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ON A GERM-GRAIN MODEL [1]

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Abstract. *A germ-grain model containing only confluent disks is considered. The intensity of the point process characterizing this model and the corresponding area fraction are given.*

1. INTRODUCTION

Let $(N, \mathcal{N})$ be a measurable space where N is a family of subsets of R^2 and $\mathcal{N}$ is the smallest σ-algebra on N. Let $\Phi = \{X_n,\ n \geq 1\}$ be a point process in R^2.

Recall that a point process Φ is a measurable mapping of the probability space $(\Omega, \mathcal{A}, P)$ into $(N, \mathcal{N})$ if N is a family of all subsets φ of R^2 satisfying the following two conditions:

(i) The set φ is locally finite (i.e. each bounded subset of R^2 contains only a finite number of the X_n).

(ii) The set is simple (so $X_i \neq X_j$, if $i \neq j$).

Then each φ in N can be regarded as a closed subset of R^2. An element φ of N can also be regarded as a measure on R^2 so that $\varphi(B)$ is the number of points of φ in B.

A point process can be considered either as random sets of discrete points or as random measures counting the numbers of points lying in spatial regions. Corresponding to this we have two different notations:

$X \in \Phi$ asserts that the point X belongs to the random sequence Φ.

$\Phi(B) = n$ asserts that the set B contains n points of Φ.

Let $\Phi = \{X_n\}$ be the stationary Poisson process of intensity λ. Process Φ is divided into several parts (subprocesses). These subprocesses are denoted by $\Phi^{(K)} = \{X_n^{(K)}\}$, $k = 1, 2, 3, \ldots$ Sulprocesses $\Phi^{(K)}$ are defined as follows:

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$\{X_n(1)\}$ is obtained by sampling a Poisson process of intensity λ and deleting any point which is distanced within $2R$ from any other point, independently whether or not this point has already been deleted. It is the first Hard-Core process of Matern (cf. [1]).

$\Phi^{(2)} = \{X_n{}^{(2)}\}$ is obtained from a Poisson process of intensity λ by retaining all pairs of points such that each of which is the unique neighbour of the other one within distance $2R$.

$\Phi(3) = \{X_n{}^{(3)}\}$ is obtained from a Poisson process of intensity λ retaining of all triplets of points for which the circles of radius $2R$ centered at each of this triplet contain only points of this triplet.

Processes $\Phi^{(k)}$ can be understood as following:

Let $\{X_n\}$ be a Poisson process of intensity λ. If X_i and X_j are distanced from each other less than $2R$, we connect them by a segment. Then we get a graph model, which consists of points (called verties of a graph) and connected segments (called sides of a graph). By this definition, the process $\Phi^{(1)} = \{X_n{}^{(1)}\}$ is a set of graphs with one vertex (none side) (cf. Fig. 1(a)), the process $\Phi^{(2)} = \{X_n{}^{(2)}\}$ is a set of graphs with 2 verties (one side (cf. Fig. 1(b)), the process $\Phi^{(3)} = \{X_n{}^{(3)}\}$ is a set of graphs with 3 verties (2 sides (cf. Fig. 1(c)) or 3 sides (cf. Fig. 1 (d)), the process $\Phi^{(4)} = \{X_n{}^{(4)}\}$ is a set of graphs with 4 verties (3 or 4 sides (cf. Fig. 1(e), (f), (g), (h)),...

