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SPLINE COLLOCATION METHODS FOR NEUMANN PROBLEM FOR ELLIPTIC EQUATIONS

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Abstract. *In this paper fast direct methods are proposed for solving linear systems arising when orthogonal collocation with piecewise Hermite bicubics is employed for the approximate solution of Neumann problems for elliptic equations in a rectangle.*

The new methods, which are matrix decomposition algorithms involving fast Fourier transforms, require $O(N_1 N_2 \log N_1 N_2)$ arithmetic operations on an $N_1 \times N_2$ partition.

1. THE PROBLEM

Many authors [1], [2], [3], [4] have been interested in approximately solving partial differential equations which can be reduced to the following form

$$(A \otimes B + B \otimes A)\,\vec{u} = \vec{f}, \tag{1.1}$$

where A, B are $N \times N$ matrices (N a natural number), $\vec{u}$, $\vec{f}$ are column vectors of N^2 coordinates

$$\vec{u} = [u_{1,1}, \dots, u_{1,N}, \dots, u_{N,1}, \dots u_{N,N}]^T,$$
$$\vec{f} = [f_{1,1}, \dots, f_{1,N}, \dots, f_{N,1}, \dots, f_{N,N}]^T,$$

and $\otimes$ denotes the tensor product.

In [8], W. Sun and N. G. Zamani have established an algorithm to solve (1.1). They have proved that the eigenvalues of matrix $B^{-1}A$ are different and real. This follows that there exists a non degenerate matrix Q such that $B^{-1}A = Q\Lambda Q^{-1}$ (Λ is a diagonal matrix).

The purpose of this paper is to establish a more effective algorithm than Zamani's one to solve a system of the form

$$(A_1 \otimes B_2 + B_1 \otimes A_2)\,\vec{u} = \vec{f} \tag{1.2}$$

arising in the study of the following problem by orthogonal spline collocation

$$-\frac{\partial^2 u}{\partial x_1^2}-p(x_2)\frac{\partial^2 u}{\partial x_2^2}+q(x_2)\frac{\partial u}{\partial x_2}+r(x_2)u=f(x_1,\,x_2)\,,$$

$$(x_1,\,x_2)\in\Omega\equiv(a_1,\,b_1)\times(a_2,\,b_2) \tag{1.3}$$

$$\frac{\partial u}{\partial n}=0\,,\;\;(x_1,\,x_2)\in\delta\Omega\,,$$

where p, q, r, f are continuous functions, $\frac{\partial u}{\partial n}$ is an outer normal derivative.

Moreover, our algorithms allows us to obtain smooth solutions in the whole domain due to which we get more informations than difference methods about the exact solutions. the operations for finding the values of approximative solutions and its derivatives on computers are simple as well. With a new algorithm we can use the fast Fourier transform.

2. SPLINE COLLOCATION EIGENVALUE PROBLEMS

Let $\{x_k\}_{k=0}^N$ be an uniform partition of $[a,\,b]$, $x_k=a+kh$, $k=0,\dots,N$ with $h=\dfrac{b-a}{N}$.

Let H_3 be the space of cubic splines on the segment $[a,\,b]$ defined as follows

$$H_3=\left\{v\in C^1[a,\,b]:v\big|_{[x_k,x_{k+1}]}\in P_3,\;k=0,\dots,N-1\right\},$$

where P_3 denotes the set of polynomials of degree at most 3. We set

$$\overset{\circ}{H}_3=\{v\in H_3:v'(a)=v'(b)=0\}\,. \tag{2.1}$$

Lemma 1.

(i) *H_3 is a real linear space of dimensions $2N+2$.*

(ii) *$\overset{\circ}{H}_3$ is a $2N$-dimensional subspace of H_3.*

Proof. (i) See [7] page 59.

(ii) It is easy to prove $\overset{\circ}{H}_3$ is a subspace of H_3.

Now we take the system $B=\{\phi_{i0},\;i=0,\dots,N,\;\phi_{i1},\;i=0,\dots,N\}$ for a basic of H_3 (see[7] page 59).

Take an arbitrary $v(x)$, $v(x)\in\overset{\circ}{H}_3$. Then $v(x)\in H_3$, hence

$$v(x) = \sum_{n=0}^{N} a_n \phi_{n0}(x) + \sum_{n=0}^{N} b_n \phi_{n1}(x)\,,\ a_n,\, b_n \in R\,.$$

Since $v'(a) = b_0$. $v'(b) = b_N$ and $v(x) \in \mathring{H}_3$, we get $b_0 = b_N = 0$. So if we set

$$\phi_n(x) = \phi_{n0}(x),\ n = 0, ..., N\,,$$
$$\phi_{N+n}(x) = \phi_{n1}(x),\ n = 1, ..., N-1\,,$$

then the system $\{\phi_n(x)\}_{n=0}^{2N-1}$ is a basic of $\mathring{H}_3$. This system for $k = 0, ..., N$, has the following properties:

$$\phi_n(x_k) = \delta_{n,k}\,,\ \ \phi_n'(x_k) = 0 \text{ with } n = 0, ..., N\,, \tag{2.2a}$$

$$\phi_{N+n}(x_k) = 0\,,\ \ \phi_{N+n}'(x_k) = h^{-1}\delta_{n,k} \text{ with } n = 1, ..., N-1\,, \tag{2.2b}$$

(here $\delta_{n,k}$ is the Kronecker delta).

