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Short Communication

ON EQUIVALENCE BETWEEN CONVEX MAXIMIZATION AND OPTIMIZATION OVER THE EFFICIENT SET [1]

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1. INTRODUCTION

Let X be a polytope in R^n and C be a $(p \times n)-$ matrix. Denote by X_E the efficient set of the vector function Cx over X. The problem under consideration in this paper is given as

$$\max\{f(x) : x \in X_E\}. \tag{1.1}$$

This problem has many applications in multiple objective decision making and has been considered by some authors [2, 3, 4, 5, 6, 7, 10]. The main difficulty arises from the fact that X_E is a nonconvex set.

In this note we examine the convex maximization and convex-concave formulations of Problem (1.1) with f being a convex function. If additionally $f(x) = \varphi(Cx)$ we show how to find a penalty parameter for which the considered problem can be reduced to a single convex maximization or convex-concave program. We also point out that the penalized problem satisfies certain decomposable property which can be used to reduce the size of the problem. Based upon these results an inner approximation algorithm has been proposed in [7] for maximizing a convex function on X_E.

2. CONVEX MAXIMIZATION AND CONVEX-CONCAVE FORMULATIONS

Let e be the row vector in R^p whose every entry is 1. Define

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$$r(x) := \max\{e(Cy - Cx) : Cy \geq Cx,\ y \in X\}. \tag{2.1}$$

It is well known [1] that $x \in X_E$ iff $x \in X, r(x) = 0$.

For each $N \geq 0$ define the problem

$$\max\{F(x) := f(x) - Nr(x) : x \in X\}. \tag{2.2}$$

Let X_E^* and $S(N)$ denote the set of all global optimal solutions of Problems (1.1) and (2.2) respectively. Let

$$N^* := \sup\{N \geq 0 : S(N) \cap X_E = \emptyset\}. \tag{2.3}$$

Then we have the following results:

Proposition 2.1. *Assume that f is convex on X. Then $N^* < \infty$ and*

(i) $S(N) \subset X_E^*$ if $N^* < N < +\infty$,

(ii) $S(N) \cap X_E = \emptyset$ if $0 \leq N < N^*$.

Proof. It is not difficult to show that

$$N^* \leq N_0 := (U_0 - L_0)/M_0,$$

where

$$M_0 := \min\{r(x) : x \in V(X) : r(x) > 0\} > 0,$$

and

$$+\infty > U_0 > \max\{f(x) : x \in X\},\ L_0 := f(x^0),\ x^0 \in X_E.$$

Now let $N > N^*$. By the definition of N^* there exists N' such that

$$N^* \leq N' < N \text{ and } S(N') \cap X_E \neq \emptyset.$$

Let $x' \in S(N') \cap X_E$ and x^N be any point in $S(N)$. Then

$$f(x^N) - Nr(x^N) \geq f(x') - Nr(x'), f(x') - N'r(x') \geq f(x^N) - N'r(x^N).$$

Adding these inequalities yields

$$(N' - N)(r(x^N) - r(x')) \geq 0.$$

Since $N' < N$ this implies $r(x^N) \leq r(x') = 0$ (because $x' \in X_E$). Hence $x^N \in X_E$.

By the same argument we can see $S(N) \cap X_E = \emptyset$ if $0 \leq N < N^*$.

□

For each vector $u = (u_1, ..., u_p)$ we define

$$|u|_- := \max\{-u_j,\ u_j \leq 0\}.$$

Proposition 2.2. *Let* $C(X) = \{u \in R^p : u = Cx, x \in X\}$ *and* $f(x) = \varphi(Cx)$ *, where* φ *is a differentiable convex function on* $C(X)$*. Then*

$$N^* \leq |\varphi'(u)|_- \ \forall u \in C(X).$$

Proof. Let x^1 and x^2 be the global optimal solutions of Problems (1.1) and (2.2) respectively, then

$$f(x^1) = f(x^1) - Nr(x^1) \leq f(x^2) - Nr(x^2). \qquad (2.4)$$

We observe that if y is a solution of the linear program

$$r(x^2) = \max\{e(Cz - Cx^2) : Cz \geq Cx^2,\ z \in X\}, \qquad (2.5)$$

then $y \in X_E$. By the well known mean theorem we have

$$f(y) + Nr(y) - [f(x^2) - Nr(x^2)] = \varphi'(\theta)(Cy - Cx^2)$$

$$+Ne(Cy - Cx^2) = (Ne + \varphi'(\theta))(Cy - Cx^2) \geq 0. \qquad (2.6)$$

Since $f(x^1) \geq f(y)$, it follows

$$f(x^1) \geq f(x^2) - Nr(x^2)$$

which together with (2.4) shows that $f(x^1) = f(x^2) - Nr(x^2)$. Moreover from (2.6) and $f(x^1) \geq f(y)$ (since $y \in X_E$) we have

$$(Ne + \varphi'(\theta))(Cy - Cx^2) = 0.$$

Remember that $Ne + \varphi'(\theta) > 0$ and $Cy \geq Cx^2$ we deduce $Cy = Cx^2$. Hence $x^2 \in X_E$. Thus by virtue of (i) Proposition 2.1 we have $N > N^*$ which implies $x^2 \in X_E^*$. □

Remark. If $\varphi(u) = \sum_{i=1,p} \xi_i u_i$ then from the above proof one can take $N^* > \max\{0, \max\{-\xi_i : \xi_i < 0\}\}$.

