VIETNAM JOURNAL OF MATHEMATICS
Volume 24, Number 3, 1996

Short Communication

THE PROPERTIES $(\widetilde{\Omega}, H_u)$ AND THE TENSOR PRODUCT

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Let E be a Frechet space with an increasing fundamental system of semi-norms $\left\{\|\cdot\|_k\right\}_{k=1}^{\infty}$. For each subset B of E define a general semi-norm $\|.\|_B^*$ on E', the dual space of E, by

$$\|u\|_B^* = \sup\left\{|u(x)| : x \in B\right\}.$$

Write

$$\|\cdot\|_q^* = \|\cdot\|_{U_q}^*$$

for $B = U_q = \left\{x \in E : \|x\|_q \leq 1\right\}$.

We say that E has the properties

$(\widetilde{\Omega})$ if $\forall \rho\ \exists q \geq p,\ d > 0\ \forall k \geq q\ \exists C > 0 : \|\cdot\|_q^{*1+d} \leq C\|\cdot\|_k^*\|\cdot\|_p^{*d}$, and

(H_u) if every entire function on E is factorized holomorphically through a Banach space.

The properties $(\widetilde{\Omega})$, (H_u) and other properties were introduced and investigated by Meise and Vogt [3] in particular, by Vogt (see [5], [6],...). It is known [3] that $(\widetilde{\Omega}) \Rightarrow (H_u)$ in the class of nuclear Frechet spaces and obviously the property $(\widetilde{\Omega})$ is closed for the Descarte and tensor product operations. The question is of wherther this statement is true for the property (H_u). In [2] N. M. Ha and N. V. Khue have proved that if $E \in (\widetilde{\Omega})$ is nuclear Frechet and $F \in (H_u)$ is Frechet - Schwartz, then $E \times F \in (H_u)$.

The aim of the present note is to prove a similar result for the tensor product with additional conditions.

Theorem. *Let E and F be nuclear Frechet spaces such that either E or F has a Schauder basis. Let $E \in (\widetilde{\Omega})$ and $F \in (H_u)$, then $E\widehat{\otimes}_\pi F \in (H_u)$.*

Proof. Given $f \in H(E\widehat{\otimes}_\pi F)$, the space of holomorphic functions on $E\widehat{\otimes}_\pi F$. We have to find balanced convex neighbourhoods U and V of 0 in E and F respectively such that f is factorized the canonical map $\omega_{(U\otimes V)} : E\widehat{\otimes}_\pi F \to E_U\widehat{\otimes}_\pi F_V$, where E_U and F_V are Banach spaces associated to U and V, respectively.

(i) Let $\{e_j\}$ be a Schauder basis of F with the coefficient functionals $e_j^* \in F^*$, the dual space of F. Consider the Taylor expansion of f at $0 \in E\widehat{\otimes}_\pi F$

$$f(w) = \sum_{n\geq 0} P_n f(w),$$

where

$$P_n f(w) = \frac{1}{2\pi i} \int_{|\lambda|=\rho} \frac{f(\lambda w)}{\lambda^{n+1}} d\lambda,$$

for $n \geq 0$ and $w \in E\widehat{\otimes}_\pi F$.

Put

$$S(f)(u,v) = \sum_{n\geq 0} \sum_{j_1,\dots,j_n\geq 1} \widehat{P_n f}(u\otimes e_{j_1},\dots,u\otimes e_{j_n})\times \times e_{j_1}^*(v)\dots e_{j_n}^*(v) \tag{1}$$

for $u \in E$, $v \in F$, where $\widehat{P_n f}$ are symmetric n-linear forms associated to $P_n f$.

First we check that (1) defines $S(f) \in H(E\times F)$. Let $K \in \mathcal{B}(E)$ and $L \in \mathcal{B}(F)$, where $\mathcal{B}(E)$ and $\mathcal{B}(F)$ denote the families of compact balanced convex subsets of E and F, respectively. Since F is nuclear, we can find $\beta(L) \in \mathcal{B}(F)$ such that $L \subseteq \beta(L)$ and

$$\delta = \sum_{j\geq 1} \|e_j^*\|_L^* \|e_j\|_{\beta(L)} \leq \frac{1}{e^2},$$

where $\|e_j\|_{\beta(L)}$ denotes the norm of e_j in the Banach space $F_{\beta(L)}$ spanned by $\beta(L)$.

