}$$

$$u(0,x) = x^2, \quad x \in \mathbb{R}. \tag{18}$$

It is easy to see that all the conditions of Corollary 2.2 are fulfilled. Therefore, a Lipschitz viscosity solution of Problem (17) - (18) is given by

$$u(t,x) = \frac{x^2}{1+2t}, \quad t \ge 0,\ x \in \mathbb{R}.$$

Example 2. Given $a > 0$,

$$\frac{\partial u}{\partial t} + e^{|\frac{\partial u}{\partial x}|} - |\frac{\partial u}{\partial x}| = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{19}$$

$$u(0,x) = ax^2, \quad x \in \mathbb{R}. \tag{20}$$

Clearly, all the conditions of Corollary 2.2 are satisfied. Hence, the formula (4) yields a viscosity solution of (19) - (20), for every $t \ge 0,\ x \in \mathbb{R}$,

$$u(t,x) = a(x - ty_0)^2 + t(|y_0|+1)(\log(|y_0|+1) - 1),$$

where y_0 is the unique solution of

$$2a(ty - x) + \operatorname{sign} y.\log(|y|+1) = 0.$$

The regularity and uniqueness of solutions are shown in the following.

Corollary 2.3. *If $g \in UC(\mathbb{R}^n)$, H is convex and satisfies (3), then the formula (4) defines a function u Lipschitz continuous on $[\epsilon, T] \times \mathbb{R}^n$ for any $\epsilon > 0$, and*

$$u \in UC([0,T] \times \mathbb{R}^n)$$

is the unique viscosity solution in $UC_x(\bar{U})$.

Proof. Under the hypotheses, Problem (1) - (2) has a unique viscosity solution in $UC_x([0,T] \times \mathbb{R}^n)$, (see Lions and Perthame [39]).

If g is uniformly continuous on $\mathbb{R}^n$, then it is easy to see that for any fixed $\delta > 0$,

$$C = \sup_{x,y \in \mathbb{R}^n} \frac{|g(x) - g(y)|}{\max\{\delta, |x-y|\}} < +\infty. \tag{21}$$

This implies (16) and therefore, by virtue of Corollary 2.2, u is a viscosity solution of Problem (1) - (2).

Next, we claim that the quantity $|x-y|, (t,x) \in (\epsilon, T] \times \mathbb{R}^n,\ y \in h(t,x)$ must be bounded and the limit (9) is uniform with respect to $x \in \mathbb{R}^n$, i.e., for some positive constant M,

$$\begin{aligned} &|x-y| \le M, \quad \forall (t,x) \in (\epsilon, T] \times \mathbb{R}^n,\ \forall y \in h(t,x), \\ &\lim_{\substack{y \in h(t,x') \\ (t,x') \to (0,x)}} y = x, \quad \text{uniformly for} \quad x \in \mathbb{R}^n. \end{aligned} \tag{22}$$

Indeed, if it is not the case, there are $(t_i, x_i) \in (0,T] \times \mathbb{R}^n,\ y_i \in h(t_i, x_i),\ |x_i - y_i| \ge \epsilon > 0, i \in \mathbb{N}$ such that

$$\frac{|x_i - y_i|}{t_i} \to +\infty, \quad \text{as} \quad i \to \infty.$$

Then,

$$g(x_i) + t_i H^*(0) \ge u(t_i, x_i) = g(y_i) + t_i H^*(\frac{x_i - y_i}{t_i}).$$

Hence, from (21) it follows that,

$$C \ge \frac{g(x_i) - g(y_i)}{|x_i - y_i|} \ge \frac{H^*\left(\frac{x_i - y_i}{t_i}\right)}{|\frac{x_i - y_i}{t_i}|} - \frac{H^*(0)}{|x_i - y_i|} \to +\infty, \quad \text{as} \quad i \to \infty,$$

a contradiction, i.e., (22) holds true. Since H^* is locally Lipschitz, the function

$$(t,x) \mapsto tH^*\Big(\frac{x}{t}\Big), \quad (t,x) \in [\epsilon, T] \times \bar{B}(0; 2M),$$

is Lipschitz. Let $L > 0$ be the Lipschitz constant for this function. For every $x, x' \in \mathbb{R}^n,\ t, t' \ge \epsilon$, let $y \in h(t,x)$ and $y' \in h(t,x')$. By (22), $|x' - y'|,\ |x-y| \le M$. So,

$$u(t', x') - u(t, x') \le t' H^*\Big(\frac{x' - y'}{t'}\Big) - tH^*\Big(\frac{x' - y'}{t}\Big) \le L|t - t'|,$$

and if $|x - x'| \leq M$, then $|x' - y| \leq |x' - x| + |x - y| \leq 2M$. Hence,

$$u(t, x') - u(t, x) \leq tH^*\Big(\frac{x' - y}{t}\Big) - tH^*\Big(\frac{x - y}{t}\Big) \leq L|x - x'|.$$

Consequently, for $|x - x'| \leq M$ and therefore for every $x, x' \in \mathbb{R}^n,\ t, t' \geq \epsilon$,

$$u(t', x') - u(t, x) \leq L(|t' - t| + |x' - x|).$$

Interchanging (t', x') and (t, x), we obtain

$$|u(t', x') - u(t, x)| \leq L(|t' - t| + |x' - x|), \quad \forall (t', x'),\, (t, x) \in [\epsilon, T] \times \mathbb{R}^n.$$

That means $u \in C^{0,1}([\epsilon, T] \times \mathbb{R}^n)$.

To prove that $u \in UC([0, T] \times \mathbb{R}^n)$, we need only indicate that the limit (2) is uniform with respect to $x \in \mathbb{R}^n$, i.e.,

$$\lim_{(t,x') \to (0,x)} u(t, x') = g(x) \quad \text{uniformly for} \quad x \in \mathbb{R}^n. \tag{23}$$

Actually,

$$u(t, x') \leq g(x') + tH^*(0), \quad \forall (t, x') \in (0, T] \times \mathbb{R}^n.$$

Hence,

$$\lim_{(t,x') \to (0,x)} \sup u(t, x') \leq g(x), \quad \text{uniformly for} \quad x \in \mathbb{R}^n. \tag{24}$$

On the other hand, since g is uniformly continuous, for every $\epsilon' > 0$, there exists $\delta_1 > 0$ such that if

$$t \in (0, T],\ x, y \in \mathbb{R}^n,\ t < \delta_1,\ |x - y| < \delta_1,$$

then

$$g(y) - tH(0) \geq g(x) - \epsilon'.$$

By virtue of (22), for above $\delta_1 > 0$, there exists $0 < \delta < \delta_1$ such that if $t + |x' - x| < \delta$ then

$$|y - x| < \delta_1,\ \forall y \in h(t, x').$$

So, for every $t \in (0, T],\ x, x' \in \mathbb{R}^n,\ t + |x' - x| < \delta$, let $y \in h(t, x')$, then

$$u(t, x') = g(y) + tH^*\Big(\frac{x' - y}{t}\Big) \geq g(y) - tH(0) \geq g(x) - \epsilon'.$$

That means

$$\lim_{(t,x') \to (0,x)} \inf u(t, x') \leq g(x), \quad \text{uniformly for} \quad x \in \mathbb{R}^n. \tag{25}$$

Combining (24) and (25), we obtain (23). ∎

Let us illustrate Corollary 2.3 by considering the two following Cauchy problems.

Example 3. Given $a > 0$,

$$\frac{\partial u}{\partial t} + \frac{1}{2}\Big(\frac{\partial u}{\partial x}\Big)^2 = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{26}$$

$$u(0,x) = a|x|, \quad x \in \mathbb{R}. \tag{27}$$

According to Corollary 2.3, the unique viscosity solution of Problem (26), (27) is given by

$$u(t,x) = \begin{cases} \frac{-a^2 t}{2} + a|x|, & |x| > at \\ \frac{x^2}{2t}, & |x| \le at. \end{cases}$$

Example 4.

$$\frac{\partial u}{\partial t} + e^{|\frac{\partial u}{\partial x}|} - |\frac{\partial u}{\partial x}| = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{28}$$

$$u(0,x) = |x|, \quad x \in \mathbb{R}. \tag{29}$$

It is easy to check that all the requirements of Corollary 2.3 are fulfilled. An elementary calculus yields

$$u(t,x) = \begin{cases} |x| - (e-1)t, & |x| > (e-1)t, \\ (|x|+t)(\log(\frac{|x|}{t}+1)-1), & |x| \le (e-1)t, \end{cases}$$

a viscosity solution of (28) - (29).

