VIETNAM JOURNAL OF MATHEMATICS
Volume 24, Number 4, 1996

HOLOMORPHIC FUNCTIONS OF UNIFORM TYPE WITH VALUES IN RIEMANN DOMAINS

THAI THUAN QUANG

Abstract. *It is studied under what conditions every holomorphic function on a (DFC)-space with values in a Riemann domain over Frechet space is of uniform type. Moreover, necessary as well as sufficient conditions for which every holomorphic function on a nuclear Frechet space with values in a Riemann domain over a Frechet space is of uniform type, are given in terms of the linear topological invariants* $\overline{\overline{\Omega}}, \tilde{\Omega}, \underline{DN}, DN$, *introduced by Vogt [18, 19, 20,...].*

Let E be a locally convex space and X a complex manifold modelled on a locally convex space. A holomorphic map f from E to X is called a map of uniform type if f can be factorized holomorphically through the canonical map ω_ϱ from E to E_ϱ for some continuous semi-norm ϱ on E. Here for each continuous semi-norm ϱ on E by E_ϱ we denote the canonical Banach space associated to ϱ and by ω_ϱ we denote the canonical map from E to E_ϱ. Now by $H(E, X)$ and $H_u(E, X)$ we denote sets of holomorphic maps and holomorphic maps of uniform type from E to X respectively. In the present paper we investigate some necessary as well as sufficient conditions for which the following equality holds:

$$H(E, X) = H_u(E, X). \tag{UN}$$

This problem for vector-valued holomorphic maps, i.e. for the case where X is a locally convex space was investigated by some authors. The first result of this problem belongs to Colombeau and Mujica. In [2] they have shown that the (UN) holds when E is a dual Frechet-Montel space and X a Frechet space. Next a necessary and sufficient condition for which (UN) holds in the class of scalar holomorphic functions on a nuclear Frechet space was established by Meise and Vogt [8]. An important sufficient condition of (UN) for scalar holomorphic functions on such a space was found by above two authors [8]. Recently [6], L. M. Hai and T. T. Quang have also considered this problem for holomorphic maps with values in the projective space associated to a

Frechet space. However until now, when X does not have a linear structure, the problem is not investigated. Here we consider this problem for holomorphic maps with values in Riemann domains.

The paper contains two sections. In the first one we prove Theorem 1.1 on the uniformility of holomorphic functions on a (DFC)-space with values in a pseudoconvex Riemann domain over a Frechet space. The result of B. D. Tac and N. T. Nga [17] which shows that every plurisubharmonic function on a separable (DFC)-space is of uniform type is also used to obtain this theorem.

In Section 2 we give a necessary and sufficient condition for which (UN) holds in the case holomorphic functions on a nuclear Frechet space with values in a Riemann domain D over a Frechet space B (Theorem 2.2). We prove that this relation defines subclass which contains all spaces E with the property $\overline{\overline{\Omega}}$ and spaces B with the property $\underline{DN}$ (resp. $E \in (\tilde{\Omega}), B \in (\underline{DN})$). Here $\overline{\overline{\Omega}}, \tilde{\Omega}, \underline{DN}, DN$ are linear topological invariants introduced by Vogt [18, 19, 20,...].

Finally we shall use standard notations from the theory of locally convex spaces as presented in the books of Schaefer [14] and Pietsch [13].

1. HOLOMORPHIC FUNCTIONS ON (DFC)-SPACES WITH VALUES IN RIEMANN DOMAINS OVER FRECHET SPACES

The following is an extension of the result of Colombeau and Mujica [2] to the non-vector valued case.

Theorem 1.1. *Let D be a Riemann domain over a Frechet space F and $f : E \to D$ be a holomorphic function on a (DFC)-space. Then f is of uniform type if one of the following two conditions holds:*

(i) *D is pseudoconvex.*

(ii) *The space $H(D)$ of holomorphic functions on D separates the points of D.*

To prove the theorem we need some auxiliary results. First we recall the result of B. D. Tac and N. T. Nga [17].