Let us denote by $\{\varsigma_m\}_{m=1}^{2N}$ the set of Gauss points of $[a, b]$ defined by

$$\varsigma_{2k+1} = x^{(k+1/2)} - \frac{h}{2\sqrt{3}}\,,\ \ \varsigma_{2k+2} = x^{(k+1/2)} + \frac{h}{2\sqrt{3}}\,, \tag{2.3}$$

where $x^{(k+1/2)} = \dfrac{x_k + x_{k+1}}{2}$, $k = 0, \ldots, N-1$.

Let us consider the following eigenfunction problem.

Find $U \in H_3$ so that:

$$-U''(\varsigma_m) = \lambda U(\varsigma_m)\,,\ \ m = 1, \ldots, 2N\,, \tag{2.4a}$$
$$U'(a) = U'(b) = 0\,. \tag{2.4b}$$

Lemma 2.

(i) *In the space* $\mathring{H}_3$ *the rule* $\langle u, v\rangle = h \sum_{m=1}^{2N} u(\varsigma_m)v(\varsigma_m)$, $u, v \in \mathring{H}_3$ *is an inner product.*

(ii) *We have:* $\langle u'', v\rangle = \langle u, v''\rangle$, $u, v \in \mathring{H}_3$.

Proof. (i) It is sufficient show that from $\langle v, v\rangle = 0$ it follows that $v = 0$. Indeed

$$\langle v, v\rangle = h\sum_{m=1}^{2N} v^2(\varsigma_m) = 0\,,$$

hence $v(\varsigma_m) = 0,\ m = 1, ..., 2N$.

Since $v \in \overset{\circ}{H}_3$ we have $v(x) = \sum\limits_{i=0}^{2N-1} x_i\,\phi_i(x)$, hence $\sum\limits_{i=0}^{2N-1} x_i\,\phi_i(\varsigma_m) = 0$, $m = 1, ..., 2N$ or $Bx = 0$, where

$$x = [x_0, ..., x_{2N-1}]^T,\;\; B = \left(b_{m,n}\right)_{m=1,\,n=0}^{2N,\,2N-1},\, b_{m,n} = \phi_n(\varsigma_m)\,.$$

Because B is non degenerate (Theorem 2.2), we obtain $x = 0$, that is $v = 0$.

(ii) Clearly that if $f \in C^4[a, b]$ then

$$\int_{x_k}^{x_{k+1}} f(x)dx = \frac{1}{2}h[f(\varsigma_{2k+1}) + f(\varsigma_{2k+2})] + \frac{f^{(4)}(\theta_k)h^5}{4320}\,,$$

where $\theta_k \in [x_k,\, x_{k+1}]$ (see [7] page 310).

For arbitrary $u,\, v \in \overset{\circ}{H}_3$, assume that on $[x_k,\, x_{k+1}]$, $k = 0, ..., N-1$ we get

$$u(x) = \sum_{i=0}^{3} a_{k,i}x^i\,,\;\; v(x) = \sum_{i=0}^{3} b_{k,i}x^i\,.$$

Denote by $(u,\, v)$ the inner product in $L_2[a,\, b]$. We have

$$(u'',\, v) = \int_a^b u''v\,dx = \sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}} u''v\,dx$$

$$= \sum_{m=1}^{2N} \frac{1}{2}hu''(\varsigma_m)v(\varsigma_m) + \sum_{k=0}^{N-1}\frac{a_{k,3}b_{k,3}h^5}{4320}\,.$$

Hence

$$(u'',\, v) = \frac{1}{2}\langle u'',\, v\rangle + \frac{h^5}{4320}\sum_{k=0}^{N-1} a_{k,3}b_{k,3}\,.$$

Similarly we obtain

$$(v'',\, u) = \frac{1}{2}\langle v'',\, u\rangle + \frac{h^5}{4320}\sum_{k=0}^{N-1} a_{k,3}b_{k,3}\,.$$

From $(u'', v) = (u, v'')$ it follows that $\langle u'', v\rangle = \langle u, v''\rangle$.