3. Viscosity Solutions for Convex Hamiltonians Depending on u, Du

Let us consider the Cauchy problem for Hamilton-Jacobi equations of the form

$$\frac{\partial u}{\partial t} + H(u, D_x u) = 0 \quad \text{in} \quad U, \tag{30}$$

$$u(0,x) = g(x) \quad \text{in} \quad \mathbb{R}^n. \tag{31}$$

In this section the following conditions are supposed:

(A1) *$H(\gamma, p)$ is continuous in $\mathbb{R}^{n+1}$, $H(., p)$ is nondecreasing in $\mathbb{R}$ for each $p \in \mathbb{R}^n$;*

(A2) *$H(\gamma, .)$ is convex and positively homogeneous of degree one in $\mathbb{R}^n$, for each $\gamma \in \mathbb{R}$, i.e.*

$$H(\gamma, \Lambda p) = \Lambda H(\gamma, p),\ \forall \Lambda \ge 0,\ \forall \gamma \in \mathbb{R}.$$

Our main result in this section is

Theorem 3.1. *Assume that* (A1) *and* (A2) *hold, and that the initial data g is continuous in $\mathbb{R}^n$. Then the function*

$$u(t,x) = \inf_{y \in \mathbb{R}^n} \Big\{ H^{\#}\Big(\frac{x-y}{t}\Big) \vee g(y) \Big\}, \quad (t,x) \in U, \tag{32}$$

determines a viscosity solution of Problem (30) - (31).

To prove Theorem 3.1 first, we need

Lemma 3.2. *The set* $Q(H^{\#}) := \{q \in \mathbb{R}^n : H^{\#}(q) = -\infty\}$ *is bounded and not empty.*

Proof. Theorem 2.1 and Lemma 2.2 in [10] imply that there exists $N > 0$ such that

$$H^{\#}(z) > 0, \quad \forall |z| \geq N.$$

This proves the boundedness of $Q(H^{\#})$. Put

$$h(z) := \min\{H^{\#}(z), 0\},$$

of course, h is l.s.c. on $\mathbb{R}^n$. If

$H^{\#}(z) > -\infty$, $\forall z$, then h is l.s.c. and finite. Hence, by virtue of Theorem 2.1 and Lemma 2.2 in [10],

$$-\infty < \min_{|z|\leq N} h(z) \leq \inf_{|z|\leq N} H^{\#}(z) = \inf_{z\in\mathbb{R}^n} H^{\#}(z) = -\infty.$$

This contradiction shows that $Q(H^{\#}) \neq \emptyset$. Lemma 3.2 is proved. ■

Proof of Theorem 3.1. Now, we will show that $u \in C(U)$. Obviously,

$$u(t,x) = \inf_{z\in\mathbb{R}^n} \{H^{\#}(z) \vee g(x - tz)\} \quad \forall (t,x) \in (0,T] \times \mathbb{R}^n. \tag{33}$$

Define

$$h(t,x) := \operatorname*{Argmin}_{y\in\mathbb{R}^n} \{H^{\#}(y) \vee g(x - ty)\}$$
$$= \Big\{y_0 \in \mathbb{R}^n : H^{\#}(y_0) \vee g(x - ty_0) = \min_{y\in\mathbb{R}^n}\{H^{\#}(y) \vee g(x - ty)\}\Big\},$$
$$(t,x) \in (0,T] \times \mathbb{R}^n.$$

Let $x_0 \in \mathbb{R}^n$ and let $r > 0$ be arbitrary. From Lemma 3.2, there exists $M > 0$ such that

$$\|Q(H^{\#})\| := \sup\{|q| : q \in Q(H^{\#})\} \leq M.$$

Let $q^* \in Q(H^{\#})$, then

$$u(t,x) \leq H^{\#}(q^*) \vee g(x - tq^*) = g(x - tq^*) \leq K < +\infty, \ \forall (t,x) \in (0,T] \times B(0;r).$$

In view of Theorem 2.1 and Lemma 2.2 in [10], we can take a constant $N > 0$ such that

$$H^{\#}(z) > K, \quad \forall |z| \geq N.$$

Hence, the infimum in (33) has to be taken over the ball $B(0;N)$ for $(t,x) \in (0,T] \times B(0;r)$. Thus

$$\begin{aligned}
u(t,x) &= \inf_{|z|\le N}\{H^{\#}(z)\vee g(x-tz)\}\wedge K \\
&= \inf_{|z|\le N}\{(H^{\#}(z)\vee g(x-tz))\wedge K\} \\
&= \min_{|z|\le N}\{(H^{\#}(z)\vee g(x-tz))\wedge K\} \\
u(t,x) &= \min_{|z|\le N}\{H^{\#}(z)\vee g(x-tz)\}, \quad \forall (t,x)\in(0,T]\times B(0;r),
\end{aligned}$$

since the function $z\mapsto (H^{\#}(z)\vee g(x-tz))\wedge K,\ z\in\bar{B}(0;N)$ is finite (bounded) and lower semicontinuous on a compact set. That means $h(t,x)\neq\emptyset$ and

$$\|h(t,x)\|\le N, \quad \forall (t,x)\in(0,T]\times B(0;r). \tag{34}$$

For any $(t,x),(t',x')\in(0,T]\times B(0;r)$, from (34) let us choose $\xi\in h(t,x)$, $|\xi|\le N$. Then,

$$\begin{aligned}
u(t',x')-u(t,x) &= \inf_{z\in\mathbb{R}^n}\{H^{\#}(z)\vee g(x'-t'z)\} - H^{\#}(\xi)\vee g(x-t\xi) \\
\le H^{\#}(\xi)\vee g(x'-t'\xi) &- H^{\#}(\xi)\vee g(x-t\xi) \le |g(x'-t'\xi)-g(x-t\xi)|.
\end{aligned} \tag{35}$$

Interchanging (t,x) and (t',x'), we obtain

$$u(t,x)-u(t',x')\le |g(x'-t'\xi')-g(x-t\xi')|. \tag{36}$$

From (35) and (36), we have

$$\lim_{(t',x')\to(t,x)} u(t',x') = u(t,x), \quad \forall (t,x)\in(0,T]\times B(0,r).$$

Therefore $u\in C(U)$.

Further, we will prove that $u(t,x)$ is actually a viscosity solution of (30). For convenience, let us show that $u(t,x)$ given by (32) can be expressed for any $x\in\mathbb{R}^n$ and $0\le s<t$ as follows

$$u(t,x) = \min_{y\in\mathbb{R}^n}\Big\{H^{\#}\Big(\frac{x-y}{t-s}\Big)\vee u(s,y)\Big\}. \tag{37}$$

Indeed, arguing as in [10, pp. 56-57], it holds true that

$$u(t,x) \le \min_{y\in\mathbb{R}^n}\Big\{H^{\#}\Big(\frac{x-y}{t-s}\Big)\vee u(s,y)\Big\}.$$

To get (37), it is sufficient to show

$$u(t,x) \ge \min_{y\in\mathbb{R}^n}\Big\{H^{\#}\Big(\frac{x-y}{t-s}\Big)\vee u(s,y)\Big\}. \tag{38}$$

Let us choose $q\in\mathbb{R}^n$ such that $u(t,x)=H^{\#}((x-q)/t)\vee g(q)$. Then

$$\begin{aligned}
&\min_{y\in\mathbb{R}^n}\Big\{H^{\#}\Big(\frac{x-y}{t-s}\Big)\vee u(s,y)\Big\} \\
&= \min_{y\in\mathbb{R}^n}\Big\{H^{\#}\Big(\frac{x-y}{t-s}\Big)\vee \min_{z\in\mathbb{R}^n}\Big(H^{\#}(\frac{y-z}{s})\vee g(z)\Big)\Big\}
\end{aligned}$$

$$\le \min_{y\in\mathbb{R}^n} \Big\{ H^{\#}\Big(\frac{x-y}{t-s}\Big) \vee H^{\#}\Big(\frac{y-q}{s}\Big) \vee g(q) \Big\}$$
$$\le \min_{\{y=x-\frac{t-s}{t}(x-q)\}} \Big\{ H^{\#}\Big(\frac{x-y}{t-s}\Big) \vee H^{\#}\Big(\frac{y-q}{s}\Big) \vee g(q) \Big\}$$
$$\le H^{\#}\Big(\frac{x-q}{t}\Big) \vee H^{\#}\Big(\frac{x-q}{t}\Big) \vee g(q) = u(t,x),$$

i.e., (38) is verified.
i) u is a subsolution, i.e., if $\varphi \in C^1(U)$ and $u-\varphi$ attains its local maximum at $(t_0,x_0) \in U$, then

$$\frac{\partial\varphi(t_0,x_0)}{\partial t} + H(u(t_0,x_0), D_x\varphi(t_0,x_0)) \le 0.$$