Proposition 1.2. *Every plurisubharmonic function on a (DFC)- space is of uniform type.*

As Colombeau and Mujica [2] we have

Lemma 1.3. *Let G be an open set in a (DFC)-space E and $f : G \to F$ a holomorphic function with F is a Frechet space. Then there exist a continuous semi-norm ϱ on E, a balanced convex closed set B in F and a holomorphic function h on a neighbourhood of $\omega_\varrho(G)$ in E_ϱ with values in $F(B)$, the Banach space spanned by B, such that $f = h\omega_\varrho$.*

Lemma 1.4. *Every pseudoconvex Riemann domain D over a Banach space B satisfies the weak disc condition. This means that every sequence $\{\sigma_n\} \subset H(\Delta, D)$, converging in $H(\Delta^*, D)$, converges in $H(\Delta, D)$, where $H(\Delta, D)$ and $H(\Delta^*, D)$ denote the spaces of holomorphic maps from the open unit disc Δ in $\mathbf{C}$ (resp. $\Delta^* = \Delta \setminus \{0\}$) into D equipped with the compact-open topology.*

Proof. Given $\{\sigma_n\} \subset H(\Delta, D)$ such that $\sigma_n \to \sigma$ in $H(\Delta^*, D)$. Put

$$B_0 = \text{Clspan} \bigcup_{n \geq 1} \theta\sigma_n(\Delta)$$

where $\theta : D \to B$ is a locally biholomorphic map defining D as a Riemann domain over B. Since B_0 is a separable Banach space there exists a continuous linear map S from ℓ^1 onto B_0. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{D}_0 & \xrightarrow{\widetilde{S}} & D_0 \\ {\scriptstyle \widetilde{\theta}_0}\downarrow & & \downarrow{\scriptstyle \theta_0} \\ \ell^1 & \xrightarrow{S} & B_0 \end{array}$$

where $D_0 = \theta^{-1}(B_0); \theta_0 = \theta|_{D_0}, \widetilde{D}_0 = \ell^1 \times_{B_0} D_0$ and $\widetilde{\theta}_0, \widetilde{S}$ are the canonical projections.

It follows that $\widetilde{D}_0$ is also pseudoconvex with the biholomorphism $\widetilde{\theta}_0$. Hence $\widetilde{D}_0$ is a domain of holomorphy [12]. This implies that $\widetilde{D}_0$ satisfies the weak disc condition. On the other hand, since the map

$$\hat{S} : H(\Delta, \ell^1) \to H(\Delta, B_0)$$

induced by S is open, there exists a sequence $\{\widetilde{\beta}_n\} \subset H(\Delta, \ell^1)$ such that $\widetilde{\beta}_n \to \widetilde{\beta}$ in $H(\Delta, \ell^1)$ and $S\widetilde{\beta}_n = \theta\sigma_n$ for $n \geq 1$. Consider $\beta_n \in H(\Delta, \widetilde{D}_0)$ given by

$$\beta_n(t) = (\widetilde{\beta}_n(t), \sigma_n(t))$$

for $t \in \Delta$. Then $\beta_n \to (\beta,\sigma)$ in $H(\Delta^*, \widetilde{D}_0)$. Consequently this sequence converges to (β,σ) in $H(\Delta, \widetilde{D}_0)$. Hence $\sigma_n \to \sigma$ in $H(\Delta, D_0)$.

Now we can prove Theorem 1.1 as follows

(i) Assume that D is pseudoconvex.