Theorem 1. *The eigenfunction problem (2.4) has $2N$ different non-negative eigenvalues defined by*

$$\lambda_j^{\pm} = 12\Big[\frac{8+\eta_j \pm \mu_j}{7-\eta_j}\Big] h^{-2},\ j = 1, \dots, N-1,$$
$$\lambda_0 = 0,\ \lambda_N = 9\,h^{-2},$$

where $\eta_j = \cos\frac{j\pi}{N}$, $\mu_j = \sqrt{43 + 40\eta_j - 2\eta_j^2}$.

With respect to $x \in [x_k, x_{k+1}]$, $k = 0, \dots, N-1$, the eigenfunctions are defined by

$$\begin{aligned} U_j^{\pm}(x) = c_j^{\pm}\Big\{\big[&-(1+\eta_j)(19-\eta_j \mp 2\mu_j) + 6(1-\eta_j)(2+\eta_j \mp \mu_j)\rho_k(x) \\ &+ 36(1-\eta_j^2)\rho_k^2(x) + 8(1-\eta_j)(4+5\eta_j \pm \mu_j)\rho_k^3(x)\big]\cos\frac{kj\pi}{N} \\ &+ \sqrt{1-\eta_j^2}\big[19-\eta_j \mp 2\mu_j + 6(2+\eta_j \mp \mu_j)\rho_k(x) \\ &- 36(1-\eta_j)\rho_k^2(x) + 8(4+5\eta_j \pm \mu_j)\rho_k^3(x)\big]\sin\frac{kj\pi}{N}\Big\}, \end{aligned} \tag{2.5}$$

$$U_0(x) = c_0,\ U_N(x) = (-1)^k c_N \rho_k(x)\big[4\rho_k^2(x) - 3\big],$$

where c_0, $c_j^{\pm}$, c_N are non zero constants, $\rho_k(x) = \dfrac{x - x^{(k+1/2)}}{h}$.

Proof. If $\lambda = 0$ then it is sufficient to choose $U_0(x) = c_0$ $(c_0 \neq 0)$. Obviously that $U_0(x)$ satisfies (2.4).

Assume that $\lambda \neq 0$ is a real eigenvalue and $U(x)$ a corresponding eigenfunction of (2.4), with $x \in [x_k, x_{k+1}]$, $k = 0, \dots, N-1$.

Let

$$U(x) = \alpha_k + \beta_k(x - x^{(k+1/2)}) + \gamma_k\frac{(x-x^{(k+1/2)})^2}{2} + \delta_k\frac{(x-x^{(k+1/2)})^3}{6}, \tag{2.6}$$

where α_k, β_k, γ_k, δ_k are constants. Putting (2.6) in (2.4a) and calculating α_k, β_k through γ_k, δ_k we obtain

$$\alpha_k = -\Big(\frac{1}{\lambda} + \frac{h^2}{24}\Big)\gamma_k,\ \beta_k = -\Big(\frac{1}{\lambda} + \frac{h^2}{72}\Big)\delta_k. \tag{2.7}$$

By the continuity of $U(x)$ and $U'(x)$ from (2.6) and (2.7) with

$$r = \frac{h^2}{12} - \frac{1}{\lambda}\,,\ s = \frac{h}{2}\Big(\frac{h^2}{36} - \frac{1}{\lambda}\Big)\,,\ t = \frac{2}{h}\Big(\frac{h^2}{9} - \frac{1}{\lambda}\Big)\,, \tag{2.8}$$

we get

$$\begin{aligned} r\gamma_{k+1} - s\delta_{k+1} &= r\gamma_k + s\delta_k\,, \\ \gamma_{k+1} - t\delta_{k+1} &= -\gamma_k - t\delta_k\,. \end{aligned} \tag{2.9}$$

Since

$$rt - s = \frac{1}{2h}\Big[\frac{h^4}{108} + \frac{2h^2}{9\lambda} + \frac{4}{\lambda^2}\Big] = \frac{1}{2h}\Big[\frac{1}{3}\Big(\frac{h^2}{6} + \frac{2}{h}\Big)^2 + \frac{8}{3\lambda^2}\Big] > 0\,,$$

form (2.9) it follows that

$$\begin{vmatrix} \gamma_{k+1} \\ \delta_{k+1} \end{vmatrix} = \frac{1}{rt - s}\begin{vmatrix} rt + s & 2st \\ 2r & rt + s \end{vmatrix}\begin{vmatrix} \gamma_k \\ \delta_k \end{vmatrix}\,. \tag{2.10}$$