Assume the contrary, that u is not a subsolution. Then, there exist $\epsilon_0 > 0$, $(t_0,x_0) \in U$, a neighborhood $V(t_0,x_0)$ of (t_0,x_0) and $\varphi \in C^1(U)$ so that $u-\varphi$ attains its maximum M at (t_0,x_0) on $V(t_0,x_0)$:

$$u-\varphi \le M, \quad (t,x) \in V(t_0,x_0), \quad u(t_0,x_0) - \varphi(t_0,x_0) = M,$$

and

$$\frac{\partial\varphi(t_0,x_0)}{\partial t} + H(u(t_0,x_0), D_x\varphi(t_0,x_0)) > \epsilon_0 > 0.$$

Therefore, using $H = H^{\#*}$ we conclude that there exists $\delta > 0$ so that

$$\frac{\partial\varphi(t_0,x_0)}{\partial t} + H(u(t_0,x_0)-\delta, D_x\varphi(t_0,x_0))$$
$$= \frac{\partial\varphi(t_0,x_0)}{\partial t} + \sup_{\{q:H^{\#}(q)\le u(t_0,x_0)-\delta\}} \big\langle D_x\varphi(t_0,x_0), q\big\rangle > \epsilon_0.$$

Thus, we can take $q_0 \in \mathbb{R}^n$, $H^{\#}(q_0) \le u(t_0,x_0) - \delta$, such that

$$\frac{\partial\varphi(t_0,x_0)}{\partial t} + \big\langle D_x\varphi(t_0,x_0), q_0\big\rangle > \epsilon_0 > 0. \tag{39}$$

Set $\mu = t_0 - t,\ x = x_0 - \mu q_0$ in the inequality obtained from (37) to get

$$u(t_0,x_0) \le H^{\#}(\frac{x_0-x}{t_0-t}) \vee u(t,x) = H^{\#}(q_0) \vee u(t_0-\mu, x_0-\mu q_0).$$

Since $H^{\#}(q_0) \le u(t_0,x_0) - \delta < u(t_0,x_0)$, by the continuity of $u(t,x)$ at (t_0,x_0), there exists a number $\mu' > 0$ so that, for all $0 < \mu < \mu'$

$$u(t_0-\mu, x_0-\mu q_0) - \delta/2 > u(t_0,x_0) - \delta \ge H^{\#}(q_0).$$

Let us choose a μ'', $0 < \mu'' < \mu'$ so that $(t_0-\mu'', x_0-\mu'' q_0)$ still belongs to $V(t_0,x_0)$. Then,

$$\varphi(t_0,x_0)+M=u(t_0,x_0)\le H^{\#}(q_0)\vee u(t_0-\mu,x_0-\mu q_0)$$
$$u(t_0-\mu,x_0-\mu q_0)\le \varphi(t_0-\mu,x_0-\mu q_0)+M,\quad \forall\mu, 0<\mu<\mu''.$$

This implies

$$\varphi(t_0-\mu,x_0-\mu q_0)-\varphi(t_0,x_0)\ge 0,\quad \forall\mu, 0<\mu<\mu''.$$

Equivalently,

$$\frac{\varphi(t_0-\mu,x_0-\mu q_0)-\varphi(t_0,x_0)}{-\mu}\le 0,\quad \forall\mu, 0<\mu<\mu''.$$

Letting $\mu\to 0+$ in the last above inequality, we have

$$\frac{\partial\varphi(t_0,x_0)}{\partial t}+\big\langle D_x\varphi(t_0,x_0),q_0\big\rangle\le 0,$$

which contradicts (39). This proves that u is a subsolution.
ii) u is a supersolution, i.e., if $\varphi\in C^1(U)$ and $u-\varphi$ attains its local minimum at $(t_0,x_0)\in U$, then

$$\frac{\partial\varphi(t_0,x_0)}{\partial t}+H(u(t_0,x_0),D_x\varphi(t_0,x_0))\ge 0.$$

Assume the contrary, then there exist $\epsilon_0>0$, $(t_0,x_0)\in U$, a neighborhood $V(t_0,x_0)$ of (t_0,x_0) and $\varphi\in C^1(U)$ so that $u-\varphi$ attains its minimum m at (t_0,x_0) on $V(t_0,x_0)$:

$$u-\varphi\ge m,\quad (t,x)\in V(t_0,x_0),\quad u(t_0,x_0)-\varphi(t_0,x_0)=m$$

and

$$\frac{\partial\varphi(t_0,x_0)}{\partial t}+H(u(t_0,x_0),D_x\varphi(t_0,x_0))<-\epsilon_0<0.$$

Thus, there is a $\delta>0$ so that

$$\frac{\partial\varphi(t_0,x_0)}{\partial t}+H(u(t_0,x_0)+\delta,D_x\varphi(t_0,x_0))<-\epsilon_0.$$

Then, using $H=H^{\#*}$, we obtain

$$\frac{\partial\varphi(t_0,x_0)}{\partial t}+\sup_{\{q:H^{\#}(q)\le u(t_0,x_0)+\delta\}}\big\langle D_x\varphi(t_0,x_0),q\big\rangle<-\epsilon_0.$$

Equivalently,

$$\begin{aligned}\frac{\partial\varphi(t_0,x_0)}{\partial t}+\big\langle D_x\varphi(t_0,x_0),q\big\rangle<-\epsilon_0<0,&\\ \forall q\in\mathbb{R}^n,\ H^{\#}(q)\le u(t_0,x_0)+\delta.&\end{aligned}\tag{40}$$

On the other hand, from (37) we can write $u(t,x)$ in the form

$$u(t,x)=\min_{z\in\mathbb{R}^n}\{H^{\#}(z)\vee u(s,x-(t-s)z)\}.$$

Hence, for every $0<t<t_0$, we may select $q\in\mathbb{R}^n$ such that

$$u(t_0,x_0)=H^{\#}(q)\vee u(t,x_0-(t_0-t)q)\ge H^{\#}(q).$$

Take an arbitrary sequence

$$(t_i)_i \subset (0, t_0),\ t_i \to t_0$$

as $i \to \infty$ and put

$$\nu_i = t_0 - t_i > 0,\ i \in \mathbb{N}.$$

For every $i \in \mathbb{N}$, we choose a $q_i \in \mathbb{R}^n$ such that

$$\begin{aligned} u(t_0, x_0) &= H^{\#}(q_i) \vee u(t_i, x_0 - (t_0 - t_i)q_i) \\ &= H^{\#}(q_i) \vee u(t_0 - \nu_i, x_0 - \nu_i q_i) \geq H^{\#}(q_i). \end{aligned}$$

It follows from Theorem 2.1 and Lemma 2.2 in [10] that the sequence $(q_i)_i$ must be bounded. Without loss of generality, we can suppose that $q_i \to q_0$ as $i \to \infty$. Therefore,

$$(t_0 - \nu_i, x_0 - \nu_i q_i) \to (t_0, x_0)$$

as $i \to \infty$. By the lower semicontinuity of $H^{\#}$ and the continuity of u, we see that

$$\begin{aligned} u(t_0, x_0) \leq H^{\#}(q_0) \vee u(t_0, x_0) &\leq \lim_{i \to \infty} \inf \{H^{\#}(q_i) \vee u(t_0 - \nu_i, x_0 - \nu_i q_i)\} \\ &\leq \lim_{i \to \infty} \inf u(t_0, x_0) = u(t_0, x_0). \end{aligned}$$

This means

$$H^{\#}(q_0) \leq u(t_0, x_0) < u(t_0, x_0) + \delta. \tag{41}$$

Since $(t_0 - \nu_i, x_0 - \nu_i q_i) \to (t_0, x_0)$ as $i \to \infty$, there exists a number $N_1 \in \mathbb{N}$ such that $(t_0 - \nu_i, x_0 - \nu_i q_i) \in V(t_0, x_0)$, for all $i > N_1$. Hence,

$$\begin{aligned} \varphi(t_0, x_0) + m = u(t_0, x_0) &= H^{\#}(q_i) \vee u(t_0 - \nu_i, x_0 - \nu_i q_i) \\ \geq u(t_0 - \nu_i, x_0 - \nu_i q_i) &\geq \varphi(t_0 - \nu_i, x_0 - \nu_i q_i) + m,\ \forall i > N_1. \end{aligned}$$

