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MOSCO CONVERGENCE OF MULTIVALUED SLLN

EZZAKI FATIMA

Abstract. *Mosco convergence for strong law of large number for convex weakly compact valued martingales and closed convex valued martingale difference in a p-smooth separable Banach space are provided. As application, we present a new proof of Mosco convergence of strong law of large numbers for independent convex weakly compact valued random sets in p-smooth separable Banach space.*

INTRODUCTION

Strong law of large numbers (SLLN) for multivalued martingales are interesting because they are closely connected with both geometric and probabilistic point of view. There are two main types of strong law of large numbers for random variables. The first one deals with independent identically distributed random variables, and the second is concerned with independent random variables with the same mean and some L^p-norm conditions. The first type of SLLN is generally valid for Banach space valued random variables (cf. [1], [9, 10, 11], [13, 14], [18, 19]). On the other hand, the second type of SLLN for Banach space valued random variables is valid under some geometric conditions (cf. [16], [20], [2], [23], [13], [14]). In the present paper, we discuss several versions of second type of SLLN for martingales in separable Banach space.

In Section 1 we present a Mosco convergence result of SLLN for convex weakly compact valued supermartingales in separable p-smooth Banach space (in Pisie's sense [20]).

In Section 2 we give a new proof of Mosco convergence result of SLLN for independent convex weakly compact valued random sets in a p-smooth separable Banach space via the techniques of supermartingales introduced in section 1.

In Section 3 we introduce the notion of multivalued martingales difference and we show the existence of martingales difference selectors for this class of multifunctions. As application we present a new result of Mosco convergence of SLLN for multivalued martingales difference whose valued may be unbounded.

1. PRELIMINARIES. LAW OF LARGE NUMBERS FOR SUPERMARTINGALES

Notation and definitions

Throughout this paper, let $(\Omega, \mathcal{F}, P)$ be a probability space, E a separable Banach space, $cc(E)$ the family of all nonempty closed convex subsets of E, $\text{cwk}(E)$ the family of nonempty convex weakly compact subsets of E and $\mathcal{E}$ the Effros tribe on $cc(E)$. Set $\mathcal{R} = \{C \in cc(E) : C \cap \overline{B}(0, r) \in \text{cwk}(E), \forall r > 0\}$ where $\overline{B}(0, r)$ denotes the closed ball of radius r, centered at 0.

In the present paper, we shall use a notion of convergence, for sequences of subsets, which has been introduced by Mosco [18, 19] and which is related to the one of Kuratovski. Let t be a topology on E and $(K_n)_{n\geq 1}$ a sequence in $c(E)$. We put

$$i\text{-}liK_n = \{x \in E,\ x = t\text{-}\lim x_n : x_n \in K_n,\ \forall n \geq 1\}$$

$$t\text{-}lsK_n = \{x \in E,\ x = t\text{-}\lim x_k : x_k \in K_{n(k)},\ \forall k \geq 1\}$$

where $(K_{n(k)})_{k\geq 1}$ is a subsequence of (K_n). The subsets $t\text{-}liK_n$ and $t\text{-}lsK_n$ are the *lower limit* and the *upper limit* on (K_n), relative to the topology t. We obviously have $t\text{-}liK_n \subset t\text{-}lsK_n$. A sequence (K_n) converges to K_∞, in the sense of Kuratovski, relatively to the topology t, if the two following equalities are satisfied:

$$t\text{-}liK_n = t\text{-}lsK_n = K_\infty\,.$$

In this case, we shall write $t\text{-}\lim\limits_n K_n = K_\infty$; this is true if and only if the two inclusion hold:

$$t\text{-}lsK_n \subset K_\infty \subset t\text{-}liK_n\,.$$

Let us denote by s (resp. w) the strong (resp. weakly) topology on E. A subset K_∞ is said to be the Mosco limit of $(K_n)_{n\geq 1}$ denoted by $M\text{-}\lim\limits_n K_n = K_\infty$, if

$$K_\infty = s\text{-}liK_n = w\text{-}lsK_n$$

which is true if and only if

$$w\text{-}lsK_n \subset K_\infty \subset s\text{-}liK_n\,.$$

The corresponding definitions, for a sequence (X_n) of multifunctions on Ω, are clear. In fact, in the previous definitions, if suffices to replace K_n by $X_n(\omega)$ for all $\omega \in \Omega$ (or for almost all ω).

Concerning Mosco's convergence, we refer to Mosco [Mo], Wets [We], and Attouch [A]. In the present paper, $\mathbf{N}^*$ will be denoted the set of strictly positive integers, $\mathbf{R}$ (resp. $\mathbf{R}^+$) the set of real numbers (resp. positive real numbers).

A closed convex valued multifunction X, i.e., an application from Ω to $cc(E)$, is said to be measurable if the application X is $(\mathcal{F}, \mathcal{E})$-measurable. A measurable multifunction is also called a random set. A function f from Ω to E is said to be a selection of X if, for any ω in Ω, $f(\omega) \in X(\omega)$. A Castaing representation of X is a sequence $(f_n)_{n\geq 1}$ of measurable selections of X such that for all ω, $X(\omega) = cl\{f_n(\omega) : n \geq 1\}$. It is known see ([5], Theorem III.9) that a closed valued multifunction F is measurable if and only if it has a Castaing representation, or if and only if the real function $d(x, X(\cdot))$ is measurable for any x in E.

