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A CLASS OF LATTICES L DETERMINED BY Sub(L) UP TO ISOMORPHISM OR DUAL ISOMORPHISM

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Abstract. *In this paper we deal with a Gratzer's problem on Sub(L), the lattice of all sublatices of a lattice L. We give a condition on a lattice L such that Sub(L) determines L up to isomorphism or dual isomorphism.*

1. INTRODUCTION

In [1] G. Grätzer has proposed the following problem: "Find conditions on a lattice L under which the lattice Sub(L) determines L up to an isomorphism". Hoang Minh Chuong [2] has proved: "Let L be a modular lattice of locally finite length which has no linear decompositions. Then Sub(L) determines L up to isomorphism or dual isomorphism".

In connection with this problem, in [4] we have proposed the concept of a contractible sublattice. In this paper, we shall use this concept to prove the following theorem: "Let L be a lattice having no contractible sublattices, then Sub(L) determines L up to isomorphism or dual isomorphism".

For application, it will be proved that Boolean lattice have no contractible sublattices. Moreover, in a forthcoming paper, we describe some different types of lattices which also have this property: such as modular lattices having no linear decompositions, atomistic lattices... Therefore the class of lattices mentioned in our theorem is sufficiently large.

2. SOME NOTIONS

First we revise some notions and results of [4]:

Definition 2.1. Let $\varphi : L \to L'$ be a square preserving bijection. On L there exists a relation ρ_0 defined as follows: a, $b \in L$, $a\rho_0 b$ if either

$a < b$, $\varphi(a) > \varphi(b)$ or $a > b$, $\varphi(a) < \varphi(b)$. The equivalence generated by ρ_0 is called "φ-determined" and denoted by ρ or $\rho(\varphi)$.

For the equivalence $\rho(\varphi$ we have:

Theorem 2.2. *Let $\varphi : L \to L'$ be a square preserving bijection and A with $|A| > 1$ and equivalence class of $\rho(\varphi)$. Then:*

(a) *A is a convex sublattice.*

(b) *If $\langle a, b; c, d\rangle$ is a square on L then $c \in A \Leftrightarrow d \in A$.*

This theorem lead us to the following notion:

Definition 2.3. A proper sublattice A of an arbitrary lattice L with $|A| > 1$ is called a contractible sublattice if A satisfies conditions (a), (b) in Theorem 2.2.

Now, a definition of invariable intervals is introduced. This concept will be needed in Section 3.

From now on, we always consider a square preserving bijection denoted by $\varphi : L \to L'$.

Definition 2.4. Let $u, v \in L$ and $u < v$

1) If $\varphi(u) < \varphi(v)$ and $x \in [u, v] \Leftrightarrow \varphi(x) \in [\varphi(u), \varphi(v)]$, then $[u, v]$ is called an invariable interval of the type (I) with respect to φ.

2) If $\varphi(u) > \varphi(v)$ and $x \in [u, v] \Leftrightarrow \varphi(x) \in [\varphi(v), \varphi(u)]$, then $[u, v]$ is called an invariable interval of the type (II) with respect to φ.

Remark. For simplicity when φ is fixed we shall drop the sentence "with respect to φ". Further, if it does not make any confusion we shall write "invariable interval" instead of "invariable interval of the type (I) (or (II))".

Example 2.5. Let $\varphi : L \to L'$ be a square preserving bijection and $\langle a, b; a \wedge b, a \vee b\rangle$ be a square in L then $[a \wedge b, a \vee b]$ is an invariable interval either of the type (I) or (II).

The proof of (2.5) follows directly from the properties of the square.

Lemma 2.6. *If $[u_i, v_i]$, $i = 1, 2$, are invariable intervals of the type (I) containing the subset $A \neq \emptyset$, then $[u_1 \wedge u_2, v_1 \vee v_2]$ is also an invariable interval of the type (I) containing A.*