a) Since $f(E)$ is separable, we can cover it by a sequence of open subsets V_j of D such that $q : V_j \to q(V_j)$ is homeomorphic with $j \geq 1$, where $q : D \to F$ is a locally biholomorphic map defining D as a Riemann domain over F. By Lemma 1.3 for each $j \geq 1$ there exist a continuous semi-norm ϱ_j in E, a closed, balanced convex set B_j in F and a holomorphic function h_j on a neighbourhood $\widetilde{U}_j$ of $\omega_{\varrho_j}(U_j)$ with $U_j = f^{-1}(V_j)$, in E_{ϱ_j} such that $f|_{U_j} = h_j\omega_{\varrho_j}$. Since E is a (DFC)-space we can find a continuous semi-norm ϱ on E such that $\omega_\varrho(U_j)$ is open in $E/\mathrm{Ker}\varrho$ for $j \geq 1$. Moreover, for each $j \geq 1$ there exists $C_j > 0$ such that $\varrho > C_j\varrho_j$. This implies that the maps $h_j (j \geq 1)$ define a holomorphic map h from a neighbourhood G of $E/\mathrm{Ker}\varrho$ in E_ϱ into $q^{-1}(F(B))$, where

$$B = \mathrm{Clconv} \bigcup_{j\geq 1} \varepsilon_j B_j$$

with $\varepsilon \searrow 0$ such that B is compact.

Put $D(B) = q^{-1}(F(B))$. Consider the domain of existence D_h of h over E_ϱ. Then, D_h is contained in E_ϱ as an open subset, because $E/\mathrm{Ker}\varrho$ is dense in E_ϱ.

b) We show that D_h is pseudoconvex. It suffices to prove that D_h satisfies the weak disc condition [15]. Given $\{\sigma_n\} \subset H(\Delta, D_h)$ such that $\sigma_n \to \sigma$ in $H(\Delta^*, D_h)$. Since D and hence $D(B)$ is pseudoconvex, the sequence $\{h\sigma_n\}$ converges to $h\sigma$ in $H(\Delta, D(B))$, by Lemma 1.4. Choose a neighbourhood U of $(h\sigma)(0)$ such that

$$q : U \cong q(U)$$

and $\varepsilon > 0, N \in \mathbf{N}$ such that

$$h\sigma_n(\varepsilon\Delta) \subset U$$

for every $n > N$. For each $n > N$ define a holomorphic function

$$\widetilde{\sigma}_n : \varepsilon\Delta \to \liminf_{k\geq 1} H^\infty(W_k, F(B))$$

by

$$\widetilde{\sigma}_n(t)(x) = h(\sigma_n(t) + x)$$

and

$$\widetilde{\sigma} : \varepsilon\Delta^* \to \liminf_{k\geq 1} H^\infty(W_k, F(B))$$

by

$$\widetilde{\sigma}(t)(x) = h(\sigma_n(t) + x)$$

where $\{W_k\}_{k\geq 1}$ is a basis of neighbourhoods of $0 \in E_\varrho$. It follows that the sequence $\{\widetilde{\sigma}_n\}$ converges to $\widetilde{\sigma}$ in $H(\varepsilon\Delta^*, \liminf_{k\geq 1} H^\infty(W_k, F(B)))$. Indeed, given K a compact set in $\varepsilon\Delta^*$ and hence $\sigma(K)$ is a compact set in D_h. Then there exists $V \subset G$ such that h is uniformly continuous on $\sigma(K) + V$, i.e. for every $\delta > 0$ there exists $V(\delta) \subset V$ such that for $x, y \in \sigma(K) + V, x - y \in V(\delta)$, we have

$$\|h(x) - h(y)\| < \delta.$$

For each $k \geq 1$ and $r > 0$ put

$$U_{kr} = \left\{f \in H^\infty(W_k, F(B)) : \|f\|_{W_k} \leq \frac{1}{r}\right\}$$

and consider $\{U_l\}$ with $l : \mathbf{N} \to \mathbf{N}$, defined by

$$U_l = \text{Clconv}\,(\bigcup_{k\geq 1} j_k(U_{k,l(k)}))$$

where $j_k : H^\infty(W_k, F(B)) \to \liminf_{k\geq 1} H^\infty(W_k, F(B))$ is the canonical embedding. It is easy to see that $\{U_l\}$ is a basis of neighbourhoods of 0 in $\liminf_{k\geq 1} H^\infty(W_k, F(B))$. Given a U_l in $\liminf_{k\geq 1} H^\infty(W_k, F(B))$. Take k_0 such that $W_{k_0} \subset V$ and N_0 sufficiently large such that

$$\sigma_n(t) - \sigma(t) \subset W_{k_0}$$

$$\sigma_n(t) - \sigma(t) \subset V\left(\frac{1}{l(k_0)}\right)$$

for every $n > N_0$ and all $t \in K$.

