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ON THE STABILITY OF SOME NON-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. *In this paper the ε-stability ([7], [8]) of the trivial solution of non-linear differential equations depending on a small parameter ε: $\frac{dx}{dt} = f(t, x) + R(t, x)$, $f(t, 0) \equiv 0$, where $f(t, x)$, $R(t, x)$ are Kamke functions ([9]) is proved.*

Consider nonlinear differential equations of the form

$$\begin{aligned} &\frac{dx}{dt} = f(t, x) + R(t, x), \\ &f(t, 0) \equiv 0, \end{aligned} \tag{1}$$

where $R(t, x)$ describes the permanent perturbations. The functions $f(t, x)$, $R(t, x)$ are defined and continuous in the set

$$\Omega = \{(t, x) : \|x\| < H;\ t \geq 0\},\ (0 < H \leq \infty),$$

and satisfy conditions of the uniqueness of solutions of the Cauchy problem in Ω.

Differential equations with perturbations are considered by many mathematicians (see [1], p. 232-263). Results of I. G. Malkin ([3]), S. I. Gorsin ([4], [5], [6]), G.N. Doubosin ([2]) are fundamental for this problem. Lyapunov function method is principal and very effective in studying the stability of equations of the form (1).

In this paper, we study the stability of the differential equations (with permanent perturbations) depending on a small parameter ε:

$$\dot{x} = f(t, x) + \varepsilon R(t, x), \tag{2}$$

where f, R satisfy the conditions of the existence and uniqueness of solution of the Cauchy problem in the set $G = \mathbf{R}^+ \times B_h$, where

$$\mathbf{R}^+ = \{t \in \mathbf{R} : t \geq 0\};\ B_h = \{x \in \mathbf{R}^n : \|x\| < h\}.$$

Definition 1 ([7,8]). The system (2) is called ε-stable iff for every $\alpha > 0$ there exists $\gamma = \gamma(\alpha) > 0$ and $\epsilon_0 = \epsilon_0(\alpha) > 0$ such that the solution of the systems (2) satisfies the condition $x(t, \varepsilon) \in B_\alpha$ for all $t \geq t_0$ whenever $x(t_0, \varepsilon) \in B_\gamma$ and $\varepsilon \in (0, \varepsilon_0]$.

We will consider, in connection with (2), the equation

$$\dot{y} = f(t, y)\,; \quad f(t, 0) \equiv 0\,. \tag{3}$$

We assume that there exists a Lypuanov function $v(t, y)$, satisfying in the set G the following inequalities

$$\begin{aligned} &a(\|y\|) \leq v(t, y) \leq b(\|y\|)\,,\\ &\frac{\partial v}{\partial t} + \frac{\partial v}{\partial y} f(t, y) \leq 0\,, \end{aligned} \tag{4}$$

where $a\,b$ are functions of the $\mathcal{K}$-class (i.e. the class of continuous, strictly increasing functions $h(r)$ in $\mathbf{R}^+$ and $h(0) = 0$. Then the solution $y \equiv 0$ of the system (2) is uniformly stable in the sense of Lypuanov.

Denote

$$z : \mathbf{R}^+ \times 2L \to \mathbf{R}^n\,, \quad (t, a) \mapsto z(t, a)$$

($2L$ is an open set in $\mathbf{R}^n$, and $a = (a_1, \ldots, a_n) \in 2L$, and call $\mathcal{M}$ the family of all functions z.

Definition 2 ([8]). The family $\mathcal{M}$ is called approximate with respect to the solution $y(t, t_0, x_0)$ of the system (3) with exact degree X as $\|x_0\| \to 0$ iff for any $t_0 \in \mathbf{R}^+$ and any $x_0 \in B_h$ $(h > 0)$ there is a vector $a_0 \in 2L$ such that for $z = z(t, a_0)$ the following conditions hold

1. $z(t_0, a_0) = x_0$;
2. There exists $T > 0$ such that for all $t_0 \in \mathbf{R}^+$ the inequality

$$\|y(t, t_0, x_0) - z(t, a_0)\| \leq X(\|x_0\|)N$$

hold for all $t \in [t_0, t_0 + T]$, where N depends only on T.