Let $L^1(\Omega, \mathcal{F}, P, E) = L^1(\Omega, E)$ denote the Banach space of (equivalence classes of) measurable function f from Ω to E such that

$$\|f\|_1 = E\|f\| = \int_\Omega \|f(\omega)\|P(d\omega)$$

is finite. For any $\mathcal{F}$-measurable random set X we put

$$S_X^1(\mathcal{F}) = \{f \in L^1(\Omega, E) : f(\omega) \in X(\omega) \ a.s.\},$$

which is a closed set of $L^1(\Omega, E)$ and is nonempty if and only if the real function $d(0, X(\cdot))$ is in $L^1(\Omega, \mathbf{R})$. In this case, we shall say that a random set X is *integrable*. On the other hand, a random set X is said to be *strongly integrable* or *integrably bounded* if the function $|X(\cdots)|$ is in L^1 where $|X(\cdots)|$ is defined for all ω in Ω by $|X(\omega)| = \sup\{\|x\| : x \in X(\omega)\}$. Given a sub-$\sigma$-field $\mathcal{B}$ or $\mathcal{F}$, and a $\mathcal{F}$-measurable integrable random set X, Hiai and Umegaki ([15]) showed the existence of a $\mathcal{B}$-measurable random set G such that

$$S_G^1(\mathcal{B}) = cl\{E^{\mathcal{B}}(f) : f \in S_X^1(\mathcal{F})\},$$

the closure being taken in $\mathcal{L}^1(\Omega, E)$. G is the multivalued conditional expectation of X relative to $\mathcal{B}$ and is denoted by $E^{\mathcal{B}}X$.

For $K \in \text{cwk}(E)$ and $x^* \in E^*$, let $F(x^*, K) = \{x \in K : \langle x^*, x\rangle = \delta^*(x^*, K)\}$. Then $F(x^*, K)$ belongs to $\text{cwk}(E)$ where $\delta^*(\cdot, K)$ is the support function of subset K of E. Let $\mathcal{L}^1_{\text{cwk}(E)}(\mathcal{F})$ be the set of all $\mathcal{F}$-measurable integrably bounded multifunctions X from Ω to $\text{cwk}(E)$, by an easy argument, we show that, for every $x^* \in E^*$, $F(x^*, X(\cdot))$ belongs to $\mathcal{L}^1_{\text{cwk}(E)}(\mathcal{F})$ too.

Let $(\mathcal{F}_n)_{n\geq 1}$ be an increasing sequence of sub σ-algebras of $\mathcal{F}$. A sequence $(X_n)_{n\geq 1}$ in $L^1_{\text{cwk}(E)}(\mathcal{F})$ is $(\mathcal{F}_n)_{n\geq 1}$ adapted if $X_n \in L^1_{\text{cwk}(E)}(\mathcal{F}_n)$ for every n. An adapted sequence $(X_n, \mathcal{F}_n)_{n\geq 1}$ in $L^1_{\text{cwk}(E)}(\mathcal{F})$ is a supermartingale if $E^{\mathcal{F}_n} X_{n+1}(\omega) \subset X_n(\omega)$ for all $n \geq 1$ and all $\omega \in \Omega$. See [9] for details.

The following result is a particular case of a result due to Hess ([9], Proposition 3.7).

Proposition 1.1. *Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be a supermartingale in $\mathcal{L}^1_{\text{cwk}(E)}(\mathcal{F})$. Then there exists an adapted sequence $(f_n, \mathcal{F}_n)_{n\geq 1}$ in $L^1_E(\mathcal{F})$ such that*

(a) $(f_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale,
(b) for all $n \geq 1$, $f_n \in S^1_{X_n}(\mathcal{F}_n)$.

Let us recall the following notion of p-smooth Banach space given by Pisier [20].

Definition 1.2. Let E be a Banach space and $p \in [1, 2]$. We say that E is a p-smooth space or the norm of E is p-smooth, if the modulus of smoothness, ρ_E defined as

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x+ty\| + \|x-ty\| - 2) : \|x\| = \|y\| = 1\right\}$$

satisfies the following condition: there exists a constant k such that $\rho_E(t) \leq kt^p$ for each $t \in]0, \infty[$.

Example. Hilbert space, super reflexive space, the L^p_E space with p in $[1, 2]$ and E p-smooth space; are p-smooth spaces. We refer to Hoffmann-Jorgensen [16], Woyczynski [23] and Pisier [20] for details concerning p-smooth Banach spaces.

Before stating the main result, let us present first a useful lemma.

Lemma 1.3. *Let E be a separable p-smooth Banach space. Let $(c_n)_{n\geq 1}$ be a decreasing sequence in $\mathbf{R}^+$ such that $\lim_{n\to\infty} c_n = 0$ and $(g_n, \mathcal{F}_n)_{n\geq 1}$ a martingale in $L^1_E(\mathcal{F})$ such that* (*a*) $\lim_{k\to\infty} c_k^p E|g_k|^p = 0$,

(*b*) $\sum_{k=1}^{\infty} (c_k^p - c_{k+1}^p) E|g_k|^p < \infty$.