Thus for all $n > N_0$ we get $\sigma_n(t), \sigma(t) \in H^\infty(W_{k_0}, F(B))$ for all $t \in K$ and

$$\sup_{t\in K} \sup_{x\in W_{k_0}} \|h(\sigma_n(t) + x) - h(\sigma(t) + x)\| < \frac{1}{l(k_0)}$$

i.e.

$$\sup_{t\in K} \sup_{x\in W_{k_0}} \|\widetilde{\sigma}_n(t)(x) - \widetilde{\sigma}(t)(x)\| < \frac{1}{l(k_0)}.$$

Then $\widetilde{\sigma}_n(t) - \widetilde{\sigma}(t) \subset U_{k_0, l(k_0)}$ for all $t \in K$. Thus we infer that $\{\widetilde{\sigma}_n\}$ converges to $\widetilde{\sigma}$ in $H(\varepsilon\Delta^*, \liminf_{k\geq 1} H^\infty(W_k, F(B)))$ and hence $\widetilde{\sigma}$ can be extended holomorphically to $\varepsilon\Delta$ and $\{\widetilde{\sigma}_n\}$ converges to $\widetilde{\sigma}$ in $H(\varepsilon\Delta, \liminf_{k\geq 1} H^\infty(W_k, F(B)))$. Since $\{\widetilde{\sigma}_n(\frac{\varepsilon\Delta}{2})\}$ is bounded in $\liminf_{k\geq 1} H^\infty(W_k, F(B))$ and the inductive limit is regular [14] there exists k_1 such that

$$\widetilde{\sigma}_n(t) \in H^\infty(W_{k_1}, F(B))$$

for every $|t| \leq \varepsilon/2$ and every $n > N_0$.

Observe that σ can be extended holomorphically to $\varepsilon\Delta$ and $\sigma_n \to \sigma$ in $H(\Delta, E_\varrho)$. It remains to check that $\sigma(0) \in D_h$. We have $\widetilde{\sigma}_n(0)(x) = h(\sigma_n(0) + x)$ for every $x \in W_{k_1}$ and $n > N_0$. This yields that $\sigma(0) \in D_h$.

c) Let $\varphi(z) = -\log d(z, \partial D_h)$ for $z \in D_h$. By b) the function φ is plurisubharmonic on D_h containing $E/\mathrm{Ker}\varrho$. From Proposition 1.2 we can find a continuous semi-norm $\varrho_1 > \varrho$ on E and a plurisubharmonic function ψ on E_{ϱ_1} such that $\varphi\omega_\varrho = \psi\omega_{\varrho_1}$.

It remains to check that $\mathrm{Im}\,\omega_{\varrho_1\varrho} \subseteq D_h$ where $\omega_{\varrho_1\varrho} : E_{\varrho_1} \to E_\varrho$ is the canonical map. Otherwise there exists $z \in E_{\varrho_1}$ such that $\omega_{\varrho_1\varrho}(z) \in \partial D_h$. Let $\{z_n\} \subset E/\mathrm{Ker}\varrho_1$ with $z_n \to z$. Then

$$+\infty = \lim_{n\to\infty} \varphi\omega_{\varrho_1\varrho}(z_n) = \lim_{n\to\infty} \psi(z_n) \leq \psi(z) < +\infty.$$

This is impossible and hence $\mathrm{Im}\,\omega_{\varrho_1\varrho} \subseteq D_h$.