We have

$$\|S(f)\|_{K\times L} := \sup\Big\{|S(f)(u,v)| : u \in K, v \in L\Big\}$$
$$\leq \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1} \sup\Big\{\big|\widehat{P_n f}(u\otimes e_{j_1},\dots,u\otimes e_{j_n})e^*_{j_1}(v)\dots e^*_{j_n}(v)\big| : u \in K, v \in L\Big\}$$
$$\leq \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1} \sup\Big\{\Big|\widehat{P_n f}\Big(u\otimes \frac{e_{j_1}}{\|e_{j_1}\|_{\beta(L)}},\dots,u\otimes \frac{e_{j_n}}{\|e_{j_n}\|_{\beta(L)}}\Big)\Big| \times$$
$$\times \|e^*_{j_1}\|^*_L \, \|e_{j_1}\|_{\beta(L)} \cdots \|e^*_{j_n}\|^*_L \, \|e_{j_n}\|_{\beta(L)} : u \in K\Big\}$$
$$\leq \|(f)\|_{K\otimes\beta(L)} \sum_{n\geq 0} \frac{\delta^n n^n}{n!}.$$

Hence $S(f) \in H(E\times F)$ and moreover, (1) defines a continuous linear map $S : H(E\widehat{\otimes}_\pi F) \to H(E\times F)$.

By [2] we can find balanced convex neighbourhoods U and V of 0 in E and F, respectively, such that $S(f) \in H_b(E_U \times F_V)$, the Frechet space of holomorphic functions which are bounded on every bounded set in $E_U \times E_V$. Since

$$H(E\times F) \cong H(E, H(F)) \cong H(E)\widehat{\otimes}_\pi H(F),$$

without loss of generality we may assume that $S(f) \in H_b(E_U)\widehat{\otimes}_\pi H_b(F_V)$.

(ii) Let $\alpha \in H_b(E_U)$ and $\beta \in H_b(F_V)$. For each $n \geq 0$, define the n-linear form $\hat{P}_n(\alpha,\beta)$ on $(E_U\otimes F_V)^n$ by

$$\hat{P}_n(\alpha,\beta)\Big(\sum_{j_1\geq 1} u_{j_1}\otimes v_{j_1},\dots,\sum_{j_n\geq 1} u_{j_n}\otimes v_{j_n}\Big) =$$
$$= \sum_{j_1,\dots,j_n\geq 1} \widehat{P_n\alpha}(u_{j_1},\dots,u_{j_n})\widehat{P_n\beta}(v_{j_1},\dots,v_{j_n}). \qquad (2)$$

Let $P_n\alpha\otimes P_n\beta$ denotes the n-homogeneous polynomial on $E_U\widehat{\otimes}_\pi F_V$ induced by $\hat{P}_n(\alpha,\beta)$.

We have

$$\begin{aligned}
&\sum_{n\geq 0} \|P_n\alpha \otimes P_n\beta\|_{\mathrm{conv}(rU\otimes sV)} \\
&= \sum_{n\geq 0} \delta^{2n} \sup\Big\{ \Big|(P_n\alpha\otimes P_n\beta)\Big(\sum_{j\geq 1}\lambda_j u_j\otimes v_j\Big)\Big| : \\
&\qquad\qquad \sum_{j\geq 1}|\lambda_j| \leq 1,\ u_j \in \frac{r}{\delta}U, v_j \in \frac{s}{\delta}V\Big\} \\
&= \sum_{n\geq 0} \delta^{2n} \sup\Big\{ \sum_{j_1,\dots,j_n\geq 1} |\lambda_{j_1}|\cdots|\lambda_{j_n}| \big|(P_n\alpha\otimes P_n\beta)\times \\
&\qquad \times (u_{j_1}\otimes v_{j_1},\dots,u_{j_n}\otimes v_{j_n})\big| : \sum_{j\geq 1}|\lambda_j|\leq 1,\ u_j\in\frac{r}{\delta}U,\ v_j\in\frac{s}{\delta}V\Big\} \\
&= \sum_{n\geq 0} \delta^{2n} \sup\Big\{ \sum_{j_1,\dots,j_n\geq 1} |\lambda_{j_1}|\dots|\lambda_{j_n}|\ \big|\widehat{P_n\alpha}(u_{j_1},\dots,u_{j_n})\big| \\
&\qquad \big|\widehat{P_n\beta}(v_{j_1},\dots,v_{j_n})\big| : \sum_{j\geq 1}|\lambda_j|\leq 1,\ u_j\in\frac{r}{\delta}U,\ v_j\in\frac{s}{\delta}V\Big\} \\
&\leq \|\alpha\|_{\frac{r}{\delta}U}\|\beta\|_{\frac{s}{\delta}V}\sum_{n\geq 0}\frac{\delta^{2n}n^{2n}}{(n!)^2} = C(\delta)\|\alpha\|_{\frac{r}{\delta}U}\|\beta\|_{\frac{s}{\delta}V}\,.
\end{aligned}$$

Thus the form

$$(\alpha,\beta)\mapsto \sum_{n\geq 0} P_n\alpha\otimes P_n\beta \tag{3}$$

defines a continuous linear map R: $H_b(E_U)\widehat{\otimes}_\pi H_b(F_V) \longrightarrow H_b(E_U\widehat{\otimes}_\pi F_V)$.