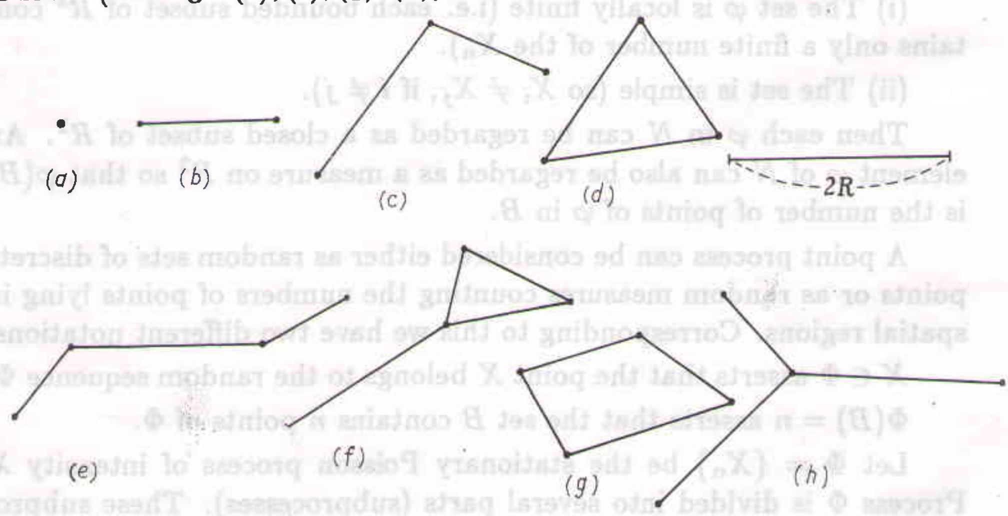

Figure 1. Types of a graph model

At each point X_n of the process Φ we put a circle of radius R centered at X_n. Then we get a Boolean germ-grain model (points X_n are germs, circles are grains). This model is stationary and isotropic (cf. [1]).

If points ${X_n}^{(k)}$ of processes $\Phi^{(k)}$ $(k = 1, 2, 3, 4...)$ are germs we get germ-grain submodels type k $(k = 1, 2, 3, 4...)$ from the Boolean model.

Let $C = \bigcup_{n=1}^{\infty} (X_n + b(0, R))$ be the Boolean model, where $X_n = \Phi$ are the germs, the circle of radius R centered at the origin 0, $b(0, R)$ is the primary grain. Then the submodel of type K will be

$$C^{(K)} = \bigcup_n (X_n^{(K)} + b(0, R)), \; k = 1, 2, 3, 4, ...$$

Recall that an intensity $\lambda^{(k)}$ $(k = 1, 2, 3...)$ is the mean number of points of process $\{{X_n}^{(k)}\}$ per unit area fraction $p^{(k)}$ $(k = 1, 2, 3...)$ of germ-grain model and $C^{(k)}$ is the mean of the area of $C^{(k)}$ in the unit area (cf. [1]).

Example. The area fraction of Boolean model C is $p = 1 - e^{-\lambda\pi R^2}$ (cf. [1]). It is of interest to find the intensities $\lambda^{(k)}$ of the processes $\Phi^{(k)}$ and the area fractions $p^{(k)}$ of the models $C^{(k)}$.

In this paper $\lambda^{(2)}$ and $p^{(2)}$ will be calculated.

Now we give a practical example of the above submodels:

A field of a membrane with red flood cells passing through its pores. The pore diameter is $5\,\mu$ (pore length about $11\,\mu$). The diameter of the average red cell is $7.5\,\mu$. To pass through a single pore the redcell has to deform (like a plastic bag not too full of water). If a cell meets a confluent pore it may not have to deform at all, it can just plop through.

The confluent pores in this membrane are germ-grain submodels considered in this paper.

It is well-known that for process $\{X_n^{(1)}\}, \lambda^{(1)} = \lambda^{(1)} = \lambda \exp(-4\pi\lambda R^2)$.

Process $\{{X_n}^{(1)}\}$ has a maximum intensity of $(4\pi e R^2)^{-1}$ as λ is varied (cf. [2], [3]).

For the model of type 1 $C^{(1)}$ the area fraction is $p^{(1)} = \lambda\pi R^2 \exp(-4\pi\lambda R^2)$ (cf. [3]).

2. MAIN RESULT

Theorem 2.