Using (2.4b), we get

$$U'(a) = \beta_0 - \gamma_0\frac{h}{2} + \delta_0\frac{h^2}{8} = 0\,,$$

$$U'(b) = \beta_{N-1} - \frac{1}{2}h\gamma_{N-1} + \frac{1}{8}h^2\delta_{N-1} = 0 \tag{2.11}$$

or

$$t\delta_0 = \gamma_0\,,\ t\delta_{N-1} = -\gamma_{N-1}\,.$$

To find eigenvalues of λ we first suppose that

$$rst < 0\,. \tag{2.12}$$

By $rst < 0$ we get $|rt + s| < |rt - s|$. It follows that there exists an angle θ so that

$$\cos\theta = \frac{rt + s}{rt - s}\,,\ \theta \in (0, \pi)\,. \tag{2.13}$$

By (2.10), (2.13) we obtain

$$\gamma_k = \gamma_0\cos(k\theta) + \delta_0\frac{st}{\sqrt{-rst}}\sin(k\theta)\,,$$

$$\delta_k = \delta_0\cos(k\theta) - \frac{\gamma_0\sqrt{-rst}}{st}\sin(k\theta)\,,\ k = 0, \dots, N-1\,. \tag{2.14}$$

The use of (2.11), (2.13), (2.14) gives us

$$\operatorname{cotg}(N-1)\theta = -\frac{rt + s}{2\sqrt{-rst}} = -\cos\theta\,.$$

Hence $\theta = \dfrac{j\pi}{N}$, $j = 1, \dots, N-1$. Setting $\eta_j = \cos\dfrac{j\pi}{N}$ and using (2.13) we obtain

$$\eta_j = \frac{rt+s}{rt-s}.$$

Putting r, s, t from (2.8) into η_j we get a second degree equation

$$(7-\eta_j)(\lambda h^2)^2 - 24(8+\eta_j)(\lambda h^2) + 432(1-\eta_j) = 0. \qquad (2.15)$$

Solving (2.15) we obtain $\lambda_j^{\pm}$'s as defined in the theorem. Now we show that all just found $\lambda^{\pm}$'s satisfy (2.12).

Since $\lambda_j^{\pm}$ are solution of (2.15) we get

$$\frac{\lambda_j^{\pm} h^2}{12} = \frac{8+\eta_j \pm \sqrt{43+40\eta_j - 2\eta_j^2}}{7-\eta_j}.$$

Consider the function $y^{\pm} = \dfrac{8+x\pm\sqrt{43+40x-2x^2}}{7-x}$, $-1 < x < 1$.

It is obvious that y^+ (y^-) is increasing (decreasing) on $(-1, 1)$. Consequently

$$y^+(-1) < y^+(x) < y^+(1)\,, \ y^-(1) < y^-(x) < y^-(-1)\,.$$

It follows that

$$12h^{-2} < \lambda_j^+ < 36h^{-2} \quad 0 < \lambda_j^- < 9h^{-2}. \qquad (2.16)$$

From (2.8) and (2.16) we immediately see $rst < 0$.

Setting $t = 0$ we obtain $\lambda_N = 9h^{-2}$. Replacing $t = 0$ in (2.11) we get $\gamma_0 = \gamma_{N-1} = 0$. Using (2.9) we have

$$\gamma_k = 0\,, \ \delta_k = (-1)^k \delta_0\,, \ k = 0, \ldots, N-1\,. \qquad (2.17)$$

The eigenfunction corresponding to λ_N is

$$U_N(x) = \beta_k(x - x^{(k+1/2)}) + \delta_k \frac{(x-x^{(k+1/2)})^3}{6}.$$

Using (2.7) and (2.17) we get

$$U_N(x) = (-1)^k c_N \rho_k(x)[4\rho_k^2(x) - 3]\,.$$

For calculating the eigenfunctions corresponding to $\lambda_j^{\pm}$ we apply formulas (2.6), (2.7), 2.14) and then by reducing we will obtain $U_j^{\pm}(x)$ as stated in Theorem 1.

Corollary 1. *Let $U_j^{\pm}(x)$, $U_0(x)$, $U_N(x)$ be the eigenfunctions of (2.4) defined by (2.5). If we choose*

$$c_j^{\pm} = \frac{3}{2}\Big[\frac{3}{2(b-a)}\Big]^{\frac{1}{2}}\theta_j^{\pm}\,,\quad c_0 = \Big[\frac{1}{2(b-a)}\Big]^{\frac{1}{2}}\,,\quad c_N = \frac{3}{4}\Big[\frac{3}{2(b-a)}\Big]^{\frac{1}{2}}\,,$$

with $\theta_j^{\pm} = [27(1+\eta_j)(8+\eta_j\mp\mu_j)^2 + (1-\eta_j)(11+7\eta_j\mp4\mu_j)^2]^{-\frac{1}{2}}$, then the system $M = \{U_j^{\pm},\ j=1,...,N-1,\ U_0(x),\ U_N(x)\}$ is orthonormal with respect to the inner product $\langle\ ,\ \rangle$.