(iii) It remains to check that

$$f = R\ S(f)\,.$$

Indeed, we have from (1), (2) and (3)

$$(R\,S)(f)\Big(\sum_{k\geq 1} u_k\otimes v_k\Big) = R(S(f))\Big(\sum_{k\geq 1} u_k\otimes v_k\Big) =$$

$$= \sum_{n\geq 0} R\Big(\sum_{j_1,\dots,j_n\geq 1} \widehat{P_n f}\big(\cdot\otimes e_{j_1},\dots,\cdot\otimes e_{j_n}\big)e^*_{j_1}(\cdot)\dots e^*_{j_n}(\cdot)\Big)\times$$

$$\times\Big(\sum_{k\geq 1} u_k\otimes v_k\Big)$$

$$= \sum_{n\geq 0}\sum_{j_1,\dots,j_n\geq 1}\sum_{k_1,\dots,k_n\geq 1}\widehat{P_n f}\big(u_{k_1}\otimes e_{j_1},\dots,u_{k_n}\otimes e_{j_n}\big)\times$$

$$\times\, e^*_{j_1}(v_{k_1})\dots e^*_{j_n}(v_{k_n})$$

$$= \sum_{n\geq 0}\sum_{k_1,\dots,k_n\geq 1}\widehat{P_n f}\times$$

$$\times\Big(u_{k_1}\otimes\sum_{j\geq 1}e^*_j(v_{k_1})e_j,\dots,u_{k_n}\otimes\sum_{j\geq 1}e^*_j(v_{k_n})e_j\Big)$$

$$= \sum_{n\geq 0}\sum_{k_1,\dots,k_n\geq 1}\widehat{P_n f}\big(u_{k_1}\otimes v_{k_1},\dots,u_{k_n}\otimes v_{k_n}\big)$$

$$= \sum_{n\geq 0} P_n f\Big(\sum_{k\geq 1}u_k\otimes v_k\Big) = f\Big(\sum_{k\geq 1}u_k\otimes v_k\Big)$$

for all

$$\sum_{k\geq 1} u_k\otimes v_k \in E\widehat{\otimes}_\pi F\,.$$

The case where E has a Schauder basis is proved similarly.

Acknowledgement. The author acknowledge Prof. Nguyen Van Khue for his kind advices and support during the preparation of the present paper.

REFERENCES

1. S. Dineen, *Complex analysis in locally convex spaces*, North-Holland Math. Stud., 57, 1981.
2. N. M. Ha and N. V. Khue, *The property* (H_u) *and* $(\widetilde{\Omega})$ *with the exponential representation of holomorphic functions*, Publicacions Matematiques, **38** (1994), 37-49.
3. R. Meise and D. Vogt, *Holomorphic functions of uniformly bounded type on nuclear Frechet spaces*, Studia. Math., **83** (1986), 147-166.
4. A. Pietsch, *Nuclear locally convex spaces*, 2nd edition Berlin, Springer-Verlag (1972) (Ergebnisse der Mathematik, 66).

5. D. Vogt, *Subspaces and quotien spaces of* (s), in Functional Analysis: Surveys and Recent Results, K-D Bierstedt, B. Funchssteiner (Eds.), North-Holland Mathematics Studies, **27** (1977), 167-187.
6. D. Vogt, *Characterisierung der Unterraume eines nuklearen stabilen Potenzreihenraume von endlichem Typ*, Studia. Math., **71** (1982), 251-270.
7. D. Vogt, *Eine Characterisierung der Potenzreihenraume von endlichem Typ und ihre Folgerungen*, Manuscripta. Math., **37** (1982), 269-301.
8. D. Vogt, *Frechtraume, zwischen denen jede stetige linear Abbildung beschraukt ist*, J. Reine Angew. Math., **345** (1983), 182-200.
9. D. Vogt, *On two classes of* (F)*-spaces*, Arch. Math., **45** (1985), 255-266.

Received July 20, 1996

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