Let $S \subset I\!R^p$ be the simplex defined in [8] and

$$g(\lambda) := \max\{\lambda^T Cy : y \in X\}.$$

As before, for each fixed positive number N define the problem

$$\max\{f(x) - N(g(\lambda) - \lambda^T Cx) : x \in X,\ \lambda \in S\}. \tag{2.7}$$

Then we have the following results

Proposition 2.1'. *Proposition 2.1 with the same assumptions is also true for Problems (2.2') and (2.7). In other words, for any global optimal solution (x^N, λ^N) of (2.7) we have*

(i) (x^N, λ^N) *is global optimal for (2.2) if* $N^* < N < +\infty$,

(ii) $x^N \notin X_E$ *if* $0 \le N < N^*$.

Proposition 2.2'. *Under the assumptions of Proposition 2.2 any global optimal solution of Problem (2.7) with N is the same as in the Proposition 2.2 is also a global optimal solution to (2.2').*

3. DECOMPOSABLE PROPERTY AND DIMENSION REDUCTION FORMS

Definition 3.1. Let Q be a $(p \times n)$ - matrix and K be a convex set in R^n. A function q defined on K is said to be *nondecreasing with respect to Q* (or briefly *Q-nondecreasing*) on K if $q(x) \le q(x')$ for every $x, x' \in K$ satisfying $Qx \le Qx'$.

The function q is said to be *increasing with respect to Q* (or briefly *Q-increasing*) on K if it is nondecreasing and $q(x) < q(x')$ whenever $x, x' \in K$, $Qx \le Qx'$ and $Qx \ne Qx'$. It follows immediately that if q is Q - increasing on K then

$$\max\{q(x) : x \in K\} = \max\{q(x) : x \in K_E\}$$

where K_E denotes the efficient set of Q over K

Proposition 3.1. (i) $\Omega := \{y \in R^n : Cy \le 0\} \subset O^+G(X)$,

(ii) $-r$ *is C-increasing on* $G(X)$,

(iii) *The constancy space of* $-r$ *is* $L(\Omega) = \{y : Cy = 0\}$.

Proof.

(i) and (ii) are immediate from the definitions.

(iii) It is easy to verify that $-r$ is closed proper convex and dom $(-r)$ $= G(X)$. Since $G(X) = \{x : -r(x) \le 0\}$, by Theorem 8.7 in [9] the constancy space of $-r$ is the lineality space of $G(X)$ which is equal to the set $-O^+G(X) \cap O^+G(X)$. From (i) we have $L(\Omega) \subset O^+G(X)$. For every $t \geq 0$, $y \in \Omega$ one has

$$Cz \geq Cx \geq Cx - tCy = C(x - ty)$$

which means that $x - ty \in G(X)$. This is true for all $t \geq 0$ and $y \in \Omega$. Hence $\Omega \subset -O^+G(X)$. □

From this proposition it follows that r is constant on $L(\Omega) = \{x : Cx = 0\}$. Note that $\dim L(\Omega) = n - k$ because $\operatorname{rank} C = k$.

Proposition 3.2. *If f is C-nondecreasing on $G(X)$ then for any $N > 0$ Problem (2.2) is equivalent to (1.1).*

Proof. Let x^N be a global optimal solution of (2.2). Then $x^N \in X_E$. Thus

$$f(x^N) = f(x^N) - Nr(x^N) \geq f(x) - Nr(x) = f(x) \quad \forall x \in X \supset X_E$$

which means that x^N solves Problem (1.1) globally.

Conversely, if x^* is a global optimal solution of (1.1), then

$$f(x^*) - Nr(x^*) = f(x^*) \geq f(x^N) = f(x^N) - Nr(x^N).$$

Hence x^* is a global optimal solution of (2.2). □

Proposition 3.3. *Under the assumptions of Proposition 2.2 the function $f(x) - Nr(x)$ with*

$$N > |\varphi'(u)|_-, \quad \forall u \in C(X)$$

is $C-$increasing on $G(X)$.

Proof. Let $x, x' \in G(X)$ such that $Cx \leq Cx'$ and $Cx \neq Cx'$. Since $Cx \leq Cx'$ we have

$$\max\{e(Cy - Cx') : Cy \geq Cx', y \in X\}$$
$$\leq \max\{e(Cy - Cx) : Cy \geq Cx, y \in X\}.$$

This and $f(x) = \varphi(Cx)$ imply

$F(x') := f(x') - Nr(x') = f(x') - N\max\{e(Cy - Cx') : Cy \geq Cx', y \in X\} = f(x') - N\max\{e(Cy - Cx + Cx - Cx') : Cy \geq Cx', y \in$

$X\} \geq f(x') - N\max\{e(Cy - Cx) : Cy \geq Cx', y \in X\} - N\max\{e(Cx - Cx') : Cy \geq Cx', y \in X\} \geq f(x) - N\max\{e(Cy - Cx) : Cy \geq Cx, y \in X\} + f(x') - f(x) - N\max\{e(Cx - Cx') : Cy \geq Cx', y \in X\} = F(x) + \varphi(Cx') - \varphi(Cx) - Ne(Cx - Cx') = F(x) + (\varphi'(\theta + Ne)(Cx' - Cx) > F(x)$. □

The more detailed proofs of the above propositions can be found in [7].

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