Proof. For arbitrary $U_i(x), U_j(x) \in M$, $i \neq j$ we have

$$U_i''(\varsigma_m) + \lambda_i U_i(\varsigma_m) = 0\,,\ \ U_j''(\varsigma_m) + \lambda_j U_j(\varsigma_m) = 0,\ \ m = 1, ..., 2N\,.$$

It implies

$$\langle U_i'' + \lambda_i U_i,\, U_j\rangle = 0\,,\ \langle U_j'' + \lambda_j U_j,\, U_i\rangle = 0\,.$$

Taking into account that $\langle U_i'', U_j\rangle = \langle U_j'', U_i\rangle$ and $\lambda_i \neq \lambda_j$, we get $\langle U_i, U_j\rangle = 0$. Orthonormalizing the system M we obtain the coefficients mentioned in Corollary 1.

Corollary 2. *Let M be an orthonormal system with coefficients mentioned in Corollary 1. For $j = 1, ..., N-1$, $k = 0, ..., N$ we have*

$$U_j^{\pm}(x_k) = \gamma\alpha_j^{\pm}\cos\frac{kj\pi}{N}\,,\quad \frac{dU_j^{\pm}(x_k)}{dx} = \gamma\beta_j^{\pm}h^{-1}\sin\frac{kj\pi}{N}\,,$$

where

$$\gamma = -5\Big[\frac{3}{2(b-a)}\Big]^{\frac{1}{2}}\,,\quad \alpha_j^{\pm} = (5+4\eta_j\mp\mu_j)\theta_j^{\pm}\,,\ \beta_j^{\pm} = -18\theta_j^{\pm}\sin\frac{j\pi}{N}$$

and

$$U_0(x_k) = -\frac{\gamma}{6\sqrt{3}}\,,\ \ U_N(x_k) = (-1)^{k+1}\frac{\gamma}{8}\,,$$

$$\frac{dU_0(x_k)}{dx} = 0\,,\ \ \frac{dU_N(x_k)}{dx} = 0\,.$$

Proof. For $k = 0, ..., N-1$ by substituting $\rho_k(x_k) = -\dfrac{1}{2}$ in (2.5) and reducing, we obtain immediately the results. For $k = N$ we have

$\rho_{N-1}(x_N) = \frac{1}{2}$. Then by substituting and reducing we get the desired formula. Similarly the other formulas are proved.

Now let $\{\phi_n\}_{n=0}^{2N-1}$ be a basic of $\overset{\circ}{H}_3$ and $U(x) = \sum\limits_{n=0}^{2N-1} z_n\phi_n(x)$. Then the eigenfunction problem (2.4) becomes the general eigenfunction one

$$A\vec{z} = \lambda = \lambda B\vec{z}, \tag{2.18}$$

where

$$\vec{z} = [z_0, ..., z_{2N-1}]^T,$$

$$\begin{aligned} A &= (a_{m,n})_{m=1\,n=0}^{2N,\,2N-1}, \quad a_{m,n} = -\phi_n''(\varsigma_m), \\ B &= (b_{m,n})_{m=1\,n=0}^{2N,\,2N-1}, \quad b_{m,n} = \phi_n(\varsigma_m). \end{aligned} \tag{2.19}$$

Theorem 2. *Let $\lambda_j^{\pm}$, $j = 1, ..., N-1$, λ_0, λ_N be eigenvalues of (2.4) given by Theorem 1 and $U_j^{\pm}$, U_0, U_N are corresponding orthonormal eigenfunctions given by Corollary 1. Let*

$$U_j^{\pm}(x) = \sum_{n=0}^{2N-1} z_{n,j}^{\pm}\phi_n(x), \; U_0(x) = \sum_{n=0}^{2N-1} z_{n,0}\phi_n(x),$$

$$U_N(x) = \sum_{n=0}^{2N-1} z_{n,N}\phi_n(x).$$

Set

$$\Lambda = diag(\lambda_1^-, ..., \lambda_{N-1}^-, \; \lambda_0, \; \lambda_1^+, ..., \lambda_{N-1}^+, \; \lambda_N), \tag{2.20a}$$

$$Z = h^{\frac{1}{2}}[\vec{Z}_1^-, ..., \vec{Z}_{N-1}^-, \; \vec{Z}_0, \; \vec{Z}_1^+, ..., \vec{Z}_{N-1}^+, \; \vec{Z}_N], \tag{2.20b}$$

where

$$\vec{Z}_j^{\pm} = [z_{0,j}^{\pm}, ..., z_{2N-1,j}^{\pm}]^T, \; \vec{Z}_0 = [z_{0,0}, ..., z_{2N-1,0}]^T,$$

$$\vec{Z}_N = [z_{0,N}, ..., z_{2N-1,N}]^T.$$

Then

$$AZ = BZ\Lambda, \;\; Z^T B^T BZ = I,$$

(I is the unit matrix of rank 2N).

Proof. Using directly Theorem 1, we can prove the first equality. By Corollary 1 we can show the second relation.