(ii) Assume now that $H(D)$ separates the points of D. By (i) there exist a continuous semi-norm ϱ on E and a holomorphic function h from E_ϱ into $\widehat{D}$, the envelope of holomorphy of D, such that $f = h\omega_\varrho$. Obviously $h(E_\varrho) \subseteq \mathrm{Cl}D$. Choose a continuous function φ on $\widehat{D}$ such that $\varphi^{-1}(0) = \mathrm{Cl}D \setminus D$. Consider the continuous function $\frac{1}{\varphi f}$ on E. As in Proposition 1.2 there exists a continuous semi-norm $\varrho_1 > \varrho$ on E such that

$$\sup\left\{\left|\frac{1}{\varphi f(z)}\right| : \varrho_1(z) \leq r\right\} < \infty$$

for every $r > 0$. Then $h\omega_{\varrho_1\varrho}(E_{\varrho_1}) \subseteq D$. Indeed, in the converse case there exists $z \in E_{\varrho_1}$ such that $h\omega_{\varrho_1\varrho}(z) \in \partial D$, i.e. $\varphi h\omega_{\varrho_1\varrho}(z) = 0$. Take a sequence $\{z_n\} \subset E/\mathrm{Ker}\varrho_1$ with $z_n \to z$. Then

$$\infty > \sup\left\{|\frac{1}{\varphi f(z_n)}| : n \geq 1\right\} = \sup\left\{|\frac{1}{\varphi h\omega_{\varrho_1\varrho}}(z_n)| : n \geq 1\right\} = \infty.$$

This contradiction shows that $h\omega_{\varrho_1\varrho}$ is holomorphic on E_ϱ, and hence (ii) is proved.

The theorem has thereby been proved.

2. HOLOMORPHIC FUNCTIONS ON NUCLEAR FRECHET SPACES WITH VALUES IN RIEMANN DOMAINS OVER FRECHET SPACES

Let E be a Frechet space with a fundamental system of semi-norms $\{\|.\|_k\}$. For each subset B of E we define a semi-norm $\|.\|_B^*$ on E^*, the strongly dual space of E, with values in $[0, +\infty]$ by

$$\|u\|_B^* = \sup\{|u(x)| : x \in B\}.$$

Instead of $\|.\|_{U_p}^*$ we write $\|.\|_p^*$, with

$$U_p = \{x \in E : \|x\|_p \leq 1\}.$$

Using these notations we define: E has the property

$$\left.\begin{array}{l}(\tilde{\Omega}) : \forall\, p\ \exists\, q\ d > 0\ \forall\, k\ \exists\, C > 0\\ (\overline{\overline{\Omega}}) : \forall\, p\ \exists\, q\ \forall\, k,\ d > 0\ \exists\, C > 0\end{array}\right\} \|\cdot\|_q^{*1+d} \leq C\|\cdot\|_k^*\|\cdot\|_p^{*d}$$

$$\left.\begin{array}{l}(DN) : \exists\, p\ \exists\, d > 0\ \forall\, q\ \exists\, k,\ C > 0\\ (\underline{DN}) : \exists\, p\ \forall\, q\ \exists\, k,\ d,\ C > 0\end{array}\right\} \|\cdot\|_q^{1+d} \leq C\|\cdot\|_k\|\cdot\|_p^{d}.$$

The above properties have been introduced and investigated by Vogt [18, 19, 20,...]. Hereafter, to be brief, whenever E has the property $\overline{\overline{\Omega}}$ (resp. $\tilde{\Omega}$) we write $E \in (\overline{\overline{\Omega}})$ (resp. $E \in (\tilde{\Omega})$).

In this section we shall find necessary and sufficient condition for which relation (UN) holds when E is a nuclear Frechet space and X is a Riemann domain over a Frechet space (Theorem 2.2). It is also a characterization for nuclear Frechet spaces having the properties $\overline{\overline{\Omega}}, \widetilde{\Omega}$. We begin with recalling [5] a result of L.M.Hai which is also useful in proving Theorem 2.2. The main tools to obtain this result are an interpolation argument as well as methods and results from the theory of nuclear Frechet spaces.

Proposition 2.1. *Let E and F be Frechet spaces and E, nuclear. Then*

$$H_u(E,F) = H(E,F)$$

if one of the following conditions holds

(i) $E \in (\overline{\overline{\Omega}})$ *and* $F \in (\underline{DN})$,

(ii) $E \in (\widetilde{\Omega})$ *and* $F \in (DN)$.