Denote such a function by $z(t, t_0, z_0)$

Definition 3 ([9]). Function

$$F : \mathbf{R}^+ \times \overline{B}_\delta \to \mathbf{R}^n\,,$$

where $\overline{B}_\delta = \{x \in \mathbf{R}^n : \|x\| \leq \delta\}$ is called a Kamke function if there exists a scalar continuous, positive function

$$\omega : \mathbf{R}^+ \times [0, \delta] \to \mathbf{R}^+\,; \ \omega(t, 0) = 0$$

such that

1) For each $(t, x_1), (t, x_2) \in \mathbf{R}^+ \times \overline{B}_\delta$,

$$\|F(t, x_1) - F(t, x_2)\| \leq \omega(t, \|x_1 - x_2\|)$$

2) The unique solution $u = u(t)$ of the differential equation

$$u' = \omega(t, u)$$

on any interval $(t_0, t_0 + \varepsilon]$ satisfying

$$u(t) \to 0 \text{ and } \frac{u(t)}{t - t_0} \to 0 \text{ as } t \to t_0 + 0$$

is $u(t) \equiv 0$.

Remark. Functions of the form

$$\omega(t, u) = g(t)\, h(u)\,, \tag{5}$$

where $g(t) \geq 0$ is continuous on any $t_0 < t \leq a$ and

$$\int_{t_0^+}^{+\infty} g(t)\, dt < +\infty\,,$$

$h(u)$ is continuous if $u \geq 0$, $h(0) = 0$, $h(u) > 0$ as $u > 0$ and

$$\int_{0^+}^{+\infty} \frac{du}{h(u)} = +\infty$$

are Kamke function. If

$$\int_{\tau}^{\tau+T} g(t)\, dt = L(T)$$

we say that function $g(t)$ satisfies condition (E).

We consider now ε-stability of the system (2). Denoting

$$\varphi(t, x) = \frac{\partial v}{\partial x}(t, x)\, R(t, x)\,,$$

we have the following theorem:

Theorem 1. *Assume that in the set G the following conditions are satisfied:*

1) *There exists a differential function satisfying the inequality (4).*

2) $f(t, x)$, $R(t, x)$, $\varphi(t, x)$ are Kamke functions of the form (5) corresponding to functions $g_1(t)\, h_1(u)$, $g_2(t)\, h_2(u)$, $g_3(t)\, h_3(u)$ respectively, where functions $g_1(t)$, $g_2(t)$, $g_3(t)$ satisfy the condition (E) with $L_1(T)$, $L_2(T)$, $L_3(T)$ and $h_2(u)$ is bounded.

3) For every $\eta > 0$ there exists $T = T(\eta) > 0$ such that for for all $\delta \in \mathcal{K}$,

$$\frac{1}{T}\int_{t_0}^{t_0+T} \varphi(t;\, z(t,\, t_0,\, x_0))\, dt \le -\delta(\|x_0\|) < 0$$

whenever $\|x_0\| \geq \eta$ and $t_0 \geq 0$.

4) The family $\mathcal{M}$ is approximate with respect to the solution $y(t;\, t_0,\, x_0)$ of the system (3) with exact degree χ as $\|x_0\| \to 0$. Moreover, $0 \le \lim\limits_{\alpha\to 0} \frac{h_3(\chi(\alpha))}{\delta(\alpha)} < \frac{T}{L_3 N}$.

Then the system (2) is ε-stable.

Proof. By the second assumption there exists a unique solution of the Cauchy problem for the equation (2) (see [9]). Let be given $\alpha > 0$ and $\alpha_1 \in (0,\, \alpha)$. Denoting $w = a(\alpha_1) \le b(\alpha_1)$ we have $\eta = b^{-1}(a(\alpha_1)) \le \alpha_1$ and

$$B_\eta \subset \Omega_t = \{x \in B_h : v(t,\, x) \le w\} \subset B_{\alpha_1}\,,$$

$$\Gamma_t = \{x \in B_h : v(t,\, x) = w\} \text{ is the bounded of } \Omega_t\,.$$

Let us consider a trajectory $x(t)$ starting from the neighbourhood B_η (i.e. $v(t_0,\, x(t_0)) \in B_\eta$). we will prove that the trajectory $x(t)$ remains in B_α if $\varepsilon = \varepsilon(\alpha) > 0$ is sufficiently small. Indeed assume that there exists a moment $\tau > 0$ such that $x(\tau) \in \Gamma_\tau$, that is $v(\tau,\, x(\tau)) = w$ and the trajectory leaves the set Ω_t as $t > \tau$. Then we have

$$\dot v(t,\, x)|_{(2)} = \dot v(t,\, x)|_{(3)} + \varepsilon\, \varphi(t,\, x(t)) \le \varepsilon\, \varphi(t,\, x)\,.$$