Now we try to find an explicit form of the matrix Z by means of a special basic of $\overset{\circ}{H}_3$. Choosing the basic system (2.2) of $\overset{\circ}{H}_3$ we get the representation, say:

$$U_j^{\pm}(x) = \sum_{n=0}^{N} z_{n,j}^{\pm}\phi_n(x) + \sum_{n=1}^{N-1} z_{n+N,j}^{\pm}\phi_{N+n}(x)\,.$$

By Corollary 2 we obtain

$$z_{N+k,j}^{\pm} = \gamma\beta_j^{\pm}\sin\frac{kj\pi}{N},\ k=1,...,N-1\,,$$

$$z_{k,j}^{\pm} = \gamma\alpha_j^{\pm}\cos\frac{kj\pi}{N},\ k=0,...,N\,,$$

$$z_{k,0} = -\frac{\gamma}{6\sqrt{3}},\ k=0,...,N\,,\ z_{N+k,0}=0,\ k=1,...,N-1\,,$$

$$z_{k,N} = (-1)^{k+1}\frac{\gamma}{8},\ k=0,...,N\,,\ z_{N+k,N}=0,\ k=1,...,N-1\,.$$

Set

$$\Lambda_\alpha^- = \mathrm{diag}(\alpha_1^-,...,\alpha_{N-1}^-)\,,\ \Lambda_\alpha^+ = \mathrm{diag}\Big(-\frac{1}{6\sqrt{3}},\alpha_1^+,...,\alpha_{N-1}^+,-\frac{1}{8}\Big)\,,$$

$$\Lambda_\beta^{\pm} = \mathrm{diag}(\beta_1^{\pm},...,\beta_{N-1}^{\pm})$$

$$S = \Big(\sin\frac{kj\pi}{N}\Big)_{k,j=1}^{N-1},\ \tilde{C} = \Big(\cos\frac{kj\pi}{N}\Big)_{k=0,\,j=1}^{N,\,N-1},\ C = \Big(\cos\frac{kj\pi}{N}\Big)_{k,j=0}^{N}.$$

Then we get

$$Z = -3\Big(\frac{6}{N}\Big)^{\frac{1}{2}}\left[\begin{array}{c|c} \tilde{C}\Lambda_\alpha^- & C\Lambda_\alpha^+ \\ \hline S\Lambda_\beta^-\ |\vec{0} & S\Lambda_\beta^+\ |\vec{0} \end{array}\right], \qquad (2.21)$$

($\vec{0}$ is a column vector zero of dimension $N-1$).

3. ALGORITHMS FOR SOLVING ORTHOGONAL SPLINE COLLOCATION EQUATION

3.1. Consider the problem

$$-\Delta u + cu = f(x_1,x_2),\ (x_1,x_2 \in \Omega \equiv (a_1,b_1)\times(a_2,b_2)\,, \qquad (3.1a)$$

$$\frac{\partial u}{\partial n} = 0,\ (x_1,x_2)\in\partial\Omega\,. \qquad (3.1b)$$

where c, a_i, b_i, $i = 1, 2$ are constants. $\frac{\partial u}{\partial n}$ is an outer normal derivative, Δ is Laplace operator, f is continuous on Ω. Moreover we assume that (3.1) has an unique solution. Let N_i, $i = 1, 2$ be positive integers $\{x_k^{(i)}\}_{k=0}^{N_i}$ an uniform of $[a_i, b_i]$, $h_i = \frac{b_i - a_i}{N_i}$, $\overset{\circ}{H}{}_3^{(i)}$ spaces of piecewise cubic splines on $[a_i, b_i]$ defined as in (2.1).

Let $\{\phi_{n_i}^{(i)}\}_{n_i=0}^{2N_i-1}$ be a basic of $\overset{\circ}{H}{}_3^{(i)}$ as in (2.2) with x_k, h, N replaced respectively by $x_k^{(i)}$, h_i, N_i. Then we have the orthogonal collocation approximation

$$U(x_1, x_2) = \sum_{n_1=0}^{2N_1-1} \sum_{n_2=0}^{2N_2-1} u_{n_1,n_2}\, \phi_{n_1}^{(1)}(x_1)\, \phi_{n_2}^{(2)}(x_2). \qquad (3.2)$$