Theorem 2.2. *A nuclear Frechet space E has the property $\overline{\overline{\Omega}}$ (resp. $\widetilde{\Omega}$) if every holomorphic function on E with values in a Riemann domain D over a Frechet space B having the property $\underline{DN}$ (resp. DN) which satisfies one of following two conditions:*

(i) *D is pseudoconvex,*

(ii) *$H(D)$ separates the points of D,*

is of uniform type.

It suffices to prove in the case $E \in (\overline{\overline{\Omega}})$ and $B \in (\underline{DN})$. We need the following lemma.

Lemma 2.3. *Let B be a Frechet space and $B \in (\underline{DN})$. Then*

$$\ell^1(B) = \Big\{(\xi_b)_{b\in B} \subset \mathbb{C}^\infty : |||(\xi_b)|||_p = \sum_{b\in B} |\xi_b| \exp \|b\|_p < \infty \quad \textit{for every} \quad p \geq 1\Big\}$$

also has the property $\underline{DN}$.

Proof. Since $B \in (\underline{DN}), \exists\, p, \forall\, q, \exists\, k, d, C > 0$ such that

$$\|b\|_q^{1+d} \leq C \|b\|_k \|b\|_p^d$$

for all $b \in B$. Upon iterating we get

$$\|b\|_q \leq C^{\frac{1}{1+d}} \|b\|_k^{\frac{1}{1+d}} \|b\|_p^{\frac{d}{1+d}}$$

for all $b \in B$. We have the following estimation:

$$\begin{aligned}
|\!|\!|(\xi_b)|\!|\!|_q &= \sum_{b\in B} |\xi_b| \exp \|b\|_q \le \sum_{b\in B} |\xi_b| \exp\left[C^{\frac{1}{1+d}} \|b\|_k^{\frac{1}{1+d}} \|b\|_p^{\frac{d}{1+d}}\right] \\
&\le \exp C^{\frac{1}{1+d}} \sum_{b\in B} |\xi_b| \exp\left[\frac{1}{1+d}\|b\|_k + \frac{d}{1+d}\|b\|_p\right] \\
&\le \exp C^{\frac{1}{1+d}} \Big[\sum_{b\in B} \Big(|\xi_b|^{\frac{1}{1+d}} \exp \frac{1}{1+d}\|b\|_k\Big)^{1+d}\Big]^{\frac{1}{1+d}} \\
&\quad \times \Big[\sum_{b\in B} \Big(|\xi_b|^{\frac{d}{1+d}} \exp \frac{d}{1+d}\|b\|_p\Big)^{\frac{1+d}{d}}\Big]^{\frac{d}{1+d}} \\
&= \exp C^{\frac{1}{1+d}} \Big(\sum_{b\in B} |\xi_b| \exp \|b\|_k\Big)^{\frac{1}{1+d}} \Big(\sum_{b\in B} |\xi_b| \exp \|b\|_p\Big)^{\frac{d}{1+d}} \\
&= \exp C^{\frac{1}{1+d}} |\!|\!|(\xi_b)|\!|\!|_k^{\frac{1}{1+d}} |\!|\!|(\xi_b)|\!|\!|_p^{\frac{d}{1+d}}
\end{aligned}$$

for all $(\xi_b) \in \ell^1(B)$. Hence $\ell^1(B) \in \underline{(DN)}$.

Now we prove Theorem 2.2.

Given f a holomorphic function on E with values in D, where D is as in the theorem and E a nuclear Frechet space with the property $\overline{\overline{\Omega}}$. Since $f(E)$ is separable without loss of generality we may assume that B is separable. Take a continuous linear map S from $\ell^1(B)$ onto B with $\ell^1(B) \in \underline{(DN)}$, by Lemma 2.3, and consider the commutative diagram:

$$\begin{array}{ccccc}
\widetilde{D} & \xrightarrow{\ \widetilde{S}\ } & D & \xleftarrow{\ f\ } & E \\
\Big\downarrow{\scriptstyle \widetilde{\theta}} & & \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle id} \\
\ell^1(B) & \xrightarrow{\ S\ } & B & \xleftarrow{\ \theta f\ } & E
\end{array}$$

as in Lemma 1.6.