Then $\lim_{n\to\infty} c_n g_n = 0$ *a.s.*

Proof. Recall first the Chow inequality for positive integrable submartingales, see ([6], p. 107). If $(h_n, \mathcal{F}_n)_{n\geq 1}$ is a positive submartingale, then for every $\varepsilon > 0$ and every $m \geq 1$, we have

$$P\big[\max_{1\leq k\leq m} c_k h_k \geq \varepsilon\big] \leq E(h_1) + \sum_{k=2}^{m} c_k E(h_k - h_{k-1}) .$$

Since

$$\varepsilon^p P\big[\max_{k\geq 1} c_k |g_k| \geq \varepsilon\big] = \varepsilon^p P\big[\max_{k\geq 1} c_k^p |g_k|^p \geq \varepsilon^p\big]$$

and $(|g_k|^p, \mathcal{F}_k)_{k\geq 1}$ is a submartingale, by the preceding inequality, we get

$$\varepsilon^p P\big[\max_{k\geq n} c_k^p |g_k|^p \geq \varepsilon^p\big] \leq c_n^p E|g_n|^p + \sum_{k=n+1}^{\infty} c_k^p E(|g_k|^p - |g_{k-1}|^p) .$$

So we have to prove that

$$\lim_{n\to\infty} c_n^p E|g_n|^p + \sum_{k=n+1}^{\infty} c_k^p E(|g_k|^p - |g_{k-1}|^p) = 0 .$$

Since $\lim_{n\to\infty} c_n^p E|g_n|^p = 0$ by (a), it is enough to check that $\sum_{k=2}^{\infty} c_k^p E(|g_k|^p - |g_{k-1}|^p) < \infty$. But for all integer $m \geq 1$, we have

$$c_1^p E|g_1|^p + \sum_{k=2}^{\infty} c_k^p E(|g_k|^p - |g_{k-1}|^p) = \sum_{k=1}^{m-1} (c_k^p - c_{k+1}^p) E|g_k|^p + c_m^p E|g_m|^p .$$

Then by (a) and (b) we get

Theorem 1.4. *Let E be a p-smooth Banach space with strongly separable dual E^*, $({}_c n)$ as in Lemma 1.3 and $(X_i)_{i\geq 1}$ a sequence of $cwk(E)$-valued random variables. If the following two conditions are satisfied*

(i) $(S_n = \sum_{i=1}^{n} X_i . \mathcal{F}_n)_{n\geq 1}$ *is a supermartingale,*

(ii) $\sum_{i=1}^{\infty} c_i^p E|X_i|^p < \infty$,

then $0 \in s\text{-}lic_n S_n(\omega)$ *a.s.*

Proof. By Valadier [22] note that $E^{\mathcal{F}_n}.S_{n+1} in \mathcal{L}^1_{\text{cwk}(E)}(\mathcal{F}_n)$. Since $(S_n, \mathcal{F}_n)_{n\geq 1}$ is a supermartingale, then for all $n \geq 1$ and for all $x^* \in E^*$, we have by integrating

$$\int_\Omega \delta^*(x^*, E^{\mathcal{F}_n} S_{n+1}(\omega))P(d\omega) \leq \int_\Omega \delta^*(x^*, S_n(\omega))P(d\omega) .$$

Equivalently by Strassen's theorem, ([5], Theorem V.14), we get

$$\delta^*(x^*, \int_\Omega E^{\mathcal{F}_n} S_{n+1} P(d\omega)) \leq \delta^*(x^*, \int_\Omega S_n P(d\omega)) .$$

Consequently

$$\delta^*(x^*, \int_\Omega S_n(\omega)P(d\omega)) + \delta^*(x^*, \int_\Omega X_{n+1}(\omega)P(d\omega)) \leq \delta^*(x^*, \int_\Omega S_n(\omega)(d\omega)) .$$

Then $\delta^*(x^*, \int_\Omega X_{n+1}(\omega)P(d\omega)) \leq 0$ which implies that $\int_\Omega X_{n+1}(\omega) P(d\omega) = \{0\}$. By Lemma 5.7 in [13], there exists $f_{n+1} \in L^1_E(\mathcal{F})$ such that $Xn_{n+1}(\omega) = \{f_{n+1}(\omega)\}$ *a.s.* Then $\forall n \geq 1$,

$$S_n(\omega) = X_1(\omega) + f_2(\omega) + \cdots + f_n(\omega) \quad a.s.$$

Since $(S_n, \mathcal{F}_n)_{n\geq 1}$ is a supermartingale in $L^1_E(\mathcal{F})$, by Proposition 1.1 there exists an adapted sequence $(g_n, \mathcal{F}_n)_{n\geq 1}$ in $L^1_E(\mathcal{F})$ such that $(g_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale and for all $n \geq 1$, $g_n \in S^1_{S_n}$. So there exists h_1^n in $S^1_{X_1}$ such that $\forall n \geq 1$

$$g_n = h_1^n + f_2 + \cdots + f_n .$$

Since $(g_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale, then we have $E^{\mathcal{F}_n} g_{n+1} = g_n$, hence $h_1^n = h_1^{n+1} + E^{\mathcal{F}_n} f_{n+1}$. Therefore $g_{n+1} - g_n = f_{n+1} - E^{\mathcal{F}_n} f_{n+1}$. Set

$d_1 = g_1$ and $d_{i+1} = g_{i+1} - g_i$. By applying the Pisier's martingale inequality [20] to $(g_n, \mathcal{F}_n)_{n\geq 1}$, there exists a constant $B > 0$ such that

$$E|g_n|^p \leq B\sum_{i=1}^{n} E|d_i|^p = B(E|g_1|^p + \sum_{i=1}^{n} E|f_i - E^{\mathcal{F}_{i-1}} f_i|^p).$$

By Hölder inequality, we get

$$E|g_n|^p \leq 2^p B \sum_{i=1}^{n} E|f_i|^p \,. \tag{1}$$

Now we must show that $\lim_{n\to\infty} c_n g_n = 0$ *a.s.* By Lemma 1.3, it is enough to show that the following properties hold:

(a) $\lim_{k\to\infty} c_k^p E|g_k|^p = 0$,

(b) $\sum_{k=1}^{\infty} (c_k^p - c_{k+1}^p) E|g_k|^p < \infty$.