Consequently,

$$\varphi(t_0 - \nu_i, x_0 - \nu_i q_i) - \varphi(t_0, x_0) \leq 0,\ \forall i > N_1.$$

Now, it follows from Mean Value Theorem that

$$\begin{aligned} &\varphi(t_0 - \nu_i, x_0 - \nu_i q_i) - \varphi(t_0, x_0) = \left\langle D\varphi(t_0 - \theta_i, x_0 - \theta_i q_i), (-\nu_i, -\nu_i q_i) \right\rangle \\ &= -\nu_i \left\{ \frac{\partial \varphi}{\partial t}(t_0 - \theta_i, x_0 - \theta_i q_i) + \left\langle D_x \varphi(t_0 - \theta_i, x_0 - \theta_i q_i), q_i \right\rangle \right\}, \end{aligned}$$

where $0 \leq \theta_i \leq \nu_i,\ \forall i \in \mathbb{N}$. This implies

$$\frac{\partial \varphi}{\partial t}(t_0 - \theta_i, x_0 - \theta_i q_i) + \left\langle D_x \varphi(t_0 - \theta_i, x_0 - \theta_i q_i), q_i \right\rangle \geq 0,\ \forall i > N_1.$$

On the other hand, we have

$$\begin{aligned} &|(t_0 - \theta_i) - t_0| + |(x_0 - \theta_i q_i) - x_0| + |q_i - q_0| \\ &= \theta_i + \theta_i |q_i| + |q_i - q_0| \leq \nu_i + \nu_i |q_i| + |q_i - q_0| \to 0 \quad \text{as} \quad i \to \infty, \end{aligned}$$

i.e., $(t_0 - \theta_i, x_0 - \theta_i q_i, q_i) \to (t_0, x_0, q_0)$ as $i \to \infty$. Since $\varphi \in C^1$, it follows that there exists a number $N_2 \in \mathbb{N}$ such that

$$\begin{aligned}&\frac{\partial\varphi(t_0, x_0)}{\partial t} + \big\langle D_x\varphi(t_0, x_0), q\big\rangle + \epsilon_0/2\\ >&\frac{\partial\varphi}{\partial t}(t_0 - \theta_i, x_0 - \theta_i q_i) + \big\langle D_x\varphi(t_0 - \theta_i, x_0 - \theta_i q_i), q_i\big\rangle, \quad \forall i > N_2.\end{aligned}$$

Putting $N = \max\{N_1, N_2\}$, for all $i > N$, we obtain

$$\begin{aligned}&\frac{\partial\varphi(t_0, x_0)}{\partial t} + \big\langle D_x\varphi(t_0, x_0), q\big\rangle + \epsilon_0/2\\ >\ &\frac{\partial\varphi}{\partial t}(t_0 - \theta_i, x_0 - \theta_i q_i) + \big\langle D_x\varphi(t_0 - \theta_i, x_0 - \theta_i q_i), q_i\big\rangle \geq 0.\end{aligned}$$

Consequently,

$$\frac{\partial\varphi(t_0, x_0)}{\partial t} + \big\langle D_x\varphi(t_0, x_0), q\big\rangle \geq -\epsilon_0/2. \tag{42}$$

From (41) and (42), we have the fact that conflicts with (40). This proves that u is a supersolution.

Finally, we will check that $u(t, x)$ satisfies the initial data (31)

$$\lim_{(t,x)\to(0,x_0)} u(t, x) = g(x_0), \quad \forall x_0 \in \mathbb{R}^n. \tag{43}$$

Indeed, for every $x_0 \in \mathbb{R}^n$, using the representation (33), for some $q^* \in Q(H^{\#})$, $|q^*| \leq M$,

$$u(t, x) \leq H^{\#}(q^*) \vee g(x - tq^*) = g(x - tq^*), \quad \forall (t, x) \in U.$$

Consequently,

$$\lim_{(t,x)\to(0,x_0)} \sup\, u(t, x) \leq \lim_{(t,x)\to(0,x_0)} g(x - tq^*) = g(x_0). \tag{44}$$

Let $r > 0,\ N > 0$ be corresponding to (34), then for $(t, x) \in (0, T] \times B(0; r)$, $|\xi| \leq N$,

$$u(t, x) = H^{\#}(\xi) \vee g(x - t\xi) \geq g(x - t\xi).$$

Letting $(t, x) \to (0, x_0)$, we have

$$\lim_{(t,x)\to(0,x_0)} \inf\, u(t, x) \geq \lim_{(t,x)\to(0,x_0)} g(x - t\xi) = g(x_0). \tag{45}$$

A combination of (44) and (45) gives us (43). The proof of Theorem 3.1 is complete. ■

Corollary 3.3. *Assume* (A1), (A2) *and* $g \in BUC(\mathbb{R}^n)$. *Then* u *defined by* (32) *is the unique viscosity solution of Problem* (30) − (31) *in the space* $BUC(\bar{U})$.

Proof. Under the above conditions, the Problem (30) - (31) has the unique viscosity solution u defined by (32), (see, for example, Ishii [23], Lions [38]). We remain to prove that $u \in BUC(\bar{U})$. First,

$$u(t,x) \leq g(x - tq^*) \leq \|g\|_{L^\infty},$$
$$u(t,x) = H^\#(\xi) \vee g(x - t\xi) \geq g(x - t\xi) \geq -\|g\|_{L^\infty}.$$

That means u is bounded with $\|u\|_{L^\infty} \leq \|g\|_{L^\infty}$. Re-examinating the proof of Theorem 3.1, we see that the constant N in the formula (34) can be chosen to be independent of $(t,x) \in U$. Concretely,

$$H^\#(z) > \|g\|_{L^\infty}, \quad \forall |z| > N.$$

Thus, we obtain the representation (34) for all $(t,x) \in U$ with a fixed constant N. Since N is common and g is uniformly continuous, the inequalities (35), (36) imply the uniform continuity of u. ∎

We illustrate our result by the following examples.

Example 5.

$$\frac{\partial u}{\partial t} + \Big|\frac{\partial u}{\partial x}\Big|.\frac{u + |u|}{2} = 0, \quad u(0,x) = |x|, \quad T > t > 0, \quad x \in \mathbb{R}. \tag{46}$$

Of course, the initial data $g(x) = |x|$ is not bounded. Problem (46) has a viscosity solution derived by the formula (32),

$$u(t,x) = \frac{|x|}{1+t}, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

Example 6.

$$\frac{\partial u}{\partial t} + \Big|\frac{\partial u}{\partial x}\Big|.\frac{u + |u|}{2} = 0, \quad u(0,x) = |x|^2, \quad T > t > 0, \quad x \in \mathbb{R}. \tag{46}$$

Clearly, $g(x) = |x|^2$ is not Lipschitz in $\mathbb{R}$. According to the formula (32), Problem (47) has an explicit viscosity solution as follows

$$u(t,x) = \frac{2x^2}{1 + 2t|x| + \sqrt{1 + 4t|x|}}, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

6. Viscosity Solutions for Hamiltonians Depending on t, u, Du with Quasiconvex Initial Data

4.1. Preliminaries

Given a function $f : \mathbb{R}^n \to \mathbb{R}$, set

$$\gamma_* = \inf_{x \in \mathbb{R}^n} f(x).$$

Consider the multifunction L defined by

$$L : (\gamma_*, +\infty) \to 2^{\mathbb{R}^n} \setminus \emptyset$$
$$a \mapsto E_{f,a},$$

which will be accordingly called the *generated multi* (by f).

Definition 4.1. *The function f is said to have L-l.s.c. property if the generated multi L is $\epsilon - \delta$-l.s.c.*

Let f satisfy the condition

$$\lim_{|x|\to\infty} f(x) = +\infty, \tag{48}$$

then $E_{f,a}$ is bounded for any $a \in \mathbb{R}$. Therefore, by virtue of Proposition 2.1 in [77] we obtain

Proposition 4.2. *If f is continuous, and satisfies the growth condition* (48)*, then the multi L is $\epsilon - \delta$-l.s.c. if and only if L is l.s.c.*

The class of functions having L-l.s.c. property is broad enough as seen by the following.

Theorem 4.3. *Let the continuous function f do not attain its local minimum in any open subset of $\mathbb{R}^n \setminus \operatorname{Argmin} f$, and let f satisfy the growth condition* (48)*. Then, f has L-l.s.c. property.*

Proof. In view of Proposition 4.2, it is sufficient to prove that L is l.s.c. To this end, assume $(\gamma_i)_i \subset (\gamma_*, +\infty)$, V is open in $\mathbb{R}^n$ and $\gamma_i \to \gamma \in (\gamma_*, +\infty)$, and $L(\gamma) \cap V \neq \emptyset$. We have to show that $L(\gamma_i) \cap V \neq \emptyset$ for all large i.