Proof. Without loss of generality, assume that $u_1 \| u_2$ (u_1 uncomparable with u_2) and $v_1 \| v_2$ (Fig. 1a). Once $a \in A$, it is easily seen that $u_1 < u_1 \vee u_2 \leq a \leq v_1 \wedge v_2 < v_1$ and thus $u_1 \vee u_2, v_1 \wedge v_2 \in [u_1, v_1]$. Since $[u_1, v_1]$

is invariable, we have $\varphi(u_1 \vee u_2) > \varphi(u_1)$ and $\varphi(v_1) > \varphi(v_1 \wedge v_2)$. For the squares $\langle u_1, u_2; u_1 \wedge u_2, u_1 \vee u_2 \rangle$ and $\langle v_1, v_2; v_1 \wedge v_2, v_1 \vee v_2 \rangle$ we also have $\varphi(u_1) > \varphi(u_1 \wedge u_2)$ and $\varphi(v_1 \vee v_2) > \varphi(v_1)$ respectively. Consequently $\varphi(v_1 \vee v_2) > \varphi(v_1) > \varphi(u_1) > \varphi(u_1 \wedge u_2)$ (Fig. 1b).

So far, we have $[u_1 \wedge u_2, v_1 \vee v_2]$ with $\varphi(u_1 \wedge u_2) < \varphi(v_1 \vee v_2)$. Denote $K = [\varphi(u_1 \wedge u_2, \varphi(v_1 \vee v_2]$ we shall show that $x \in [u_1 \wedge u_2, v_1 \vee v_2] \Leftrightarrow \varphi(x) \in K$.

(i) Let $x \in [u_1 \wedge u_2, v_2 \vee v_2]$. We shall prove $\varphi(x) \in K$.

Case 1: If x is uncomparable with at least one of the two elements u_1, u_2 then $\varphi(x) > \varphi(u_1 \wedge u_2)$; because if $\varphi(x) < \varphi(u_1 \wedge u_2)$ we must have xSu_1 (x is comparable with u_1) and xSu_2, but this is impossible.

To prove $\varphi(x) < \varphi(v_1 \vee v_2)$ we consider the relation between x and v_1, v_2.

(1) If x is uncomparable with at least one of the two elements v_1, v_2 then we obviously have $\varphi(x) < \varphi(v_1 \vee v_2)$.

(2) If xSv_1 and xSv_2 then $x < v_1, v_2$ so $\varphi(x) < \varphi(v_1), \varphi(v_2)$ and thus $\varphi(x) < \varphi(v_1 \vee v_2)$.

Case 2: If xSu_1 and xSu_2 then $x > u_1, u_2$, that leads to $\varphi(x) > \varphi(u_1), \varphi(u_2)$ and thus $\varphi(x) > \varphi(u_1 \wedge u_2)$.

Considering similarly the relation between x and v_1, v_2, we can easily deduce that $\varphi(x) < \varphi(v_1 \vee v_2)$.

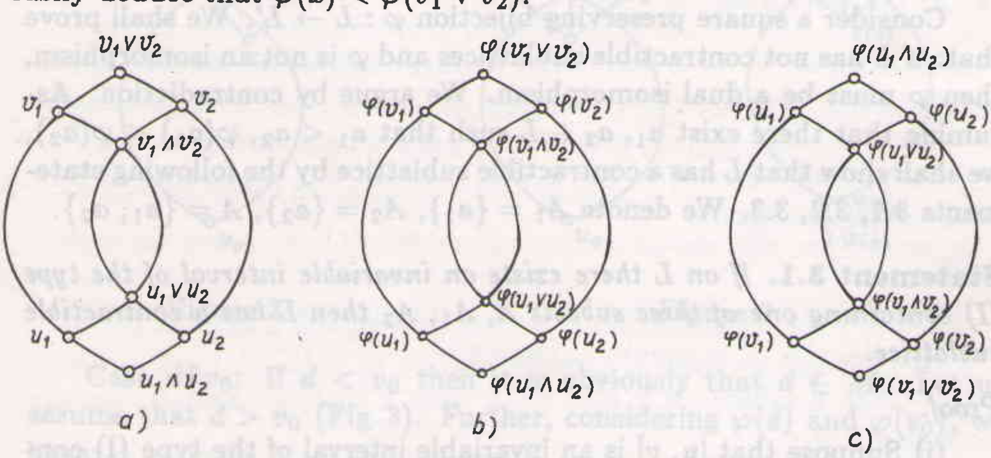

Figure 1

(ii) Let $\varphi(x) \in K$. We will prove $x \in [u_1 \wedge u_2, v_1 \vee v_2]$ by contradiction.