1. *Let λ be the intensity of stationary Poisson process. Then the*

intensity $\lambda^{(2)}$ of process $\Phi^{(2)}$ is

$$\lambda^{(2)} = 8\pi\lambda^2 R^2 e^{-8\lambda\pi R^2} \int_{2\pi/3}^{\pi} \frac{e^{4\lambda R^2 u}}{e^{4\lambda R^2 \sin u}} \sin u \, du$$

2. *The area fraction of model $C(2)$ with the germs being process $\Phi^{(2)}$ and the grains being confluent double circles is*

$$p^{(2)} = e^{-8\lambda\pi R^2} \lambda^2 \pi^4 R^8 \Big\{ 24 \int_{2\pi/3}^{\pi} \frac{e^{4\lambda R^2 u}}{e^{4\lambda R^2 \sin u}} \sin u \, du - \frac{5}{2} E e^{\lambda \gamma_{2R}(T)} \Big\}$$

where $Ee^{\lambda\gamma_{2R}(T)}$ is the expectation of $e^{\lambda\gamma_{2R}(T)}$, and T is the random variable with probability distribution:

$$\mathrm{Prob}\Big\{T < \frac{t}{R}\Big\} = 1 + \frac{1}{\pi}\Big\{2\Big(\frac{t^2}{R} - 1\Big)\arccos\frac{t}{2R} - \frac{t}{R}\Big(1 + \frac{t^2}{2R^2}\Big)\sqrt{1 - \frac{t^2}{4R^2}}\Big\}$$

$(0 < t \leq 2R)$.

Proof. First of all we introduce some concepts and notation:

$b(X, R)$ is the circle of radius R centered at X

$$b_1 = v_2(b(0, 2R)) = \pi(2R)^2,$$

$$b_2 = v_2(b(0, 4R)) = \pi(4R)^2,$$

v_2 is the Lebesgue measure in R^2.

A probability distribution P of the point process Φ is defined as follows:

$$P(Y) = P(\Phi \in Y) = P(\{\omega \in \Omega : \Phi(\omega) \in Y\}), \; Y \in \mathcal{N}.$$

The Palm distribution at X of P is a distribution defined on $[N, \mathcal{N}]$ by

$$P_X(Y) = P(\Phi \in Y \,\|_X), \; Y \in \mathcal{N},$$

which is the conditional probability of $(\Phi \in Y)$ that given that Φ has a point at X. If Φ is a stationary then

$$P(\Phi \in Y \,\|_X) = P(\Phi \in Y_X \,\|_0),$$

where $Y_X = \{\varphi_x = \varphi + x \in N : \varphi \in N\}$.

The Palm distribution at 0 of P is

$$P_0(Y) = P(\Phi \in Y \,\|_0),\; Y \in \mathcal{N}.$$

It is obvious that $P_0(Y) = P_X(Y_X)$, $\forall X$.

The reduced Palm distribution is defined as follows:

$$P_0!(Y) = P(\Phi - \{0\} \in Y \,\|_0),\; Y \in \mathcal{N},$$

$$P_X!(Y) = P(\Phi - \{X\} \in Y \,\|_X),\; Y \in \mathcal{N}.$$

For a stationary Poisson process we get $P_X! = P$, $\forall X$ (cf. [1]).

Now we construct an intensity measure $\Lambda^{(2)}$. For $\varphi \in N$ and $X \in R^2$, denote $Q(\varphi, x) = \bigcup_{Y \in b(X,2R)\cap\varphi} b(Y, 2R) \cap \varphi$. For a point process Φ, $Q(\Phi, X)$ is defined similarly. Then $\Lambda^{(2)}_{(B)}$ is the mean number of points of $\Phi^{(2)}$ in a Borel set $B \subseteq R^2$, i.e.

$$\Lambda^{(2)}_{(B)} = E\{\Phi^{(2)}_{(B)}\} = E \sum I_B(X_i)\, I_A(X_i),$$

where $A = \{X \in \Phi : \Phi(b(X, 2R)) = 1,\; Q(\Phi, X) = \{X\}\}$ and $I_A(X)$ is the indicator function of set A.

