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Short Communication

THE LOCALLY MOST POWERFUL RANK TESTS FOR TESTING RANDOMNESS AND SYMMETRY

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In this note we construct the locally most powerful rank tests for testing the randomness and the symmetry, respectively, against very general alternatives. We recover from these tests the well-known results of Hájek and Šidák [3], Lehmann [5] and Gibbons [1].

1. Notations. Let

$$\mathcal{F}_0 = \{F : F \text{ is an absolute continuous d.f. on } \mathbb{R}\}, \tag{1.1}$$

$$\mathcal{F}_1 = \{F : F \in \mathcal{F}_0,\ F(-x) = 1 - F(x),\ x \in \mathbb{R}\}. \tag{1.2}$$

Let $X = (X_1, ..., X_N)$ be a vector of N i.r.v's. The hypothesis of randomness $\mathcal{H}_0$ (of symmetry $\mathcal{H}_1$) means that $X_1, ..., X_N$ have a common d.f. $F \in \mathcal{F}_0$ ($F \in \mathcal{F}_1$.

Let us consider the following alternatives, for $h = 0, 1$:

$$\mathcal{K}_h^1(\Delta) = \{X \text{ has a density } q_\theta(x) = \prod_{i=1}^{N} f_i(x_i,\, \theta),\ \theta \in \Delta\}, \tag{1.3}$$

$$\mathcal{K}_h^2(\Delta) = \{X \text{ has a d.f. } Q_\theta(x) = \prod_{i=1}^{N} G_i(F(x_i);\, \theta),\ F \in \mathcal{F}_h,\ \theta \in \Delta\}, \tag{1.4}$$

where $\Delta = \Delta^+ = (0,\, a)$ or $\Delta = \Delta^- = (-a,\, 0)$, $a > 0$, and for $\theta \in \widetilde{\Delta} \doteq \Delta \cup \{0\}$, $1 \le i \le N$, $f_i(x,\, \theta)$ are densities on R such that $f_i(x,\, 0) = f(x)$, and $G_i(x,\, \theta)$ are d.f. on $\mathbb{R}$ such that $G_i(x,\, 0) = y$, and moreover for $h = 1 : f(-x) = f(x)$.

Let $R = (R_1, ..., R_N)$, $V = (V_1, ..., V_N)$ and $X_{(\cdot)} = (X_{(1)}, ..., X_{(N)})$

be the rank vector, the sign vector and the vector of order statistics of $X = (X_1, ..., X_N)$, respectively. The rank vector and the vector of order statistics of $|X| = (|X_1|, ..., |X_N|)$ will be denoted by $R^+ = (R_1^+, ..., R_N^+)$ and $|X|_{(\cdot)} = (|X|_{(1)}, ..., |X|_{(N)})$, respectively.

2. The locally most powerful rank tests (LMPRT). The methods of proof of Theorem II.4.8 [3] may be ameliorated to get the following result.

Theorem 2.1. *Let, for* $1 \leq i \leq N$,

(i) $f_i'(x, \theta) \doteq \partial f_i(x, \theta)/\partial\theta$ *exist,* $\theta \in \tilde{\Delta}$, *and be continuous at* $\theta = 0$ *for a.e.* $x \in \mathbb{R}$, *where* $f_i'(x, 0)$ *is understood to be one-sided.*

(ii) $\lim_{\Delta \ni \theta \to 0} \int |f_i'(x, \theta)| dx = \int |f_i'(x, 0)| dx < \infty$.

Denote

$$A(k, i) = E\{f_i'(X_{(k)}, 0)/f(X_{(k)})\}, \tag{2.1}$$

where $X_{(1)}, ..., X_{(N)}$ *are order statistics of* N *i.r.v's with the common density* $f(x)$.

Then the test with critical region

$$S(R) = \sum_{i=1}^{N} A(R_i, i) \geq \lambda \ (resp. \ \leq \lambda) \tag{2.2}$$

is the LMPRT for testing $\mathcal{H}_0$ *against* $\mathcal{K}_0^1(\Delta^+)$ *(resp. against* $\mathcal{K}_0^1(\Delta^-)$*) at the corresponding level.*

Theorem 2.2. *Suppose, for* $1 \leq i \leq N$,

(iii) $g_i(y, \theta) \doteq \partial G_i(y, \theta)/\partial y$ *exists for* $\theta \in \tilde{\Delta}$, $0 < y < 1$,

(iv) $g_i'(y, \theta) \doteq \partial g_i(y, \theta)/\partial\theta$ *exists for* $\theta \in \tilde{\Delta}$, $0 < y < 1$,

(v) $\lim_{\Delta \ni \theta \to 0} \int_0^1 |g_i'(y, \theta)| dy = \int_0^1 |g_i'(y, 0)| dy < \infty$.