Suppose that $x \in [u_1 \wedge u_2, v_1 \vee v_2]$ then $x < u_1 \wedge u_2$ or $x > v_1 \vee v_2$.

(1) If $x < u_1 \wedge u_2$ then $x < u_1, u_2$. Moreover, $\varphi(x) > \varphi(u_1 \wedge u_2)$ implies that $\varphi(x) > \varphi(u_1), \varphi(u_2)$. From the invariability of $[u_i, v_i]$, $i = 1, 2$, we have $\varphi(x) > \varphi(v_1), \varphi(v_2)$ and therefore $\varphi(x) > \varphi(v_1 \vee v_2)$, which contradicts the fact that $\varphi(x) \in K$.

(2) If $x > v_1 \vee v_2$, by the same arguments we have $\varphi(x) < \varphi(u_1 \wedge u_2)$ which contradicts the inclusion $\varphi(x) \in K$.

In short, $\varphi(x) \in K$ then $x \in [u_1 \wedge u_2, v_1 \vee v_2]$.

The proof of the lemma is completed.

Lemma 2.7. *If $[u_i, v_i]$, $i = 1, 2$, are invariable intervals of the type (II) containing the subset $A \neq \emptyset$, then $[u_1 \wedge u_2, v_1 \vee v_2]$ is also an invariable interval of the type (II) containing A.*

Proof. Suppose $u_1 \| u_2$ and $v_1 \| v_2$. By examining the squares $\langle u_1, u_2;\ u_1 \wedge u_2, u_1 \vee u_2 \rangle$ and $\langle v_1, v_2;\ v_1 \wedge v_2, v_1 \vee v_2 \rangle$ we have $\varphi(u_1 \wedge u_2) > \varphi(u_1) > \varphi(v_1) > \varphi(v_1 \vee v_2)$ (Fig. 1a, 1c). Thus we have $[u_1 \wedge u_2, v_1 \vee v_2]$ with $\varphi(u_1 \wedge u_2) > \varphi(v_1 \vee v_2)$. Acting similarly as is the proof of Lemma 2.6, parts (i), (ii), we have $x \in [u_1 \wedge u_2, v_1 \vee v_2] \Leftrightarrow \varphi(x) \in [\varphi(v_1 \vee v_2), \varphi(u_1 \wedge u_2)]$. This completes the proof of Lemma 2.7.

3. MAIN THEOREM

Consider a square preserving bijection $\varphi : L \to L'$. We shall prove that: if L has not contractible sublattices and φ is not an isomorphism, then φ must be a dual isomorphism. We argue by contradiction. Assuming that there exist $a_1, a_2 \in L$ such that $a_1 < a_2$, $\varphi(a_1) < \varphi(a_2)$, we shall show that L has a contractible sublattice by the following statements 3.1, 3.2, 3.3. We denote $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $A = \{a_1, a_2\}$.

Statement 3.1. *If on L there exists an invariable interval of the type (I) containing one of three subsets A, A_1, A_2 then L has a contractible sublattice.*

Proof.

(i) Suppose that $[u, v]$ is an invariable interval of the type (I) containing A. Since L has no contractible sublattices and φ is not an isomorphism, L is an equivalence class and $u\rho v$ (Theorem 2.2). According to Definition 2.1 there exist $x_1, x_2, \ldots, x_n \in L$ such that $u\rho_0 x_1$, $x_1 \rho_0 x_2, \ldots, x_n \rho_0 v$, where, without loss of generality, we may assume that $u < x_1$, $\varphi(u) > \varphi(x_1)$. Since $\varphi(v) > \varphi(u) > \varphi(x_1)$, therefore vSx_1. Moreover, $x_1 > v$ because $[u, v]$ is invariable. Putting $k = x_1$ we obtain $u < v < k$ and $\varphi(k) < \varphi(u) < \varphi(v)$.

Consider M_1 as a family of all invariable intervals $[u', v']$ of the type (I) containing A and $v' < k([u, v] \in M_1)$. Denote $M = \cup M_1$, therefore $A \subseteq M$ and $k \notin M$. Consider $x, y \in M$ then $x \in [u_1, v_1]$, $y \in [u_2, v_2]$ $(\exists [u_1, v_1], [u_2, v_2] \in M_1)$.