Using the refined Campbell theorem (cf. [1]) we get

$$\begin{aligned}
\Lambda^{(2)}_{(B)} &= \lambda \iint\limits_{R^2 N} I_{(B)}(x) I_{\{\varphi_{(x)}(b(x,2R)=2)\}} I_{\{Q(\varphi_{(x)},x)=\{X,Y\}\}} P_0(d\varphi)dx \\
&= \lambda \iint\limits_{R^2 N} I_{(B)}(x) I_{\{\varphi(b(0,2R))=2)\}} I_{\{Q(\varphi,0)=\{0,Y\}\}} P_0(d\varphi)dx \\
&= \lambda v_{(2)}(B) P_0\{\varphi(b(0,2R)) = 2,\; Q(\varphi, 0) = \{0, Y\}\} \\
&= \lambda v_{(2)}(B) P_0^!\{\varphi(b(0,2R)) = 1,\; Q(\varphi_0, 0) = \{Y\}\} \\
&= \lambda v_{(2)}(B) P\{\varphi(b(0,2R)) = 1,\; Q(\varphi, 0) = \{Y\}\} \\
&= \lambda v_{(2)}(B) \mathbb{P}\{\Phi(b(0,2R)) = 1,\; Q(\Phi, 0) = \{Y\}\}
\end{aligned}$$

where $\varphi_0 = \varphi - \{\theta\}$.

In order to calculate $P\{\Phi(b(0,2R)) = 1,\ Q(\Phi,0) = \{Y\}\}$ we consider the circle of radius $4R$ centered at 0. Suppose in the $b(0,4R)$ there are k points of process Φ $(k = 2, 3...)$. These k points are uniformly distributed in $b(0,4R)$. Denote domain $D = b(0,4R) - (b(0,2R) \cup \mathcal{D})$, where $\mathcal{D} = (b(0,2R) \cup b(Y,2R) \setminus b(0,2R)$ (cf. Fig. 2).

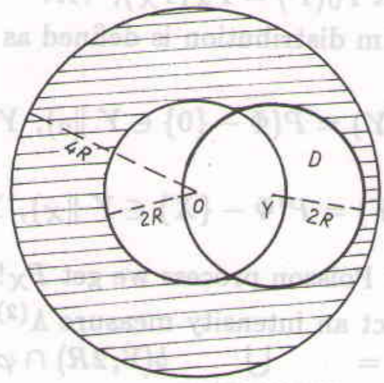

Figure 2. Domain D is the dashed region

We have

$$\begin{aligned}
&P\{\Phi(b(0,2R)) = 1,\ Q(\Phi,0) = \{Y\}\} \\
&= \sum_{k=1}^{\infty} e^{-\lambda b_2} \frac{(\lambda b_2)^k}{k!} C_k^1 \int\limits_{b(0,2R)} \int\limits_{D} \cdots \int\limits_{D} \frac{1}{(b_2)^k} dy dy_1 dy_2 ... dy_{k-1} \\
&= \sum_{k=1}^{\infty} e^{-\lambda b_2} \frac{\lambda^k}{(k-1)!} b_1 \int\limits_{b(0,2R)} \frac{dy}{b_1} \{v_2(D)\}^{k-1} \\
&= \sum_{k=1}^{\infty} e^{-\lambda b_2} \frac{\lambda^k}{(k-1)!} b_1 \int\limits_{b(0,2R)} \frac{dy}{b_1} [b_2 - 2b_1 + \gamma_{2R}(|Y|)]^{k-1} \\
&= \sum_{k=1}^{\infty} e^{-\lambda b_2} \frac{\lambda^k}{(k-1)!} b_1 E\big(b_2 - 2b_1 + \gamma_{2R}(|Y|)\big)^{k-1} \\
&= \lambda b_1 e^{-\lambda b_2} E \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \big(b_2 - 2b_1 + \gamma_{2R}(|Y|)\big)^{k-1} \\
&= \lambda b_1 e^{-\lambda b_2} E e^{\lambda(b_2 - 2b_1 + \gamma_{2R}(|Y|))} \\
&= \lambda b_1 e^{-2\lambda b_1} E e^{\lambda \gamma_{2R}(|Y|)} \qquad (1)
\end{aligned}$$