Indeed by (1) we have $c_k^p E|g_k|^p \leq 2^p B c_k^p \sum_{i=1}^{k} E|f_i|^p$. But

$$\sum_{i=1}^{\infty} c_i^p E|f_i|^p \leq \sum_{i=1}^{\infty} c_i^p E|X_i|^p < \infty$$

then by Kronecker's lemma ([17], p. 238) we have $\lim_{k\to\infty} c_k^p \sum_{i=1}^{k} E|f_i|^p = 0$. It follows that $\lim_{k\to\infty} c_k^p E|g_k|^p = 0$. This proves (a). While (b) follows from the inequalities

$$\begin{aligned}\sum_{k=1}^{\infty}(c_k^p - c_{k+1}^p) E|g_k|^p &\leq 2^p B \sum_{k=1}^{\infty} (c_k^p - c_{k+1}^p) \sum_{i=1}^{k} E|f_i|^p \\ &= 2^p B \sum_{i=1}^{\infty} E|f_i|^p \sum_{k=i}^{\infty} (c_k^p - c_{k+1}^p) \\ &= 2^p B \sum_{i=1}^{\infty} c_i^p E|f_i|^p < \infty \,.\end{aligned}$$

Then $\lim_{n\to\infty} c_n g_n(\omega) = 0$ *a.s.* Since $g_n(\omega) \in S_n(\omega)$ *a.s.*, so $0 \in s\text{-}lic_n S_n(\omega)$ *a.s.*

Corollary 1.5. *Let E be a p-smooth Banach space with strongly separable dual E^*. Let $(X_i)_{i\geq 1}$ be a sequence of cwk(E)-valued random sets as in Lemma 1.3. If the following two conditions are satisfied*

(i) $(S_n = \sum_{i=1}^{n} X_i, \mathcal{F}_n)_{n\geq 1}$ *is a martingale,*

(ii) $\sum_{i=1}^{\infty} c_i^p E|X_i|^p < \infty$.

Then M-$\lim_{n\to\infty} c_n S_n = \{0\}$ a.s..

Proof. Since $(S_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale and $\sum_{i=1}^{\infty} c_i^p E|X_i|^p < \infty$, by Theorem 1.4 it follows that $0 \in s\text{-}lic_n S_n(\omega)$ *a.s.* Now we show that $w\text{-}lsc_n S_n(\omega) = \{0\}$ *a.s.* Let $x^* \in E^*$, then $(\delta^*(x^*, S_n), \mathcal{F}_n)_{n\geq 1}$ is a real valued martingale and $\delta^*(x^*, S_n) = \sum_{i=1}^{n} \delta^*(x^*, Xi)$. By Lemma 1.3 it is enough to show that the two following conditions are satisfied

(a) $\lim_{k\to\infty} c_k^p E|\delta^*(x^*, S_k)|^p = 0$.

(b) $\sum_{k=1}^{\infty} (c_k^p - c_{k+1}^p) E|\delta^*(x^*, S_k)|^p < \infty$.

By Burkholder's inequality for a real martingales [3], there exists a constant A such that $\forall n \in \mathbf{N}^*$, we have

$$E|\delta^*(x^*, S_n)|^p \leq A \sum_{i=1}^{n} E|\delta^*(x^*, X_i)|^p .$$

Then for all $k \geq 1$ we have

$$c_k^p E|\delta^*(x^*, S_k)|^p \leq A c_k^p \sum_{i=1}^{k} E|\delta^*(x^*, X_i)|^p .$$

Since

$$\sum_{i=1}^{\infty} c_i^p E|\delta^*(x^*, S_i)|^p \leq \sum_{i=1}^{\infty} |x^*| c_i^p E|X_i|^p < \infty$$

by Kronecker's lemma ([17], p. 238), it follows that

$$\lim_{k\to\infty} c_k^p \sum_{i=1}^{k} E|\delta^*(x^*, X_i)|^p = 0 .$$

So
$$\lim_{k\to\infty} c_k^p E|\delta^*(x^*, S_k)|^p = 0 .$$

We have

$$\begin{aligned}\sum_{k=1}^{\infty}(c_k^p - c_{k+1}^p)E|\delta^*(x^*, S_k)|^p &\le A\sum_{k=1}^{\infty}(c_k^p - c_{k+1}^p)\sum_{i=1}^{k} E|\delta^*(x^*, X_i)|^p \\ &= A\sum_{i=1}^{\infty} E|\delta^*(x^*, X_i)|^p \sum_{k=i}^{\infty}(c_k^p - c_{k+1}^p) \\ &= A\sum_{i=1}^{\infty} c_i^p E|\delta^*(x^*, X_i)|^p < \infty .\end{aligned}$$

Whence (b) is proved. By Lemma 1.3 we conclude that

$$\lim_{n\to\infty} \delta^*(x^*, c_n S_n(\omega)) = 0 \ a.s.$$

Let $D = \{x_i^*,\ i \ge 1\}$ be a countable dense subset of E^*. Then for all $i \ge 1$, $\lim_{n\to\infty} \delta^*(x_i^*, c_n S_n(\omega)) = 0$ *a.s.* so there exists a negligible set $N \in \mathcal{F}$ such that $\forall\omega \in (\Omega \setminus N)$ and $\forall i \ge 1$, $\lim_{n\to\infty} \delta^*(x_i^*, c_n S_n(\omega)) = 0$ *a.s.* Let $\omega \in (\Omega \setminus N)$ and $x \in w\text{-}ls c_n S_n(\omega)$. Then there exists $x_{nk} \in c_{n_k} S_{nk}(\omega)$, $\forall k \ge 1$, such that

$$\begin{aligned}\forall i \ge 1,\ \langle x_i^*, x\rangle &= \lim_{k\to\infty}\langle x_i^*, x_{nk}\rangle \\ &\le \lim_{n\to\infty} \delta^*(x_i^*, c_n S_n(\omega)) = 0 \ a.s.\end{aligned}$$

Whence $w\text{-}ls c_n S_n(\omega) = \{0\}$. Finally $M\text{-}\lim_{n\to\infty} c_n S_n(\omega) = \{0\}$ *a.s.*

2. APPLICATION: LAW OF LARGE NUMBERS FOR INDEPENDENT cwk(E)-VALUED RANDOM VARIABLES

Before stating the main application, let us mention the following lemma.