By lemma 2.6 we have $[u_1 \wedge u_2, v_1 \vee v_2] \in M_1$. Note that $v_1 \vee v_2 < k$ because if $v_1 \vee v_2 = k$ then $\varphi(k) > \varphi(v_1) > \varphi(a_1)$ which contradicts the relation $\varphi(k) < \varphi(u_1) < \varphi(a_1)$. Therefore $x \wedge y, x \vee y \in [u_1 \wedge u_2, v_1 \vee v_2] \subseteq M$ and this implies that M is a sublattice of L.

Now we prove that M is contractible.

(a) Obviously M is convex.

(b) Let $\langle a, b; c, d\rangle$ with $c < d$ be a square on L. We shall prove that $c \in M \Leftrightarrow d \in M$.

1) Necessity: As $c \in M$, $c \in [u_0, v_0](\exists [u_0, v_0] \in M_1)$.

Case $d \| v_0$ (Fig. 2): Consider the square $\langle v_0, d; c_1, d_1\rangle$ where $c_1 = v_0 \wedge d$, $d_1 = v_0 \vee d$. As $\varphi(c_1) < \varphi(v_0)$ then $\varphi(d_1) > \varphi(v_0)$. Thus we have $[u_0, d_1]$ with $\varphi(u_0) < \varphi(d_1)$. It is easy to prove that $u_0, d_1] \in M_1$ and therefore we have $d \in M$.

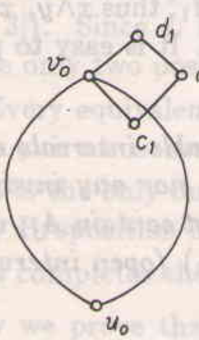

Figure 2

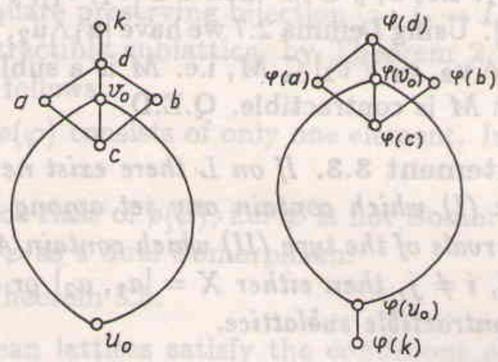

Figure 3

Case dSv_0: If $d < v_0$ then it is obviously that $d \in M$. Let us assume that $d > v_0$ (Fig. 3). Further, considering $\varphi(d)$ and $\varphi(v_0)$, we have:

If $\varphi(d) < \varphi(v_0)$ then $\varphi(d) < \varphi(u_0)$ because $[u_0, v_0]$ is invariable. Moreover, $[c, d]$ is also invariable (see Example 2.5). From $\varphi(u_0) > \varphi(d)$ and $v_0 \in [c, d]$ it follows $\varphi(u_0) > \varphi(v_0)$, but it is impossible.

Thus necessarily $\varphi(d) > \varphi(v_0)$ $(> \varphi(u_0))$ and we have an interval

$[u_0, d]$ with $\varphi(u_0) < \varphi(d)$. We can easily show that $[u_0, d] \in M_1$ and thus $d \in M$.

2) Sufficiency: By symmetry we have: $d \in M \Rightarrow c \in M$.

(ii) Now, suppose that there exists an invariable interval $[u, v]$ of the type (I) containing A_1. If $A_2 \subseteq [u, v]$ we have the case (i). Let us assume that $[u, v]$ does not contain A_2. Take M_1 as family of all invariable intervals of the type (I) which contain A_1 and do not contain A_2 ($[u, v] \in M_1$). Denote $M = \cup M_1$, it can be proved that M is a contractible sublattice.

Symmetrically, if on L there exists an invariable interval of the type (I) containing A_2 then L has a contractible sublattice. This completes the proof of the statement.

Statement 3.2. *If on L there exists an invariable interval on the type (II) containing A_i and which does not contain A_j, $i, j = 1, 2, i \neq j$, then L has a contractible sublattice.*

Proof. It is sufficient to consider the case where $i = 2$, $j = 1$.