Actually, since $L(\gamma) \cap V \neq \emptyset$, there exists $x \in L(\gamma) \cap V$. By the assumption that f does not attain its minimum in V, there is $x' \in V$ such that

$$f(x') < f(x) \leq \gamma.$$

Since $\gamma_i \to \gamma$, for $\gamma - f(x') > 0$, there exists $N \in \mathbb{N}$ such that for all $i > N$ we have

$$\gamma - \gamma_i < \gamma - f(x').$$

Equivalently,

$$\gamma_i > f(x') \quad \text{for all} \quad i > N,$$

which means $x' \in E_{f,\gamma_i} = L(\gamma_i)$ for all $i > N$. Hence, $L(\gamma_i) \cap V \neq \emptyset$ for all $i > N$. ∎

The next corollary is an immediate consequence of Theorem 4, Sec. 2, Chap. 9 in [42], its Remark and Theorem 4.3.

Corollary 4.4. *Given a continuous function f satisfying* (48)*. Assume that, either f is strictly quasiconvex or f is convex. Then, f has L-l.s.c. property.*

Lemma 4.5. *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex and has L-l.s.c. property. The family*

$$(f^*(\gamma, p))_{|p|=1}$$

is equilower semicontinuous on $(\gamma_*, +\infty)$, and $f^*(\gamma, p) = -\infty$ if $\gamma < \gamma_*$.

Proof. Let $\gamma_0 \in (\gamma_*, +\infty)$ and let $\epsilon > 0$ be arbitrary. By the $\epsilon - \delta$-lower semicontinuity of L, there exists $0 < \delta < \gamma_0 - \gamma_*$ such that

$$L(\gamma_0) \subset L(\gamma) + B_\epsilon(0) \quad \text{on} \quad (\gamma_0 - \delta, \gamma_0 + \delta).$$

Hence, it holds for $|p| = 1$,

$$\begin{aligned} f^*(\gamma_0, p) &= \sup(\langle p, x\rangle, f(x) \le \gamma_0) \\ &= \sup(\langle p, x\rangle, x \in L(\gamma_0)) \\ &\le \sup(\langle p, x\rangle, x \in L(\gamma) + B_\epsilon(0)) \\ &= \sup(\langle p, x\rangle, x = y + z,\ y \in L(\gamma),\ z \in B_\epsilon(0)) \\ &\le \sup(\langle p, y\rangle, y \in L(\gamma)) + \epsilon|p| \\ f^*(\gamma_0, p) &\le f^*(\gamma, p) + \epsilon, \quad \forall \gamma \in (\gamma_0 - \delta, \gamma_0 + \delta). \end{aligned}$$

The last inequality implies that $(f^*(\gamma, p))_{|p|=1}$ is equilower semicontinuous at γ_0.

The second statement is obvious. ∎

4.2. The Formula for Quasiconvex Data

We consider the Cauchy problem for some first order nonlinear PDEs, where the Hamiltonians depend on t, u and $D_x u$, namely

$$\frac{\partial u(t,x)}{\partial t} + H(t, u(t,x), D_x u(t,x)) = 0, \quad (t,x) \in U = (0,T) \times \mathbb{R}^n, \qquad (49)$$

$$u(0,x) = g(x), \quad x \in \mathbb{R}^n. \qquad (50)$$

The following conditions will be imposed upon the Hamiltonian and the initial data.

(B1) The initial function $g \in C(\mathbb{R}^n)$ is quasiconvex, has L-l.s.c. property, and satisfies the growth condition (48):

$$g(x) \to +\infty \quad \text{as} \quad |x| \to +\infty$$

(B2) The Hamiltonian $H : [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous and

(i) $H(t, \gamma, \Lambda p) = \Lambda H(t, \gamma, p)$ for all $(t, \gamma, p) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n, \Lambda \ge 0$;

(ii) $H(t, \gamma, p)$ is nondecreasing in $\gamma \in \mathbb{R}$ for each $(t,p) \in [0,T] \times \mathbb{R}^n$.

(B3) The Hamiltonian H satisfies one of the two following conditions:

(i) For every fixed $t_0 \in (0,T)$, there exists a function $h : [0,T] \times \mathbb{R} \to \mathbb{R}$, $h(t,\gamma)$ is positive for almost every $t \in (0,T)$, $h(.,\gamma)$ is integrable for any γ, such that

$$H(t, \gamma, p) = h(t, \gamma) H(t_0, \gamma, p), \quad \forall (t, \gamma, p) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n.$$

(ii) If $0 \le a_i \le 1,\ |p_i| = 1,\ i = 1, ..., m$ and $\sum_{i=1}^m a_i = 1$, then

$$H(t, \gamma, \sum_{i=1}^m a_i p_i) \ge \sum_{i=1}^m a_i H(t, \gamma, p_i),$$

for all $(t, \gamma) \in [0,T] \times \mathbb{R}$.

The expected viscosity solution is given by

$$u(t,x) := \inf\left\{\gamma \in \mathbb{R} : \sup_{p\in\mathbb{R}^n}\Big(\langle p,x\rangle - g^*(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\Big) \le 0\right\}, \tag{51}$$

$(t,x) \in U$.

Remark 3. Observe that the continuity of g and the growth condition (48) yield that $\gamma_* := \inf\{g(x) : x \in \mathbb{R}^n\}$ is finite.

Remark 4. In view of Lemma 4.5, it is easy to see that $u(t,x) \ge \gamma_*$ for all $(t,x) \in [0,T]\times\mathbb{R}^n$.

By virtue of Prop. 2.1 and Cor. 2.6 in [9] and the assumption (B2), the function

$$p \mapsto \langle p,x\rangle - g^*(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau$$

is positively homogeneous of degree 1. Hence, we may restrict the class of p's over which the supremum is taken to the closed ball $\bar{B}(0;1) = \{p \in \mathbb{R}^n : |p| \le 1\}$, or to the spherical surface $S(0;1) = \{p \in \mathbb{R}^n : |p| = 1\}$. That is,

$$\begin{aligned} u(t,x) &= \inf\left\{\gamma \in \mathbb{R} : \sup_{|p|\le 1}\Big(\langle p,x\rangle - g^*(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\Big) \le 0\right\} \\ &= \inf\left\{\gamma \in \mathbb{R} : \sup_{|p|=1}\Big(\langle p,x\rangle - g^*(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\Big) \le 0\right\}. \end{aligned} \tag{52}$$

The facts for $\gamma \ge \gamma_*, g^*(\gamma,.)$ is finite, convex and so conditions, and $g^*(\gamma,p) \to +\infty$ as $\gamma \to +\infty$ guarantee that $u(t,x)$ is not equal to $+\infty$ and so well-defined. Let us affirm the continuity of $u(t,x)$ as in the following lemma.

Lemma 4.6. *Let* (B2) *hold, and let $g \in C(\mathbb{R}^n)$ be quasiconvex and satisfy* (48). *Then, the function $u(t,x)$ defined by* (51) *is continuous and satisfies*

$$u(0,x_0) = \lim_{(t,x)\to(0,x_0)} u(t,x) = g^{*\#}(x_0) = g(x_0).$$

Proof. From Prop. 2.1 and Cor. 2.6 in [9], $g^*(\gamma,p)$ is upper semicontinuous in $\gamma \in \mathbb{R}$, and nondecreasing. Hence, it follows from Theorem 6, App. C 1 in [42], that the function

$$\gamma \mapsto \sup_{|p|=1}\Big(\langle p,x\rangle - g^*(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\Big)$$

is nonincreasing and lower semicontinuous, and so continuous from the right. Using the representation (52), we will show that, for any $\gamma_0 \in \mathbb{R}$

$$E_{u,\gamma_0} = \big\{(t,x) \in [0,T]\times\mathbb{R}^n : u(t,x) \le \gamma_0\big\}$$

$$= \left\{(t,x) : \inf\left\{\gamma \in \mathbb{R} : \sup_{|p|=1}\Big(\langle p,x\rangle - g^*(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\Big) \le 0\right\} \le \gamma_0\right\}$$

$$= A := \Big\{(t,x) : \sup_{|p|=1} \Big(\big\langle p,x\big\rangle - g^*(\gamma_0,p) - \int\limits_0^t H(\tau,\gamma_0,p)d\tau\Big) \le 0\Big\}. \qquad (53)$$