Applying Proposition 2.1 to θf, we can find $p \in \mathbb{N}$ and a holomorphic function h on E_p such that $h\omega_p = \theta f$. Since E is nuclear, there exists $q > p$ such that the canonical map $\omega_{qp} : E_q \to E_p$ is nuclear. Thus there exist nuclear maps $\beta : \ell^1 \to E_p$ and $\alpha : E_q \to \ell^1$ such that $\beta\alpha = \omega_{qp}$. Consider the holomorphic function $h\beta$ from ℓ^1 into B. This function is bounded on every bounded set in ℓ^1. By using Taylor series expansion at zero, it is easy to see that there exists a holomorphic function $g : \ell^1 \to \ell^1(B)$ such that $Sg = h\beta$. Now define a holomorphic function $f_1 : E \to \widetilde{D}$ by

$$f_1(x) = (g\alpha\omega_q(x), f(x))$$

for $x \in E$. Then f is of uniform type if f_1 so is. Hence without loss of generality we may assume that $D = \widetilde{D}$ and $B = \ell^1(B)$. Since D satisfies (i) or (ii) the canonical map

$$T : D \to F =: \prod\{\mathbf{C}_f : f \in H(D)\}$$

where $\mathbf{C}_f = \mathbf{C}$ for $f \in H(D)$, is a homeomorphism onto the image [7]. Consider the holomorphic function $h_1 = Tf : E \to F$. Let W be a neighbourhood of zero in E such that $f(W)$ is contained in an open subset U of D for which $\theta : U \cong \theta(U)$. Choose a neighbourhood V of $\theta f(0)$ such that $T(\theta|_U)^{-1}$ is bounded on V. Then Tf is bounded on a neighbourhood of $0 \in E$. By an argument as Meise and Vogt [8] we can find $p \in \mathbf{N}$ and a holomorphic function g_1 on E_p with values in F such that

$$h_1 = g_1\omega_p.$$

This yields from the relation $g_1(E/\mathrm{Ker}\|.\|_p) \subset \mathrm{Im}\, T$ that g_1 induces a holomorphic function $\widetilde{g} : E/\mathrm{Ker}\|.\|_p \to D$. Extend $\widetilde{g}$ to a holomorphic function g on a neighbourhood of $E/\mathrm{Ker}\|.\|_p$ in E_p. Let Ω_g be the domain of existence of g.

(i) Ω_g is a domain of holomorphy. Since the topology of E is defined by Hilbert semi-norms without loss of generality we may assume that E_q is a Hilbert space. Choose $q > p$ such that the canonical map $\omega_{qp} : E_q \to E_p$ is compact. Let τ denote the linear metric topology on $H(\Omega_g)$ generated by the uniform convergence on sets:

$$K_r = \Big\{\omega_{qp}(z) : \|z\| \le r, \omega_{qp}(z) \in \Omega_g, \mathrm{dist}(\omega_{qp}(z), \partial\Omega_g) \ge \frac{1}{r}\Big\}.$$

Note that the canonical map $[H(\Omega_g), \tau] \to H(E)$ is continuous and

$$H(E)_{bor} \cong \liminf_{k} H_b(E_k)$$

(see [8]). Here $H(E)_{bor}$ denotes the bornological space associated to $H(E)$ and for every $k \ge 1, H_b(E_k)$ denotes the Frechet space of holomorphic functions on E_k which are bounded on every bounded set in

E_k. Therefore, we can find $k > q$ such that $H(\Omega_g) \subseteq H_b(E_k)$. It remains to check that $\operatorname{Im}\omega_{kp} \subset \Omega_g$. In the converse case there exists $z \in E_k$ such that $\omega_{kp}(z) \in \partial\Omega_g$. Choose a sequence $\{z_n\} \subset E/\mathrm{Ker}\|.\|_k$ which converges to z. Since E_p is a Hilbert space we can find $f \in H(\Omega_g)$ such that

$$\sup |f\omega_{kp}(z_n)| = \infty.$$

This is impossible because $f\omega_p \in H(E_k)$.