Hence

$$v(t,\, x(t)) \le w + \varepsilon \int_\tau^t [\varphi(t_1,\, x(t_1)) - \varphi(t_1,\, y(t_1))]\, dt_1 + \\ + \varepsilon \int_\tau^t \varphi(t_1,\, y(t_1))\, dt_1$$

where $y(t) = y(t;\, \tau,\, x(\tau))$

We have

$$\left\| \int_\tau^t [\varphi(t_1, x(t_1)) - \varphi(t_1, y(t_1))] dt_1 \right\| \le \int_\tau^t \left\| \varphi(t_1, x(t_1)) - \varphi(t_1, y(t_1)) \right\| dt_1 \le \int_\tau^t g_3(t_1)\, h_3\big(\|x(t_1) - y(t_1)\|\big) dt_1 \tag{6}$$

Moreover, the following equality holds

$$\dot{x}(t) - \dot{y}(t) = f(t, x(t)) - f(t, y(t)) - \varepsilon R(t, x(t)).$$

Therefore,

$$\|x(t) - y(t)\| \le \int_\tau^t g(t_1)\, h_1\big(\|x(t_1) - y(t_1)\|\big) dt_1 + \varepsilon \int_\tau^{\tau+T} g_2(t_1)\, h_2\big(\|x(t_1)\|\big) dt_1$$

for every $t \in [\tau, \tau + T]$.

Since $h_2(u) \le c_2$ and $\int_\tau^{\tau+T} g_2(t_1) dt_1 = L_2(T)$, it follows that

$$\|x(t) - y(t)\| \le \varepsilon c_2 L_2(T) + \int_\tau^t g(t_1)\, h_1\big(\|x(t_1) - y(t_1)\|\big) dt_1$$

By the Bihari lemma ([1]), one has

$$\|x(t) - y(t)\| \le \sigma^{-1}\Big\{G(a) + \int_\tau^{\tau+T} g(t_1) dt_1\Big\},$$

where $G(u) = \int \frac{du}{h_1(u)}$, $a = \varepsilon c_2 L_2(T)$.

Therefore

$$\|x(t) - y(t)\| \le \sigma^{-1}\big\{G(a) + L_1(T)\big\} - k \text{ (const)}$$

By substituting this in (6) we have

$$\varepsilon \left\| \int_\tau^t [\varphi(t_1, x(t_1)) - \varphi(t_1, y(t_1))] dt_1 \right\| \le \varepsilon c_3 L_3 \to 0, \text{ as } \varepsilon \to 0$$

On the other hand, we have

$$\int_{\tau}^{\tau+T}[\varphi(t_1, y(t_1)) - \varphi(t_1, z(t_1))]dt_1$$
$$\leq \int_{\tau}^{\tau+T}\|\varphi(t_1, y(t_1)) - \varphi(t_1, z(t_1))\|dt_1$$
$$\leq \int_{\tau}^{\tau+T} g_3(t_1)\, h_3(\|y(t_1) - z(t_1)\|)\, dt_1 \tag{6}$$

where $z(t) = z(t, x(\tau))$.

This leads to the inequality

$$\varepsilon \int_{\tau}^{t} \varphi(t_1, y(t_1))dt_1 \leq \varepsilon \int_{\tau}^{\tau+T} \varphi(t_1, z(t_1))dt_1 + \varepsilon \int_{\tau}^{\tau+T} g_3(t_1) h_3\big(\|y(t_1) - z(t_1)\|\big)$$
$$\leq \varepsilon\big(- T\delta(\|x(\tau)\|) + NL_3(T)h_3(\chi(\|(\tau)\|N))\big) < 0,$$

the last inequality being immediate form the condition 4) Theorem 1.

Thus, for α and ε sufficiently small $v(t+T, x(t+T)) < w$ is true, i.e. $x(t)$ return into the set Ω after a moment less than T, since all the above estimates hold uniformly w.r.t. τ. The theorem is proved.

Remark. If the function is differentiable with respect to x the following theorem is true.

Theorem 2. *Let $\varphi(t, x)$ be differentiable with respect to x and $\|\varphi_x(t, x)\| \leq \psi(\|x\|)$ $(\psi \in K)$.*

Assume all conditions of Theorem 1 are satisfied, moreover the function χ mentioned in condition 4) satisfies the inequality

$$\lim_{\alpha \to 0} \frac{\psi(\alpha)\, \chi(\alpha)}{\delta(\alpha)} = 0\,.$$

Then the system (2) is ε-stable.

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