Evidently, $A \subset E_{u,\gamma_0}$. The opposite inclusion can be obtained as follows. For $(t,x) \in E_{u,\gamma_0}$, if

$$\sup_{|p|=1} \Big(\big\langle p,x\big\rangle - g^*(\gamma_0+\epsilon_0,p) - \int\limits_0^t H(\tau,\gamma_0+\epsilon_0,p)d\tau\Big) > 0,$$

for some $\epsilon_0 > 0$, then, for all $\gamma < \gamma_0 + \epsilon_0$,

$$\sup_{|p|=1} \Big(\big\langle p,x\big\rangle - g^*(\gamma,p) - \int_0^t H(\tau,\gamma,p)d\tau\Big) > 0.$$

That is,

$$u(t,x) \ge \gamma_0 + \epsilon_0,$$

a contradiction. Hence, for all $\epsilon > 0$, we have

$$\sup_{|p|=1} \Big(\big\langle p,x\big\rangle - g^*(\gamma_0+\epsilon,p) - \int\limits_0^t H(\tau,\gamma_0+\epsilon,p)d\tau\Big) \le 0.$$

By the continuity from the right of the function on the left-hand side mentioned above, letting $\epsilon \to 0+$, we obtain

$$\sup_{|p|=1} \Big(\big\langle p,x\big\rangle - g^*(\gamma_0,p) - \int\limits_0^t H(\tau,\gamma_0,p)d\tau\Big) \le 0,$$

i.e., $(t,x) \in A$, and thus $E_{u,\gamma_0} \subset A$. Hence, (53) holds.

If $\gamma_0 < \gamma_*$, then $E_{u,\gamma_0} = \emptyset$, a closed set. Otherwise, by virtue of Theorem 6, App C1 in [42] and Berge's Maximum Theorem in [13, p.123], the function

$$v(t,x) := \sup_{|p|=1} \Big(\big\langle p,x\big\rangle - g^*(\gamma_0,p) - \int_0^t H(\tau,\gamma_0,p)d\tau\Big), \quad (t,x) \in [0,T] \times \mathbb{R}^n$$

is continuous, since $\gamma_0 \ge \gamma_*$. Hence, from (53), the level set E_{u,γ_0} is closed. So, u is lower semicontinuous as well.

We have left to prove u is upper semicontinuous. Denote

$$\partial E_{g,\gamma} = \{x \in \mathbb{R}^n : g(x) = \gamma\},$$

which is the boundary of the level set $E_{g,\gamma}$. Since $g(x) \to +\infty$ as $|x| \to \infty$ and g is continuous, the level sets $E_{g,\gamma}$ and the boundaries $\partial E_{g,\gamma}$ are compact sets in $\mathbb{R}^n$ for all $\gamma \in [\gamma_*, +\infty)$, and $E_{g,\gamma} = \emptyset$ if $\gamma < \gamma_*$.

To every fixed $(t_0,x_0) \in [0,T] \times \mathbb{R}^n$, we put $\gamma_0 = u(t_0,x_0)$. Pay attention to Remark 4 that $\gamma_0 \ge \gamma_*$, so the level sets $E_{g,\gamma} \ne \emptyset$ for all $\gamma \ge \gamma_0$. For $\delta > 0$, since $E_{g,\gamma_0}, \partial E_{g,\gamma_0}$ are compact sets, we have

$$d := d\Big(E_{g,\gamma_0}, \partial E_{g,\gamma_0+\delta}\Big) = \inf_{x\in E_{g,\gamma_0}\ \ y\in\partial E_{g,\gamma_0+\delta}} |x-y|$$
$$= \min_{x\in E_{g,\gamma_0}\ \ y\in\partial E_{g,\gamma_0+\delta}} |x-y|.$$

If $d = 0$, then, there exists $\xi \in E_{g,\gamma_0} \cap \partial E_{g,\gamma_0+\delta}$, i.e., $g(\xi) \le \gamma_0$ and $g(\xi) = \gamma_0 + \delta$, a contradiction. Hence, $d > 0$. We will show that

$$E_{g,\gamma_0} + B(0,d) \subset E_{g,\gamma_0+\delta}. \tag{54}$$

Actually, if $x \in E_{g,\gamma_0} + B(0,d)$, then $x = y + z$, where $y \in E_{g,\gamma_0}$ and $|z| < d$. The facts $g(y + tz) \to +\infty$ as $t \to +\infty$, $g(y) \le \gamma_0$, and g is continuous imply that there exists $t' > 0$ such that

$$g(y + t'z) = \gamma_0 + \delta.$$

This means

$$y + t'z \in \partial E_{g,\gamma_0+\delta}.$$

Set $x' := y + t'z$ to get $t'd > t'|z| = d(y, x') \ge d$, and so $t' > 1$. Therefore, we have

$$x = y + z \in (y, y + t'z) \subset E_{g,\gamma_0+\delta},$$

where $(y, y + t'z)$ denotes the open line segment from y to $y + t'z$, since $E_{g,\gamma_0+\delta}$ is a convex set. Hence, (54) is verified.

Since E_{g,γ_0} is a compact set, the supremum in the definition of $g^*(\gamma_0, p)$ is attained, i.e., for every $p \in S(0;1)$, there exists $q \in \mathbb{R}^n$ such that

$$\begin{aligned} g^*(\gamma_0, p) &= \sup\big(\langle p, x\rangle : x \in E_{g,\gamma_0}\big) \\ &= \max\big(\langle p, x\rangle : x \in E_{g,\gamma_0}\big) \\ &= \langle p, q\rangle, \quad q \in E_{g,\gamma_0}. \end{aligned}$$

So, from (54) we have

$$\begin{aligned} g^*(\gamma_0 + \delta, p) &= \sup\big(\langle p, x\rangle : x \in E_{g,\gamma_0+\delta}\big) \\ &\ge \sup\big(\langle p, x\rangle : x \in E_{g,\gamma_0} + B(0,d)\big) \\ &\ge \sup\big(\langle p, q\rangle + \langle p, z\rangle : |z| < d\big) \\ &\ge g^*(\gamma_0, p) + d, \quad \forall p \in S(0;1). \end{aligned}$$

Hence, by the continuity of the function v mentioned above there exists a neighborhood V of (t_0, x_0), $V \subset [0,T] \times \mathbb{R}^n$ such that for all $(t,x) \in V$,

$$\begin{aligned} 0 &\ge \sup_{|p|=1} \Big(\langle p, x_0\rangle - g^*(\gamma_0, p) - \int_0^{t_0} H(\tau, \gamma_0, p)d\tau\Big) \\ &\ge \sup_{|p|=1} \Big(\langle p, x\rangle - g^*(\gamma_0, p) - \int_0^{t} H(\tau, \gamma_0, p)d\tau\Big) - d \\ &\ge \sup_{|p|=1} \Big(\langle p, x\rangle - g^*(\gamma_0 + \delta, p) - \int_0^{t} H(\tau, \gamma_0, p)d\tau\Big) \end{aligned}$$

$$\geq \sup_{|p|=1} \Big(\langle p, x \rangle - g^*(\gamma_0 + \delta, p) - \int_0^t H(\tau, \gamma_0 + \delta, p) d\tau \Big). \tag{55}$$

Now, it follows from (55) that

$$u(t, x) \leq \gamma_0 + \delta = u(t_0, x_0) + \delta, \quad \forall (t, x) \in V.$$

This means that $u(t, x)$ is upper semicontinuous at (t_0, x_0). Since (t_0, x_0) is arbitrarily chosen, $u(t, x)$ is upper semicontinuous on $[0, T] \times \mathbb{R}^n$.

Finally, by Prop. 2.1 and Cor. 2.6 in [9],

$$u(0, x) = (g^*(\gamma, p))^{\#}(x) = g^{*\#}(x) = g(x). \qquad \blacksquare$$

We are now in a position to state the main result of this section.

Theorem 4.7. *Let* (B1) - (B3) *hold. Then, the function* $u(t, x)$ *defined by* (51) *is a viscosity solution of Problem* (49) − (50).