Lemma 2.1. *Let E be a separable Banach space, Ω, $\mathcal{F}$, P) a probability space and $(X_i)_{i\ge1}$ a sequence of independent random variables in*

$L^1_E(\mathcal{F})$. Let $\mathcal{F}_n = \sigma(X_1, ..., X_n)$ be the sub σ-field of $\mathcal{F}$ generated by $(X_i)_{i\leq n}$. Then for all $i > n$, $E^{\mathcal{F}_n} X_i = E(X_i)$.

Proof. Let $\mathcal{C} = \{\bigcap_{i=1}^{n} A_i,\ A_i \in \sigma(X_i)\}$. Then $\mathcal{F}_n = \sigma(\mathcal{C})$. In fact we known that $\mathcal{F}_n = \sigma(\bigcup_{i=1}^{n} \sigma(X_i))$ and for all $A \in \sigma(X_i)$, $A \in \mathcal{C}$, so $\bigcup_{i=1}^{n} \sigma(X_i) \subset \mathcal{C}$, then $\sigma(\bigcup_{i=1}^{n} \sigma(X_i)) \subset \sigma(\mathcal{C})$. Now let $A \in \mathcal{C}$ then $A = \bigcap_{i=1}^{n} A_i$ with $A_i \in \sigma(X_i)$. So $A_i \in \sigma(\bigcup_{i=1}^{n} \sigma(X_i))$ for all $i \leq n$, then $\bigcap_{i=1}^{n} A_i = A \in \sigma(\bigcup_{i=1}^{n} \sigma(X_i))$. This implies that $\mathcal{C} \subset \sigma(\bigcup_{i=1}^{n} \sigma(X_i))$. Whence $\sigma(\mathcal{C}) \subset \sigma(\bigcup_{i=1}^{n} \sigma(X_i))$.

Now we show that for all $A \in \mathcal{F}_n$, $x^* \in E^*$ and $i > n$, 1_A and $\langle x^*, X_i\rangle$ are independent. From ([12], Property 9-1, p. 77) it sufficient to show this for all $A \in \mathcal{C}$. Let $A \in \mathcal{C}$, then $A = \bigcap_{i=1}^{n} A_i$ with $A_i \in \sigma(X_i)$ so there exists $D_1, ..., D_n$ in $\mathcal{B}(E)$ such that $A = \bigcap_{i=1}^{n} X_i^{-1}(D_i)$. Let B_1 and B_2 in $\mathcal{B}(\mathbf{R})$ we claim that

$$P[1_A \in B_1,\ \langle x^*, X_i\rangle \in B_2] = P[1_A \in B_1]\, P[\langle x^*, X_i\rangle \in B_2].$$

Set $H_1 = \{\omega : 1_A(\omega) \in B_1\}$ and $H_2 = \{\omega : \langle x^*, X_i(\omega)\rangle \in B_2\}$. Then

$$H_1 = \begin{cases} A & \text{if } 1 \in B_1 \text{ and } 0 \notin B_1 \\ A^c & \text{if } 1 \notin B_1 \text{ and } 0 \in B_1 \\ \Omega & \text{if } 1 \in B_1 \text{ and } 0 \in B_1 \\ \emptyset & \text{if } 1 \notin B_1 \text{ and } 0 \notin B_1 \end{cases}$$

if $H_1 = \Omega$ we have $P[H_1 \cap H_2] = P[H_1].P[H_2]$,
if $H_1 = \emptyset$ we have $P[H - 1 \cap H_2] = 0 = P[H_1].P[H_2]$,
if $H_1 = A$, we have $H_1 = \{\omega : X_1(\omega) \in D_1, ..., X_n(\omega) \in D_n\}$.
Since $\langle x^*, .\rangle$ is $(\mathcal{B}(E), \mathcal{B}(\mathbf{R})$-measurable, then $D = x^{*-1}(B_2) \in \mathcal{B}(E)$, and $H_2 = X_i^{-1}(D)$. Then

$$P[H_1 \cap H_2] = P[\omega : X_1(\omega) \in D_1, ..., X_n(\omega) \in D_n,\ X_i(\omega) \in D].$$

Since X_i is independent of $X_1, ..., X_n$, so

$$P[H_1 \cap H_2] = P[\omega : X_1(\omega) \in D_1, ..., X_n(\omega) \in D_n].P[\omega : X_i(\omega) \in D]$$
$$= P[H_1].P[H_2]$$

If $H_1 = A^c$, similarly we show that 1_A and $\langle x^*, X_i\rangle$ are independent. Now for all $x^* \in E^*$, $A \in \mathcal{F}_n$ and $i > n$ we have

$$\begin{aligned}\langle x^*, \int_A E^{\mathcal{F}_n} X_i(\omega)\rangle P(d\omega) &= \int_A \langle x^*, E^{\mathcal{F}_n} X_i(\omega)\rangle P(d\omega) \\ &= \int_A E^{\mathcal{F}_n} \langle x^*, X_i(\omega)\rangle P(d\omega) \\ &= \int_\Omega 1_A \langle x^*, X_i(\omega)\rangle P(d\omega) .\end{aligned}$$