where Y is a random variable with the uniform distribution in $b(0,2R)$

and $\gamma_{2R}(t) = 8R^2\arccos\frac{t}{4R} - \frac{t}{2}\sqrt{16R^2-t^2}$, $0 \le t \le 4R$ (cf. [1]).
Then we have

$$Ee^{\lambda\gamma_{2R}(|Y|)} = \int\limits_{b(0,2R)} \frac{e^{\lambda 8R^2\arccos\frac{|Y|}{4R}}}{e^{\frac{\lambda|Y|}{2}\sqrt{16R^2-|Y|^2}}}\frac{v_2(dy)}{4\pi R^2}$$

$$= \int_0^{2\pi} d\varphi \int_0^{2\pi} \frac{e^{\lambda 8R^2\arccos\frac{1}{4R}}}{e^{\lambda\frac{1}{2}\sqrt{16R^2-t^2}}}\frac{tdt}{4\pi R^2} = 2\int_{\frac{2\pi}{3}}^{\pi} \frac{e^{4\lambda R^2 u}}{e^{4\lambda R^2 \sin u}}\sin u du\,. \tag{2}$$

By (1) and (2) we have

$$\Lambda^{(2)}(B) = 8\pi\lambda^2R^2v_2(B)e^{-8\lambda\pi R^2}\int_{\frac{2\pi}{3}}^{\pi}\frac{e^{4\lambda R^2 u}}{e^{4\lambda R^2\sin u}}\sin u du\,.$$

Therefore the intensity $\lambda^{(2)}$ is

$$\lambda^{(2)} = 8\pi\lambda^2R^2e^{-8\lambda\pi R^2}\int_{\frac{2\pi}{3}}^{\pi}\frac{e^{4\lambda R^2 u}}{e^{4\lambda R^2\sin u}}\sin u du\,,$$

and the proof of the first part of Theorem is complete.

It is known that, for the stationary and isotropic germ-grain model, in particular, for $C^{(2)}$,

$$p^{(2)} = Ev_2(C^{(2)}\cap C_0) = P(X \in C^{(2)}) = P(0 \in C^{(2)}) = 1 - P(0 \in C^{(2)})$$

where C_0 is the unit square (cf. [1]).

We will calculate $p^{(2)} = P(0 \in C^{(2)})$. For the model $C^{(2)}$ to cover the origin 0 the point 0 has to belong to a grain of $C^{(2)}$. The origin 0 can belong either to one circle or to two circles of a grain of $C^{(2)}$ (Fig. 3).

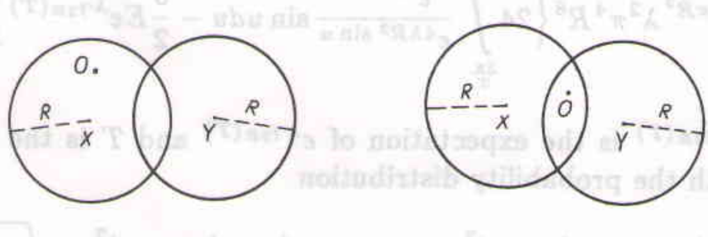

Figure 3. The point 0 belongs to a grain of model $C^{(2)}$.