Proof. It remains to prove that $u(t, x)$ in (51) satisfies the two inequalities. To do this, we need some lemmas. Assume that the point $(t_0, x_0) \in U$ is arbitrarily chosen, we set $\gamma_0 = u(t_0, x_0)$. From (53) we have

$$\begin{aligned} E_{u,\gamma_0} &= \Big\{ (t, x) \in [0, T] \times \mathbb{R}^n : u(t, x) \leq \gamma_0 \Big\} \\ &= \Big\{ (t, x) : \sup_{|p|=1} \Big(\langle p, x \rangle - g^*(\gamma_0, p) - \int_0^t H(\tau, \gamma_0, p) d\tau \Big) \leq 0 \Big\} \\ &= \bigcap_{|p|=1} \Big\{ (t, x) : \langle p, x \rangle - g^*(\gamma_0, p) - \int_0^t H(\tau, \gamma_0, p) d\tau \leq 0 \Big\}. \end{aligned}$$

Now we extend continuously the Hamiltonian H to the whole $n+1$-dimensional space as follows

$$\begin{aligned} H(t, \gamma_0, p) &= H(0, \gamma_0, p) \quad \text{if} \quad t < 0; \\ H(t, \gamma_0, p) &= H(T, \gamma_0, p) \quad \text{if} \quad t > T. \end{aligned}$$

We therefore consider the set γ defined by constraints of the form

$$G(t, x; p) := \langle p, x \rangle - g^*(\gamma_0, p) - \int_0^t H(\tau, \gamma_0, p) d\tau \leq 0, \ p \in S(0; 1), (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

Since (t_0, x_0) is an interior point of U, there exists a neighborhood V' of (t_0, x_0), $V' \subset U$. Hence,

$$E_{u,\gamma_0} \cap V' = \gamma \cap V'.$$

This means that $E_{u,\gamma}$ and γ have the same tangent cone and so share the same outer normal cone at (t_0, x_0). It is clear to check that the function $G(t, x; p)$ of (t, x) and their gradients $D_{(t,x)} G(t, x; p)$ are continuous in $(t, x), (t, x) \in \mathbb{R} \times \mathbb{R}^n$

and $p \in S(0;1)$. Let S_0 be the set of *active indices* at $(t_0, x_0) \in U$, that is, the set of indices $p \in S(0;1)$ such that $G(t_0, x_0; p) = 0$ (see Appendix in [76]). Further, we need some results concerning the set S_0.

Lemma 4.8. $S_0 \neq \emptyset$, *whenever* $\gamma_* \neq \gamma_0$.

Proof. Evidently, $\gamma_0 = u(t_0, x_0) \geq \gamma_*$. Hence, $\gamma_0 > \gamma_*$. Assume, for the sake of contradiction, that for some $\epsilon_0 > 0$

$$\sup_{|p|=1} \Big(\big\langle p, x_0 \big\rangle - g^*(\gamma_0, p) - \int_0^{t_0} H(\tau, \gamma_0, p) d\tau \Big) < -\epsilon_0 < 0. \qquad (56)$$

Since H is continuous, H is uniformly continuous on the compact set $[0, T] \times [\gamma_0 - 1, \gamma_0 + 1] \times S(0;1)$. Therefore the function

$$F(\gamma, p) := \int_0^{t_0} H(\tau, \gamma, p) d\tau$$

is uniformly continuous on $[\gamma_0 - 1, \gamma_0 + 1] \times S(0;1)$ and so the family

$$(F(\gamma, p))_{|p|=1}$$

is equicontinuous in $\gamma \in [\gamma_0 - 1, \gamma_0 + 1]$. Thus, in view of Lemma 4.5, the family

$$(T(\gamma, p))_{|p|=1}, T(\gamma, p) := \big\langle p, x_0 \big\rangle - g^*(\gamma, p) - \int\limits_0^{t_0} H(\tau, \gamma, p) d\tau$$

is equiupper semicontinuous in $\gamma \in [\gamma_0 - \delta', \gamma_0 + \delta']$ for some $0 < \delta' < \min\{1, \gamma_0 - \gamma_*\}$. It therefore follows from (56) that there exists $0 < \delta < \delta'$ such that

$$\begin{aligned} &\big\langle p, x_0 \big\rangle - g^*(\gamma, p) - \int_0^{t_0} H(\tau, \gamma, p) d\tau \\ \leq &\big\langle p, x_0 \big\rangle - g^*(\gamma_0, p) - \int_0^{t_0} H(\tau, \gamma_0, p) d\tau + \epsilon_0/2 \\ < &-\epsilon_0/2 < 0, \ \forall p \in S(0,1), \forall \gamma : \gamma_0 - \delta < \gamma < \gamma_0 + \delta. \end{aligned}$$

That is,

$$\begin{aligned} &\sup_{|p|=1} \Big(\big\langle p, x_0 \big\rangle - g^*(\gamma, p) - \int\limits_0^{t_0} H(\tau, \gamma, p) d\tau \Big) \\ \leq &-\epsilon_0/2 < 0, \ \forall \gamma : \gamma_0 - \delta < \gamma < \gamma_0 + \delta. \end{aligned}$$

In particular,

$$\sup_{|p|=1} \Big(\big\langle p, x_0 \big\rangle - g^*\Big(\gamma_0 - \frac{\delta}{2}, p\Big) - \int\limits_0^{t_0} H\Big(\tau, \gamma_0 - \frac{\delta}{2}, p\Big) d\tau \Big) \leq 0,$$

where $\delta > 0$. This contradicts the definition of $u(t_0, x_0) = \gamma_0$. The proof is complete. ∎

Lemma 4.9. (t_0, x_0) *is a regular point of* γ.

Proof. If $S_0 = \emptyset$ then

$$\sup_{|p|=1} \Big(\langle p, x_0 \rangle - g^*(\gamma_0, p) - \int_0^{t_0} H(\tau, \gamma_0, p) d\tau \Big) < 0.$$

As seen the continuity of the function v in the proof of Lemma 4.5, there exists a neighborhood V of (t_0, x_0) such that

$$\sup_{|p|=1} \Big(\langle p, x \rangle - g^*(\gamma_0, p) - \int_0^{t} H(\tau, \gamma_0, p) d\tau \Big) < 0,$$

whenever $(t, x) \in V$. Hence,

$$u(t, x) \leq \gamma_0, \quad \forall (t, x) \in V.$$

Consequently, $V \subset E_{u,\gamma_0}$. Therefore, (t_0, x_0) is an interior point of E_{u,γ_0} as well as γ. This leads to the conclusion.

Otherwise, we have $DG(t_0, x_0; p) = (-H(t_0, \gamma_0, p), p) \neq 0$ for all $|p| = 1$. This means that for the vector $h = (H(t_0, \gamma_0, p_0), -p_0)$, for some $p_0 \in S_0$, it holds

$$\big\langle (-H(t_0, \gamma_0, p_0), p_0), h \big\rangle \leq -|p_0|^2 = -1 < 0,$$

i.e., (t_0, x_0) is a normal point of γ. In view of Theorem 11.2 in [20], (t_0, x_0) therefore is a regular point of γ. ■

From Lemma 4.8, the outer normal cone of γ and so of E_{u,γ_0} at (t_0, x_0) is given by

$$\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}) = \Big\{ \sum_{i=1}^m \Lambda_i DG(t_0, x_0; p_i) : \Lambda_i \geq 0, p_i \in S_0, 1 \leq i \leq m \leq n+1 \Big\}.$$

■

Lemma 4.10. *Let*

$$S^* = \{p \in \mathbb{R}^n : G(t_0, x_0; p) = \sup_{|q|=1} G(t_0, x_0; q)\}$$

denote the set where the sup *is achieved. Then $S_0 \subset S^*$, and if $(\alpha_i)_i$ is any set of m numbers with $\alpha_i \geq 0$ and $\sum_{i=1}^m \alpha_i = 1$, we have*

$$\sum_{i=1}^m \alpha_i H(t_0, \gamma_0, p_i) \leq H\Big(t_0, \gamma_0, \sum_{i=1}^m \alpha_i p_i\Big). \tag{57}$$

Proof. Obviously, (57) holds if H satisfies the condition (B3 (ii)). Assume (B3 (i)), let

$$p' = \sum_{i=1}^m \alpha_i p_i.$$

Then

$$|p'| \leq \sum_{i=1}^m \alpha_i |p_i| = \sum_{i=1}^m \alpha_i = 1.$$

Since each p_i achieves the supremum, we have for $1 \leq i \leq m$,

$$\begin{aligned} 0 &= \langle p_i, x_0 \rangle - g^*(\gamma_0, p_i) - \int_0^{t_0} H(\tau, \gamma_0, p_i) d\tau \\ &\geq \langle p', x_0 \rangle - g^*(\gamma_0, p') - \int_0^{t_0} H(\tau, \gamma_0, p') d\tau. \end{aligned}$$