Let M_1 be a set of all invariable intervals $[u, v]$ of the type (II) containing A_2 and which do not contain A_1. Obviously $a_1 < u$. Take $M = \cup M_1$. Let $x, y \in M$ then $x \in [u_1, v_1]$, $y \in [u_2, v_2]$ ($\exists [u_1, v_1], [u_2, v_2] \in M_1$). Using Lemma 2.7 we have $[u_1 \wedge u_2, v_1 \vee v_2] \in M_1$, thus $x \wedge y, x \vee y \in [u_1 \wedge u_2, v_1 \vee v_2] \subseteq M$, i.e. M is a sublattice on L. It is easy to prove that M is contractible. Q.E.D.

Statement 3.3. *If on L there exist neither invariable intervals of the type (I) which contain any set among A, A_1, A_2 nor any invariable intervals of the type (II) which contain A_i and do not contain A_j, $i, j = 1, 2, i \neq j$, then either $X = [a_1, a_2]$ or $Y = (a_1, a_2)$ (open interval) is a contractible sublattice.*

Proof.

(i) If $X \neq L$ we shall show that X is a contractible sublattice.

(a) Evidently X is convex sublattice.

(b) Take a square $\langle a, b; c, d \rangle$ on L with $c < d$, we have to prove: $c \in X \Leftrightarrow d \in X$.

1) Necessity: Consider d and a_2:

If $d \| a_2$ then a_2 belongs to $[d \wedge a_2, d \vee a_2]$ which is an invariable interval (see Example 2.5). Moreover, if it is of the type (II) then it does not contain A_1. This contradicts the conditions of the statement.

If $d S a_2$ and $d > a_2$ then $a_2 \in [c, d]$. But $[c, d]$ is also invariable.

Furthermore, if it is of the type (II) then it does not contain A_1. This is impossible. Therefore $d < a_2$, i.e. $d \in X$.

2) Sufficiency: Follows by symmetry.

(ii) If $X = L$ we prove that $Y = (a_1, a2)$ is a contractible sublattice.

As $X = L$ it is clear that $|Y| > 1$. Evidently Y satisfies conditions (a), (b) of definition 2.3. Q.E.D.

In short, by Statements 3.1, 3.2, 3.3. we come to the desired contradition i.e. we have proved.

Proposition 3.4. *Let L be a lattice having no contractible sublatices. If the square preserving bijection $\varphi : L \to L'$ is not isomorphic then φ is a dual isomorphism.*

Now we are ready to state the main theorem:

Theorem 3.5. *If L is a lattice having no contractible sublattices then Sub(L) determines L up to isomorphism of dual isomorphism.*

Proof. Suppose that L is lattice having no contractible sublattices and f: Sub(L) $\to$ Sub(L') is a lattice isomorphism. We have to prove either $L \cong L'$ or $L \overset{*}{\cong} L'$ (dually isomorphic).

As well-known, f induces a square preserving bijection $\varphi : L \to L'$ (see [2, 3]). Since L has no contractible sublattice, by Theorem 2.2 there are only two possibilities as follows:

1) Every equivalence class of $\rho(\varphi)$ consists of only one element. In this case φ is an isomorphism.

2) L is the only one equivalence class of $\rho(\varphi)$, i.e. φ is not isomorphic. By Proposition 3.4 we have φ as a dual isomorphism.

This completes the proof of Theorem 3.5.

Now we prove that the Boolean lattices satisfy the conditions of Theorem 3.5.

We recall that a distributive lattice B containing 0 and 1 is called Boolean if $\forall a \in B$, $\exists c \in B$ (which is called a complement of a) such that: $a \wedge c = 0$, $a \vee c = 1$.

Remark 3.6. The Boolean lattice B has no contractible sublatices.

Proof. We argue by contradiction, assuming that A is a contractible sublattice of B. Since $|A| > 1$, there exists $a, b \in A$ such that $a < b$. Clearly $0, 1 \overline{\in} A$. Take c as a complement of $a : a \wedge c = 0$, $a \vee c = 1$.

Consider the element $x = b \wedge c$. Since $a \vee x = a \vee (b \wedge c) =$

$(a \vee b) \wedge (a \vee c) = b$ we have $a \| x$. Applying (b) of Definition 2.3 to $\langle a, x; a \wedge x, b\rangle$ we have $0 = a \wedge x \in A$, but this is a desired contradiction.

The proof is completed.

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