Denote

$$a(k, i) = E\{g_i'(U_{(k)}, 0)\}, \ 1 \leq i, \ k \leq N, \tag{2.3}$$

where $U_{(1)}, ..., U_{(N)}$ *are order statistics of* N *i.r.v's with common uniform distribution* $\mathcal{U}(0, 1)$.

Then the test with critical region

$$S(R) = \sum_{i=1}^{N} a(R_i, i) \geq \lambda \ (resp. \ \leq \lambda) \tag{2.4}$$

is the LMPRT for testing $\mathcal{H}_0$ against $\mathcal{K}_0^2(\Delta^+)$ (resp. against $\mathcal{K}_0^2(\Delta^-)$) at the corresponding level.

To prove this theorem, note that for $f_i(x, \theta) \doteq g_i(F(x), , \theta) f(x)$, where $f(x) = dF(x)/dx$, (iv)-(v) are equivalent to (i)-(ii).

Remarks. (a) For

$$G_i(y, \theta) = \begin{cases} (1-\theta)y + \theta y^2, & 1 \le i \le m, \ 0 < \theta < 1, \\ y, & m+1 \le i \le N, \end{cases}$$

Theorem 2.2 implies the result of Lehmann [5].

(b) For

$$G_i(y, \theta) = \begin{cases} y^{1+\theta}, & 1 \le i \le m, \\ 1 - (1-y)^{1+\theta}, & m+1 \le i \le N, \ \theta \ge 0, \end{cases}$$

Theorem 2.2 implies the result of Gibbons [1].

3. The locally most powerful signed rank tests (LMPSRT). The methods of [4] may be generalized to obtain the following results for testing $\mathcal{H}_1$ against $\mathcal{K}_1^1$ and $\mathcal{K}_1^2$.

Theorem 3.1. *With f_i satisfying (i)-(ii), denote*

$$f_{j,i}(x) = \frac{1}{2}[f_i'(x, 0) + (-1)^j f_i'(-x, 0)] \tag{3.1}$$

$$A_j(k, i) = E\{f_{j,i}(|X|_{(k)})/f(|X|_{(k)})\}, \tag{3.2}$$

$1 \le i \le N$, $j = 1, 2$, *where $|X|_{(1)}, \ldots, |X|_{(N)}$ are order statistics in absolute value of N i.r.v's with the common symmetric density $f(x)$. Then the test with critical region*

$$S(R^+, V) = \sum_{i=1}^{N} [A_1(R_i^+, i)V_i + A_2(R_i^+, i)] \ge \lambda \ (\le \lambda) \tag{3.3}$$

is the LMPSRT for testing $\mathcal{H}_1$ against $\mathcal{K}_1^1(\Delta^+)$ (against $\mathcal{K}_1^1(\Delta^-)$) at the corresponding level.

Theorem 3.2. *Let the condition (iii)-(v) be satisfied. Denote for* $j = 1, 2,\ 1 \le i,\ k \le N,\ u \in (0; 1)$,

$$g_{j,i}(u) = \frac{1}{2}\Big[g_i'\Big(\frac{1+u}{2}; 0\Big) + (-1)^j g_i'\Big(\frac{1-u}{2}; 0\Big)\Big] \tag{3.4}$$

$$a_j(k, i) = E\{g_{j,i}(U_{(k)})\}\,. \tag{3.5}$$

Then the test with critical region

$$S(R^+, V) = \sum_{i=1}^{N}[a_1(R_i^+, i)V_i + a_2(R_i^+, i)] \ge \lambda\ (\le \lambda) \tag{3.6}$$

is the LMPSRT for testing $\mathcal{H}_1$ *against* $\mathcal{K}_1^2(\Delta^+)$ *(against* $\mathcal{K}_1^2(\Delta^-)$*) at the respective level.*

Remarks. (c) For $f_i(x, \theta) \equiv f(x - \theta)$, Theorem 3.1 leads to Theorem II.4.9 [3].

(d) Theorem II.4.10 [3] follows from Theorem 3.1 if q_θ considered in $\mathcal{K}_1^1$ is of the form

$$q_\theta(x) = \prod_{i=1}^{m} e^{-\theta} f(e^{-\theta} x_i) \prod_{i=m+1}^{N} f(x_i)\,.$$

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