From ([7], Theorem 3.3.3, p. 51) it follows that

$$\begin{aligned}\int_\Omega 1_A \langle x^*, X_i(\omega)\rangle P(d\omega) &= P(A) \int_\Omega \langle x^*, X_i(\omega)\rangle P(d\omega) \\ &= P(A)\langle x^*, \int_\Omega X_i(\omega) P(d\omega)\rangle \\ &= \langle x^*, P(A) \int_\Omega X_i(\omega) P(d\omega)\rangle .\end{aligned}$$

Put $\int_\Omega X_i(\omega)P(d\omega) = X$. Then

$$\begin{aligned}\langle x^*, \int_A E^{\mathcal{F}_n} X_i(\omega) P(d\omega)\rangle &= \langle x^*, P(A) \int_\Omega X_i(\omega) P(d\omega)\rangle \\ &= \langle x^*, P(A) X\rangle \\ &= \langle x^*, \int_A X P(d\omega)\rangle .\end{aligned}$$

Then $E^{\mathcal{F}_n} X_i = X = E(X_i)$.

Corollary 2.2. *Let E be a separable Banach space, $(\Omega, \mathcal{F}, P)$ a probability space and $(X_i)_{i\geq 1}$ a sequence of independent random variables in $L^1_E(\mathcal{F})$. Put $\mathcal{F}_n = \sigma(X_1, ..., X_n)$ and $S_n = \sum_{i=1}^{n} X_i$. If $E(X_i) = 0$ for all $i \geq 1$, then $(S_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale in $L^1_E(\mathcal{F})$.*

Proof. By construction S_n is $\mathcal{F}_n$-measurable and $E^{\mathcal{F}_n} S_{n+1} = E^{\mathcal{F}_n} S_n + E^{\mathcal{F}_n} X_{n+1}$. By Lemma 2.1, it follows that $E^{\mathcal{F}_n} X_{n+1} = E(X_{n+1})$. Since $E(X_{n+1} = 0$ then $E^{\mathcal{F}_n} S_{n+1} = E^{\mathcal{F}_n} S_n$, so $(S_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale.

Now we have the following application

Corollary 2.3. *Let E be a p-smooth Banach space with strongly separable dual E^* and $(X_i)_{i\geq 1}$ a sequence of independent random sets in $\mathcal{L}^1_{\text{cwk}(E)}(\mathcal{F})$ such that*

(i) $\sum\limits_{i=1}^{\infty} \frac{E|X_i|^p}{i^p} < \infty$,

(ii) $\int_\Omega X_i(\omega)P(d\omega) = C$ *for all* $i \geq 1$.

Then $M\text{-}\lim\limits_{n\to\infty} \frac{S_n(\omega)}{n} = C$ *a.s.*

Proof. Let us prove first the following inclusion, $C \subset s\text{-}li\frac{S_n(\omega)}{n}$ *a.s.*

In fact $C = \int_\Omega X_i(\omega)P(d\omega)$ is convex weakly compact, see Castaing-Valadier [5], Theorem V.14). Then by ([8], p. 106), C is the closed convex hull of its strongly exposed points. Since $s\text{-}li\frac{S_n(\omega)}{n}$ is closed and convex, it suffices to show that any exposed point of C is contained in $s\text{-}li\frac{S_n(\omega)}{n}$ for a.s. $\omega \in \Omega$. Let x be any exposed point of C, then there is $x^* \in E^*$ with $F(x^*, C) = \{x\}$. Let $\mathcal{A}_{X_i}$ be the σ-field generated by X_i. Then it is easy to check that, for all $i \geq 1$, $F(x^*, X_i(\cdot))$ is in $\mathcal{L}^1_{\text{cwk}(E)}(\mathcal{A}_{X_i})$. Let $f \in S^1_{F(x^*, X_i)}(\mathcal{A}_{X_i})$, then $f \in S^1_{X_i}(A_{X_i})$ and $\int_\Omega f(\omega)P(d\omega) \in \int_\Omega X_i(\omega)P(d\omega) = C$. But $\langle x^*, f(\omega)\rangle = \delta^*(x^*, X_i(\omega))$ *a.s.*, then

$$\langle x^*, \int_\Omega f(\omega)P(d\omega)\rangle = \delta^*(x^*, \int_\Omega X_i(\omega))P(d\omega) = \delta^*(x^*, C)$$

so

$$\int_\Omega f(\omega)P(d\omega) \in F(x^*, C) = \{x\}.$$

This implies that

$$\int_\Omega f(\omega)P(d\omega) = x.$$

Whence for all i, we get

$$\int_\Omega F(x^*, X_i(\omega))P(d\omega) = \{x\}.$$

From Hiai ([13, 14]) there exists $f_i \in S^1_X(\mathcal{A}_{X_i})$ such that

$$F(x^*, X_i(\omega)) = \{f_i(\omega)\} \ a.s.$$

Consequently, we have $\int_\Omega f_i(\omega)P(d\omega) = x$ for all $i \geq 1$. Set $g_i = f_i - x$ and $\mathcal{F}_n = \sigma(g_1, ..., g_n)$ and $h_n = g1 + \cdots + g_n$. since $(X_i)_{i\geq 1}$ is independent, by Hess ([10]) it follows that $(g_i)_{i\geq 1}$ is independent. By Lemma 2.1 and Corollary 2,2, we conclude that $(h_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale. Moreover, we have

$$\sum_{i=1}^{\infty} \frac{E|g_i|^p}{i^p} \leq 2^{p-1} \sum_{i=1}^{\infty} \frac{E|f_i|^p + |x|^p}{i^p}$$

$$= 2^{p-1}|x|^p \sum_{i=1}^{\infty} \frac{1}{i^p} + 2^{p-1} \sum_{i=1}^{\infty} \frac{E|f_i|^p}{i^p}$$

$$\leq 2^{p-1}|x|^p \sum_{i=1}^{\infty} \frac{1}{i^p} + 2^{p-1} \sum_{i=1}^{\infty} \frac{E|X_i|^p}{i^p} < \infty .$$