The problem (3.1) will be solved by solving the following system

$$-\Delta U\big(\varsigma_{m_1}^{(1)}, \varsigma_{m_2}^{(2)}\big) + cU\big(\varsigma_{m_1}^{(1)}, \varsigma_{m_2}^{(2)}\big) = f\big(\varsigma_{m_1}^{(1)}, \varsigma_{m_2}^{(2)}\big), \qquad (3.3)$$

where $m_i = 1, ..., 2N_i$, $\{\varsigma_{m_i}^{(i)}\}_{m_i=1}^{2N_i}$ is a set of Gauss points on $[a_i, b_i]$ defined by (2.3) with x_k, h, N replaced respectively by $x_k^{(i)}$, h_i, N_i. Set:

$$\vec{u} = [u_{0,0}, ..., u_{0,2N_2-1}, ..., u_{2N_1-1,0}, ..., u_{2N_1-1,2N_2-1}]^T,$$

$$\vec{f} = [f_{1,1} ..., f_{1,2N_2}, ..., f_{2N_1,1}, ..., f_{2N_1,2N_2}]^T,$$

$$f_{m_1,m_2} = f\big(\varsigma_{m_1}^{(1)}, \varsigma_{m_2}^{(2)}\big).$$

Then (3.3) can be written in the form of a linear system of equations

$$(A_1 \otimes B_2 + B_1 \otimes A_2 + cB_1 \otimes B_2)\,\vec{u} = \vec{f}, \qquad (3.4)$$

where the matrices A_i, B_i are defined as in (2.19) with ϕ_n, ς_m, N replaced respectively by $\phi_{n_i}^{(i)}$, $\varsigma_{m_i}^{(i)}$, N_i.

Let Λ_i, Z_i be defined as in (2.20) and (2.21) with $[a, b]$, N and h replaced respectively by $[a_i, b_i]$, N_i and h_i. By Theorem 2 and by using some properties of tensor products it follows that the equation (3.4) is equivalent to

$$(\tilde{\Lambda}_1 \otimes I_2 + I_1 \otimes \Lambda_2)(Z_1^{-1} \otimes Z_2^{-1})\,\vec{u} = (Z_1^T B_1^T \otimes Z_2^T B_2^T)\,\vec{f}, \qquad (3.5)$$

where $\tilde{\Lambda}_1 = \Lambda_1 + cI_1$. Form (3.5) we obtain the following algorithm to solve (3.4).

Algorithm I

1. Calculate $\vec{g} = (Z_1^T\, B_1^T \otimes Z_2^T\, B_2^T)\, \vec{f}$.
2. Solve the system $(\tilde{\Lambda}_1 \otimes I_2 + I_1 \otimes \Lambda_2)\, \vec{v} = \vec{g}$.
3. Calculate $\vec{u} = (Z_1 \otimes Z_2)\, \vec{v}$.

From the work [6] (page 132) we see that the operation number to calculate the sums

$$f_1(k) = \sum_{n=0}^{N-1} C_1(n) \cos \frac{kn\pi}{N}\,,\; f_2(k) = \sum_{n=0}^{N-1} C_2(n) \sin \frac{kn\pi}{N}\,,$$

$$k = 0, ..., N-1\,.$$

(with $C_1(n)$, $C_2(n)$, $\cos \frac{kn\pi}{N}$, $\sin \frac{kn\pi}{N}$ given, $N = 2^n$) is $O(N \log N)$.

By the expansion of the product $(Z_1 \otimes Z_2)\, \vec{v} = \vec{g}$ we can assert that the operation number for this product depends on the calculation of four following sequences of the form

$$\begin{cases} f_{i,l} = \sum\limits_{n=0}^{N_2-1} C_{i,n} a_{n,l}\,, \\ h_{l,i} = \sum\limits_{k=0}^{N_1-1} f_{k,l} b_{i,k}\,, \end{cases}$$

where $i = 0, ..., N_1 - 1$, $l = 0, ..., N_2 - 1$,

$$a_{n,l} = \begin{cases} \cos \dfrac{nl\pi}{N_2}\,, \\ \sin \dfrac{nl\pi}{N_2}\,, \end{cases} \quad b_{i,k} = \begin{cases} \cos \dfrac{ik\pi}{N_1}\,, \\ \sin \dfrac{ik\pi}{N_1}\,. \end{cases}$$

For calculating $f_{i,l}$ we need $O(N_1\, N_2 \log N_2)$ operations. For $h_{l,i}$ we need $O(N_1\, N_2 \log N_1)$ operations. So the needed operation number is $O(N_1\, N_2 \log N_1 N_2)$. We have

$$\vec{g} = (Z_1^T\, B_1^T \otimes Z_2^T\, B_2^T)\, \vec{f} = (Z_1^T \otimes Z_2^T)(B_1^T \otimes B_2^T)\, \vec{f}\,.$$

Since the lines of B_1^T and B_2^T have four non zero elements at most therefore the operation number for calculating $(B_1^T \otimes B_2^T)\, \vec{f}$ is $O(N_1 N_2)$. By the use of this result for calculating $\vec{u}$ we see that the operation number for the product $(Z_1^T \otimes Z_2^T)(B_1^T \otimes B_2^T)\, \vec{f}$ is $O(N_1 N_2 \log N_2)$.

The second step is to solve a diagonal system of equations of rank $4\,N_1N_2$ so the operation number for this step is $O(N_1N_2)$. Consequently the operation number of the algorithm I is $O(N_1\,N_2\log N_1N_2)$.