Then we get

$$p^{(2)} = P\{\Phi(b(0,R)) = 1,\ \Phi(b(X,2R)\setminus b(0,R)) = 1,\ X\in b(0,R),$$
$$\Phi(b(Y,2R)\setminus \gamma_{2R}(|Y-X|)) = 0,\ Y\in b(X,2R)\setminus b(0,R)\}$$
$$+\ P\{\Phi(b(0,R)) = 2,\ \Phi(b(X,2R)\setminus b(0,R)) = 0,\ X\in b(0,R),$$
$$\Phi(b(Y,2R)\setminus \gamma_{2R}(|Y-X|)) = 0,\ Y\in b(0,R)\}$$

$$p^{(2)} = e^{-\lambda\pi R^2}\lambda\pi R^2 e^{-3\lambda\pi R^2}3\lambda\pi R^2 \int\limits_{x\in b(0,R)} v_2(dx)$$
$$\times \int\limits_{y\in b(X,2R)\setminus b(0,R)} e^{-\lambda(4\pi R^2-\gamma_{2R}(|Y-X|))}v_2(dy)$$
$$+\, e^{-\lambda\pi R^2}\frac{(\lambda\pi R^2)^2}{2!}e^{-3\lambda\pi R^2}\int\limits_{x\in b(0,R)} v_2(dx)$$
$$\times \int\limits_{y\in b(X,2R)\setminus b(0,R)} e^{-\lambda(4\pi R^2-\gamma_{2R}(|Y-X|))}v_2(dy)$$

$$p^{(2)} = e^{-8\lambda\pi R^2}9\lambda^2\pi^4R^8\int\limits_{x\in b(0,R)}\frac{v_2(dx)}{\pi R^2}\int\limits_{y\in b(X,2R)\setminus b(0,R)}\frac{e^{\lambda\gamma_{2R}(|Y-X|)}}{3\pi R^2}v_2(dy)$$
$$+\, e^{-8\lambda\pi R^2}\frac{1}{2}\lambda^2\pi^4R^8\int\limits_{x\in b(0,R)}\frac{v_2(dx)}{\pi R^2}\int\limits_{y\in b(0,R)}\frac{e^{\lambda\gamma_{2R}(|Y-X|)}}{\pi R^2}v_2(dy)$$
$$+\, e^{-8\lambda\pi R^2}\lambda^2\pi^4R^8\Big\{12\int\limits_{x\in b(0,R)}\frac{v_2(dx)}{\pi R^2}\int\limits_{y\in b(X,2R)}\frac{e^{\lambda\gamma_{2R}(|Y-X|)}}{4\pi R^2}v_2(dy)$$
$$-\frac{5}{2}\int\limits_{x\in b(0,R)}\frac{v_2(dx)}{\pi R^2}\int\limits_{y\in b(0,R)}\frac{e^{\lambda\gamma_{2R}(|Y-X|)}}{\pi R^2}v_2(dy)\Big\}$$
$$= e^{-8\lambda\pi R^2}\lambda^2\pi^4R^8\Big\{24\int\limits_{\frac{2\pi}{3}}^{\pi}\frac{e^{4\pi R^2u}}{e^{4\lambda R^2\sin u}}\sin u\,du - \frac{5}{2}Ee^{\lambda\gamma_{2R}(T)}\Big\},$$

where $Ee^{\lambda\gamma_{2R}(T)}$ is the expectation of $e^{\lambda\gamma_{2R}(T)}$ and T is the random variable with the probability distribution

$$\mathrm{Prob}\{T<\frac{t}{R}\} = 1+\frac{1}{\pi}\Big\{2\big(\frac{t^2}{R^2}-1\big)\arccos\frac{t}{2R}-\frac{t}{R}\big(1+\frac{t^2}{2R^2}\big)\sqrt{1-\frac{t^2}{4R^2}}\Big\}$$

$(0 < t \leq 2R)$ (cf. [3]).

This completes the proof of the theorem.

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