By Theorem 1.4 we have $\lim_{n\to\infty} \frac{h_n(\omega)}{n} = 0$ *a.a.* so $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{g_i(\omega)}{n} = 0$ *a.s.* Then $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{f_i(\omega)-x}{n} = 0$ *a.s.*. So $x \in s\text{-}li \frac{S_n(\omega)}{n}$ *a.s.* Then

$$C \subset s\text{-}li \frac{S_n(\omega)}{n} \subset C \ a.s.$$

Now we show that $w\text{-}ls \frac{S_n(\omega)}{n}$ *a.s.*. Let $D = \{x_j^*, \ j \geq 1\}$ be a countable dense subset of E^*. Then for every $j \geq 1$, $(\delta^*(x_j^*, X_n))_{n\geq 1}$ is a sequence of independent integrable random variables such that for all $n \geq 1$,

$$\int_\Omega \delta^*(x_j^*, X_n(\omega))P(d\omega) = \delta^*(x_j^*, \int_\Omega X_n(\omega))P(d\omega) = \delta^*(x_j^*, C)$$

and that

$$\sum_{i=1}^{\infty} \frac{E|\delta^*(x_j^*, X_i) - \delta^*(x_j^*, C)|^p}{i^p} \leq 2^{p-1}|\delta^*(x_j^*, C)|^p \sum_{i=1}^{\infty} \frac{1}{i^p}$$

$$+ 2^{p-1} \sum_{i=1}^{\infty} \frac{E|\delta^*(x_j^*, X_i)|^p}{i^p} < \infty .$$

By ([16]) it follows that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\delta^*(x_j^*,\, X_i(\omega))}{n} = \delta^*(x_j^*,\, C)\,.$$

Then there exists a negligible set $N \in \mathcal{F}$ such that for all $\omega \in (\Omega \setminus N)$ and $j \geq 1$ we have:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\delta^*(x_j^*,\, X_i(\omega))}{n} = \delta^*(x_j^*,\, C)\,.$$

Let $\omega \in (\Omega \backslash N)$ and $x \in w\text{-}ls\frac{S_n(\omega)}{n}$, then there exists $x_{nk} \in \frac{S_{nk}(\omega)}{n_k}$, $\forall k \geq 1$, such that

$$\langle x_j^*,\, x\rangle = \lim_{k\to\infty}\langle x_j^*,\, x_{nk}\rangle \leq \limsup_{n\to\infty}\delta^*\big(x_j^*,\, \frac{S_n(\omega)}{n}\big) = \delta^*(x_j^*,\, C),$$

then $x \in C$, so w-ls$\frac{S_n(\omega)}{n} \subset C$ *a.s.* and finally we get

$$M\text{-}\lim_{n\to\infty}\frac{S_n(\omega)}{n} = C \quad a.s.$$

3. EXISTENCE OF MARTINGALE DIFFERENCE SELECTORS OF MULTIVALUED MARTINGALE DIFFERENCE. APPLICATION TO MULTIVALUED SLLN

In this section we aim to prove the existence of martingales difference selectors for multivalued martingale difference which help us to state a new result of Mosco convergence of SLLN for unbounded multivalued martingales difference.

We begin by recalling some notations and definitions.

Let E be a Banach space. Let Ω, $\mathcal{F}$, $P)$ be a probability space and $(\mathcal{F}_n)_{n\geq 1}$ an increasing sequence of sub σ-algebra of $\mathcal{F}$. Let $(f_n,\, \mathcal{F}_n)_{n\geq 1}$ be an adapted sequence L_E^1, i.e., $\forall n \geq 1$, f_n is $\mathcal{F}_n$-measurable and integrable. $(f_n,\, \mathcal{F}_n)_{n\geq 1}$ is a martingale difference if $\forall n \geq 1$, $E^{\mathcal{F}_n} f_{n+1} = 0$ *a.s.*. This suggests the following definition.

Definition 3.1. Let $(X_i)_{i\geq 1}$ be a sequence of integrable multifunctions with nonempty closed convex valued and such that $\forall n \geq 1$, X_n is $\mathcal{F}_n$-measurable. We say that $(X_n, \mathcal{F}_n)_{n\geq 1}$ is a *multivalued martingale difference* if, $\forall n \geq 1$, $0 \in S^1_{E^{\mathcal{F}_n}X_{n+1}}(\mathcal{F}_n)$. An adapted sequence $(f_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale difference selector of multivalued martingale difference $(X_n, \mathcal{F}_n)_{n\geq 1}$, if $(f_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale difference and $\forall n \geq 1$, $f_n \in S^1_{X_n}(\mathcal{F}_n)$.

Before stating the Mosco convergence for SLLN of bounded multivalued martingales difference, let us first mention the following theorem of existence of martingale difference selectors of multivalued martingale difference.

Theorem 3.2. *Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be a martingale difference with valued in $\mathcal{R}$ such that*

$$\forall n \geq 1, \int_\Omega \sup_{f\in S^1_{X_{n+1}}(\mathcal{F}_{n+1})} |E^{\mathcal{F}_n} f| dP < \infty .$$

Then there exists a martingale difference selector of $(X_n, \mathcal{F}_n)_{n\geq 1}$.