3.2. Consider the Neumann problem for elliptic equations (1.3)

$$-\frac{\partial^2 u}{\partial x_1^2}-p(x_2)\frac{\partial^2 u}{\partial x_2^2}+q(x_2)\frac{\partial u}{\partial x_2}+r(x_2)u=f(x_1,\,x_2),\;(x_1,\,x_2)\in\Omega$$

$$\equiv(a_1,\,b_1)\times(a_2,\,b_2)\,,$$

$$\frac{\partial u}{\partial n}=0,\;(x_1,\,x_2)\in\partial\Omega\,,$$

where $p,\ q,\ r,\ f$ are continuous functions, $\frac{\partial u}{\partial n}$ is an outer normal derivative.

Assume that (1.3) has an unique solution. Then the collocation solution (3.2) of (1.3) can be obtained by solving the following linear system:

$$(A_1\otimes B_2+B_1\otimes A_2)\,\vec{u}=\vec{f}\,, \tag{3.6}$$

where

$$A_2=\left(a^{(2)}_{m,n}\right)^{2N_2,\,2N_2-1}_{m=1,\,n=0}\,,$$

$$a^{(2)}_{m,n}=p(\varsigma^{(2)}_m)\left[-\phi^{(2)''}_n(\varsigma^{(2)}_m)\right]+q(\varsigma^{(2)}_m)\phi^{(2)''}_n(\varsigma^{(2)}_m)+r(\varsigma^{(2)}_m)\phi^{(2)}_n(\varsigma^{(2)}_m)\,.$$

The matrices $A_1,\ B_1,\ B_2$ are defined as in (2.19). Replacing $\phi_n,\ \varsigma_n,\ N$ by $\phi^{(i)}_{n_i},\ \varsigma^{(i)}_m,\ N_i$ and by using Theorem 2 we can get

$$(\Lambda_1\otimes B_2+I_1\otimes A_2)(Z_1^{-1}\otimes I_2^{-1})\,\vec{u}=(Z_1^T\,B_1^T\otimes I_2)\,\vec{f}\,. \tag{3.7}$$

From (3.7) we get the following algorithm to solve (3.6).

Algorithm II

1. Calculate $\vec{g}=(Z_1^T\,B_1^T\otimes I_2)\,\vec{f}$.
2. Solve the system $(\Lambda_1\otimes B_2+I_1\otimes A_2)\,\vec{v}=\vec{g}$.
3. Calculate $\vec{u}=(Z_1\otimes I_2)\,\vec{v}$.

The first and third steps are similar to the ones of algorithm I. Choosing the basis system of $\overset{\circ}{H}{}^{(2)}_3$ as

$$\psi_0=\phi^{(2)}_0\,,\ \psi_1=\phi^{(2)}_1\,,\ \psi_3=\phi^{(2)}_2\,,\ \psi_{2N_2-1}=\phi^{(2)}_{N_2}\,,$$

$$\psi_{2k+1} = \phi_{k+1}^{(2)},\ k = 2..., N_2 - 2,\ \psi_{2k} = \phi_{N_2+k}^{(2)},\ k = 1, ..., N_2 - 1.$$

The second step is reduced to solve $2N_1$ systems of linear equations of rank $2N_2$. The coefficients of these $2N_1$ systems take the form $A_2 + \lambda_1 B_2$ (the lines and columns of this matrix have got at most four non zero elements). These matrices are of four diagonal type. Using the Gauss substitution we see that the operation number of the second step is $O(N_1 N_2)$. Consequently the algorithm II needs $O(N_1\, N_2 \log N_1 N_2)$ operations.

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REFERENCES

1. R. E. Bank, *Efficient algorithms for solving tensor product element equations*, Numer. Math., **31** (1978), 49 - 61.
2. B. Bialecki, G. Faiweather, and K. R. Bannett, *Fast direct solvers for piecewise Hermite bicubic orthogonal spline collocation equations*, SIAM J. Number. Anal. (to appear).
3. W. R. Dyksen, *Tensor product generalized ADI methods for separable elliptic problems*, SIAM J. Numer. Anal., **24** (1987), 59 - 76.
4. Yu. S. Koschliakov, E. B. Glinhier, and M. M. Smirnov, *Mathematical physic partial differential equations*, Moskva, 1970.
5. R. E. Lynch, J. R. Rice, and D. H. Thomas, *Direct solution of partial difference equation by tensor product methods*, Numer. Math., **6** (1964), 185 - 199.
6. G. L. Marchuk, *Methods of numerical mathematics*, Springer - Verlag, New York, 1975.
7. P. M. Prenter, *Spline and variational methods*, Wiley - Interscience, New York, 1975.
8. W. Sun and N. G. Zamani, *A fast algorithm for solving the tensor product collocation equations*, J. Franklin Inst., **326** (1989), 295 - 307.

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