Multiplying by α_i and summing on i, we obtain

$$\begin{aligned} &\langle p', x_0 \rangle - \sum_{i=1}^m \alpha_i g^*(\gamma_0, p_i) - \int_0^{t_0} \sum_{i=1}^m \alpha_i H(\tau, \gamma_0, p_i) d\tau \\ \geq\ &\langle p', x_0 \rangle - g^*(\gamma_0, p') - \int_0^{t_0} H(\tau, \gamma_0, p') d\tau. \end{aligned}$$

Cancel the first line terms and use Jensen's inequality on $g^*(\gamma_0, p)$, since this function is convex in p, to get

$$- \int_0^{t_0} \sum_{i=1}^m \alpha_i H(\tau, \gamma_0, p_i) d\tau \geq - \int_0^{t_0} H(\tau, \gamma_0, p') d\tau = - \int_0^{t_0} H(\tau, \gamma_0, \sum_{i=1}^m \alpha_i p_i) d\tau.$$

Therefore, by (B3(i)) we have

$$- \int_0^{t_0} \sum_{i=1}^m \alpha_i h(\tau, \gamma_0) H(t_0, \gamma_0, p_i) d\tau \geq - \int_0^{t_0} h(\tau, \gamma_0) H(t_0, \gamma_0, p') d\tau.$$

Equivalently,

$$\begin{aligned} &- \Big(\sum_{i=1}^m \alpha_i H(t_0, \gamma_0, p_i) \Big) \int_0^{t_0} h(\tau, \gamma_0) d\tau \\ \geq\ &- H(t_0, \gamma_0, p') \int_0^{t_0} h(\tau, \gamma_0) d\tau \\ =\ &- H(t_0, \gamma_0, \sum_{i=1}^m \alpha_i p_i) \int_0^{t_0} h(\tau, \gamma_0) d\tau. \end{aligned}$$

Dividing out by $\int_0^{t_0} h(\tau, \gamma_0) d\tau > 0$, we obtain (57). The lemma is completely proved. ∎

We now continue to finish the proof of Theorem 4.7. In order to show that u is a supersolution, let φ be a real smooth function on U, so that $u - \varphi$ achieves a zero minimum at the point $(t_0, x_0) \in U$. We will prove that

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} + H(t_0, u(t_0, x_0), D_x \varphi(t_0, x_0)) \geq 0. \tag{58}$$

Since $u - \varphi$ has a zero minimum at (t_0, x_0), there exists a neighbourhood V'' of (t_0, x_0) such that

$$E_{u,\gamma_0} \cap V'' \subset E_{\varphi,\gamma_0} \cap V''$$

and so

$$\mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0}) = \mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0} \cap V'') \subset \mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0} \cap V'') = \mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}).$$

By virtue of [20, Lem. 7.1, Ch. 4], we have for the smooth function φ

$$\mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0}) = \{\Lambda D\varphi(t_0,x_0), \Lambda \geq 0\}.$$

Consequently,

$$D\varphi(t_0,x_0) \in \mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}).$$

Thus, $D\varphi(t_0,x_0)$ can be expressed in the terms of $\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0})$,

$$D\varphi(t_0,x_0) = \Big(\frac{\partial\varphi(t_0,x_0)}{\partial t}, D_x\varphi(t_0,x_0)\Big) = \sum_{i=1}^{m} \Lambda_i(-H(t_0,\gamma_0,p_i),p_i), \tag{59}$$

where $p_i \in S_0, \Lambda_i \geq 0, 1 \leq i \leq m$. Now, we set $\alpha_i = \Lambda_i/\Lambda'$, where

$$\Lambda' = \sum_{i=1}^{m} \Lambda_i.$$

Assume $\Lambda' > 0$, then

$$1 \geq \alpha_i \geq 0, \sum_{i=1}^{m} \alpha_i = 1.$$

By the homogeneity of degree 1 of $H(t,\gamma,.)$, from (59) we get

$$\Big(\frac{\partial\varphi(t_0,x_0)}{\partial t}, D_x\varphi(t_0,x_0)\Big) = \sum_{i=1}^{m} \alpha_i(-H(t_0,\gamma_0,\Lambda' p_i),\Lambda' p_i).$$

Hence, $D_x\varphi(t_0,x_0) = \sum_{i=1}^{m} \Lambda'\alpha_i p_i$. Applying Lemma 4.10, we have

$$\begin{aligned}\frac{\partial\varphi(t_0,x_0)}{\partial t} &= -\sum_{i=1}^{m} \alpha_i H(t_0,\gamma_0,\Lambda' p_i)\\ &\geq -H(t_0,\gamma_0,\sum_{i=1}^{m} \Lambda'\alpha_i p_i) = -H(t_0,\gamma_0,D_x\varphi(t_0,x_0)).\end{aligned}$$

After rearranging the terms, we obtain (58).

If $\Lambda' = 0$, then $D\varphi(t_0,x_0) = 0$. Since $H(t,\gamma,.)$ is positively homogeneous, $H(t,\gamma,0) = 0$ for all $(t,\gamma) \in [0,T] \times \mathbb{R}$. Hence,

$$\frac{\partial\varphi(t_0,x_0)}{\partial t} + H(t_0,u(t_0,x_0),D_x\varphi(t_0,x_0)) = 0 + H(t_0,\gamma_0,0) = 0.$$

Lastly, we remain to prove u is a subsolution. Let $u-\varphi$ achieve a zero maximum at the point (t_0,x_0) with a smooth function φ. We will show that

$$\frac{\partial\varphi(t_0,x_0)}{\partial t} + H(t_0,u(t_0,x_0),D_x\varphi(t_0,x_0)) \leq 0. \tag{60}$$

Since $u-\varphi$ has a zero maximum at (t_0,x_0), by an argument similar to the above, we have

$$\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}) \subset \mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0}).$$

If $u(t_0, x_0) = \gamma_*$, then (t_0, x_0) is a global minimum point of u. Hence,

$$\varphi(t_0, x_0) = u(t_0, x_0) \leq u(t, x) \leq \varphi(t, x), \quad \forall (t, x) \in W,$$

for some neighborhood W of (t_0, x_0), i.e., (t_0, x_0) is also a local minimum point of the smooth function φ and so $D\varphi(t_0, x_0) = 0$. As seen above, in this case, we get already the equality in (60).

Otherwise, in view of Lemma 4.9, the outer normal cone

$$\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}) \neq \{0\}.$$

On the other hand, the fact φ is a smooth function means that the outer normal cone consists of at most one ray. This implies that

$$\mathcal{N}_{(t_0,x_0)}(E_{u,\gamma_0}) = \mathcal{N}_{(t_0,x_0)}(E_{\varphi,\gamma_0})$$

and $D\varphi(t_0, x_0) \neq 0$. Hence, we obtain the expression of $D\varphi$:

$$\begin{aligned} D\varphi(t_0, x_0) &= \Big(\frac{\partial \varphi(t_0, x_0)}{\partial t}, D_x\varphi(t_0, x_0)\Big) \\ &= \Lambda_0(-H(t_0, \gamma_0, p), p) = (-H(t_0, \gamma_0, \Lambda_0 p), \Lambda_0 p), \end{aligned}$$

for some $\Lambda_0 > 0$. Then,

$$\frac{\partial \varphi(t_0, x_0)}{\partial t} = -H(t_0, \gamma_0, \Lambda_0 p) = -H(t_0, \gamma_0, D_x\varphi(t_0, x_0)).$$

We therefore obtain (60). The proof of Theorem 4.7 is complete. ∎

As a consequence of Corollary 4.4 and Theorem 4.7, we have

Corollary 4.11. *Let $g \in C(\mathbb{R}^n)$ be strictly quasiconvex, and let g satisfy* (48). *Let* (B1) - (B2) *hold. Then the formula* (51) *determines a viscosity solution of* (49) - (50).

Corollary 4.12. *Let the finite function g be convex and satisfy* (48). *Let* (B1) - (B2) *hold. Then the formula* (51) *defines a viscosity solution of* (49) $-$ (50).

Example 7. Consider the following Cauchy problem:

$$\frac{\partial u(t, x)}{\partial t} - (1 + t)^{-u(t,x)}|D_x u(t, x)| = 0, \quad (t, x) \in (0, T) \times \mathbb{R}, \tag{61}$$

$$u(0, x) = |x|, \quad x \in \mathbb{R}. \tag{62}$$

The formula (51) then gives us a viscosity solution of Problem (61) - (62) as follows, for $(t, x) \in (0, T) \times \mathbb{R}$,

$$u(t, x) = \gamma_0,$$

where $\gamma_0 \geq 0$ is the unique solution of $\gamma - \int_0^t (1 + \tau)^{-\gamma} d\tau - |x| = 0$.

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