(ii) Arguments are similar as in the part (ii) of the proof of the Theorem 1.1.

The Theorem 2.2 is completely proved.

REFERENCES

1. J. F Colombeau, *Differential calculus and holomorphy*, North-Holland Math. Stud., **64** (1982).
2. J. F Colombeau and J. Mujica, *Holomorphic and differentiable mappings of uniform bounded type*, Holomorphy and Approximation Theory, (J. A. Barroso, Ed.), North-Holland Math. Stud., **71** (1982), 179-200.
3. S. Dineen, R. Meise, and D. Vogt, *Polar subsets of locally convex space*, Aspect of Math. and its appl. Ed. J. Barroso, Elsevier Sci. Pul. B.V., 295-319.
4. N. M. Ha and T. T. Quang, *Property $\widetilde{\Omega}$ and holomorphic functions with values in a pseudoconvex space having a Stein morphism into a complex Lie group*, (To appear in Rendi. Math.).
5. L. M. Hai, *Weak extension of Frechet-valued holomorphic functions on compact sets and linear topological invariants*, (To appear in Acta Math. Vietnam).
6. L. M. Hai and T. T. Quang, *Holomorphic maps of uniform type*, Collog. Math., **69** (1995), 81-86.
7. Y. Hervier, *On the Weierstrass problem in Banach space*, Proc. on Infinite dimensional Holomorphy, Lecture Notes in Math., **364** (1974), 157-167.
8. R. Meise and D. Vogt, *Holomorphic function of uniformly bounded type on nuclear Frechet space*, Studia Math., **83** (1986), 147-175.
9. R. Meise and D. Vogt, *Extension of entire functions on locally convex spaces*, Proc. Amer. Math. Soc., **92** (1984), 495-500.
10. J. Mujica, *Domains of holomorphy in DFC-space*, Functional Analysis, Holomorphy and Approximation Theorem Ed. S. Machado, Lecture Notes in Math., **834** (1981), 500-533.
11. P. Noverraz, *Pseudoconvex completions of locally convex topological vector spaces*, Math. Ann., **208** (1974), 59-69.
12. P. Noverraz, *Pseudo-convexite, Convexite-polynomiale et domaines d'holomorphie en dimension infinie*, North-Holland Math. Stud., North-Holland Amsterdam, 1973.

13. A. Pietsch, *Nuclear locally convex spaces,* Ergeb. Math. Grenzgeb. 66. Springer Verlag, 1972.

14. H. H. Schaefer, *Topological vector spaces,* Springer Verlag, 1971.

15. B. Shiffman, *Extension of holomorphic maps into Hermitian manifolds,* Math. Ann., **194** (1971), 249 - 258.

16. B. D. Tac, *Extending holomorphic maps in infinite dimensions*, Ann. Polon. Math., **54** (1991), 241 - 253.

17. B. D.Tac and N. T.Nga, *Oka-Weil theorem and plurisubharmonic function of uniform type*, Acta Math. Vietnam, **16** (2) (1991), 133 - 145.

18. D. Vogt, *Eine Charakterisierung der Potenzeihenräumevon endlichem Typ und ihre Folgerungen*, Manuscripta Math., **37** (1982), 261 - 301.

19. D. Vogt, *Frecheträume zwischen denen jede stetige lineare Abbidung beschränkt ist*, J. rein. angew. Math., **345** (1983), 182 - 200.

20. D. Vogt, *Charakterisierung der Unterräume eines nuklearen Stabilen Potenzreihenräume von endlichem Type*, Studia Math., **71** (1982), 251 - 270.

Received November 6, 1995
Revised January 30, 1996

Department of Mathematics
Pedagogical Institute Hanoi I
Hanoi, Vietnam.