Proof. Since $0 \in S^1_{E^{\mathcal{F}_n}X_{n+1}}(\mathcal{F}_n) = cl\{E^{\mathcal{F}_n}f | f \in S^1_{X_{n+1}}(\mathcal{F}_{n+1})\}$, then there exists a sequence $(E^{\mathcal{F}_n} f^i_{n+1})_{i\geq 1}$ with $f^i_{n+1} \in S^1_{X_{n+1}}(\mathcal{F}_{n+1})$ such that $\lim_{i\to\infty} E^{\mathcal{F}_n} f^i_{n+1} = 0$ for the norm of L^1_E. For all $\omega \in \Omega$, set

$$r_{n+1}(\omega) = d(0, X_{n+1}(\omega)) + \sup_{i\geq 1} |E^{\mathcal{F}_n} f^i_{n+1}(\omega)| .$$

Then r_{n+1} is $\mathcal{F}_{n+1}$-measurable and integrable. For all $\omega \in \Omega$, set

$$Y_{n+1}(\omega) = X_{n+1}(\omega) \cap \overline{B}(0, r_{n+1}(\omega))$$

Y_{n+1} is $\mathcal{F}_{n+1}$-measurable (see Hess [11], Proposition 3.3.3) and we have

$$\int_\Omega |Y_{n+1}|(\omega) P(d\omega) \leq \int_\Omega r_{n+1}(\omega) P(d\omega) < \infty$$

Then Y_{n+1} is $\mathcal{F}_{n+1}$-measurable integrably bounded multifunction, with values in cwk(E). By James-Pryce's theorem [21], it follows that $S^1_{Y_{n+1}}(\mathcal{F}_{n+1})$ is convex and $\sigma(L^1_E, (L^1_E)')$ compact. Then $C_n = \{E^{\mathcal{F}_n} f : f \in S^1_{Y_{n+1}}(\mathcal{F}_{n_1})\}$ is $\sigma(L^1_E, (L^1_E)')$ compact and convex. We claim that $0 \in C_n$. It is obvious that for all $i \geq 1$, $E^{\mathcal{F}_n} f^i_{n+1} \in C_n$. As $\lim_{i\to\infty} E^{\mathcal{F}_n} f^i_{n+1} = 0$ for the norm in L^1_E, then $0 \in C_n$. Hence there exists $g_{n+1} \in S^1_{Y_{n+1}}(\mathcal{F}_{n+1})$ such that $E^{\mathcal{F}_n} g_{n+1} = 0$. Let $g_1 \in S^1_{X_1}(\mathcal{F}_1)$, then $(g_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale difference selector of $(X_n, \mathcal{F}_n)_{n\geq 1}$.

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Theorem 3.3. *Let E be a Banach space with separable dual and such that the norm of E is equivalent to a p-smooth norm. Let $(X_n, \mathcal{F}_n)_{n\geq 1}$ be an integrable martingale difference with values in $\mathcal{R}$ such that*

(i) $\forall n \geq 1,\ \int_\Omega \sup_{f\in S^1_{X_{n+1}}(\mathcal{F}_{n+1})} |E^{\mathcal{F}_n} f| dP < \infty,$

(ii) $\sum_{i=1}^{\infty} \frac{E|X_i|^p}{i^p} < \infty.$

Then $0 \in s\text{-}li \frac{1}{n} \sum_{i=1}^{n} X_i(\omega)$ *a.s.*

Proof. By Theorem 3.2, there exists a selection martingale difference $(f_n, \mathcal{F}_n)_{n\geq 1}$ of $(X_n, \mathcal{F}_n)_{n\geq 1}$. Set

$$g_n = \sum_{i=1}^{n} f_i$$

then $(g_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale. By a Pisier's result in [20] there exists a positive constant B, such that

$$\forall n \geq 1,\ E|g_n|^p \leq B \sum_{i=1}^{n} E|f_i|^p .$$

By assumption (ii), it follows that

$$\sum_{i=1}^{\infty} \frac{E|f_i|^p}{i^p} < \infty .$$

We claim that $\lim_{n\to\infty} \frac{g_n}{n} = 0$ *a.s.* We have

$$\frac{E|g_k|^p}{k^p} \leq \frac{B}{k^p} \sum_{i=1}^{k} E|f_i|^p. \tag{2}$$

Since $\sum_{i=1}^{\infty} \frac{E|f_i|^p}{i^p} < \infty$ by (ii), then by Kronecker's lemma ([17], p. 144) and (2), it follows that

$$\lim_{k\to\infty} \frac{B}{k^p} \sum_{i=1}^{k} E|f_i|^p = 0 .$$

Moreover we have

$$\lim_{k\to\infty} \frac{E|g_k|^p}{k^p} = 0$$

and

$$\sum_{k=1}^{\infty} \Big(\frac{1}{k^p} - \frac{1}{(k+1)^p}\Big) E|g_k|^p \leq B \sum_{i=1}^{\infty} \frac{E|f_i|^p}{i^p} < \infty .$$

Then by Lemma 1.3 it follows that $\lim_{n\to\infty} \frac{g_n}{n} = 0$ *a.s.* Since $g_n(\omega) \in \sum_{i=1}^{n} X_i(\omega)$ *a.s.* then $0 \in s\text{-}li \frac{1}{n} \sum_{i=1}^{n} X_i(\omega)$ *a.s.*

Remark. Corollary 2.2 and Theorem 3.3 are actually valid when we replace $(\frac{1}{n})$ by any (c_n) where (c_n) is a positive decreasing sequence with $\lim_n c_n = 0$.

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Departement de Mathematiques
Universite Mohamed V
Faculte des Sciences
Agdal Rabat Maroc.