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MINIMIZATION OF AN ALMOST-CONVEX AND ALMOST-CONCAVE FUNCTION[1]

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Abstract. *We present two algorithms for solving programming problems with the objective function being almost-convex and quasi-concave (not necessarily differentiable). The first algorithm is proposed for solving the problems over a linear constraint set. The second one deals with the case when the constraint set is supposed to be compact and convex. Some results of computational experiments are supplied.*

1. INTRODUCTION

In this paper we consider the programming problem with an almost-convex and quasi-concave objective function (not necessarily differentiable). Several well known programming problems, such as linear programs or linear fractional programs, are among this class.

In the first section we will recall and introduce some concepts and definitions concerning this problem. The second section deals with the case when the constraint set is supposed to be linear. In this case an algorithm, similar to the one of dual simplex method is proposed (algorithm 1). The more general case when the constraint set is compact and convex is treated in section 3. For solving this kind of problem we develop the algorithm 2 which is a combination of algorithm 1 and the outer-approximation scheme introduced in [3], [4], [6] and [7].

2. PRELIMINARIES

Let us recall some definitions.

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Definition 1. A function $f : R^n \to R^1$ is said to be quasi-concave if for any pair of points $x, y \in R^n$, and any real number $\alpha \in [0,1]$ the following inequality is satisfied:

$$f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}.$$

Definition 2. A function $f : R^n \to R^1$ is called quasi-convex if for any points x, $y \in R^n$, and any real number $\alpha \in [0, 1]$, it always satisfies

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}.$$

Definition 3. A function $f : R^n \to R^1$ is said to be almost-convex if it is quasi-convex and satisfies $f(\alpha x + (1-\alpha)y) \langle \max\{f(x), f(y)\}$, for all $x, y \in R^n$, $f(x) \neq f(y)$, and $\forall\alpha \in (0, 1)$.

The following properties of a quasi-convex and quasi-concave function f are immediates from their definitions:

1) $\min\{f(x), f(y)\} \leq f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$, for all $x, y \in R^n$ and $\forall\alpha \in (0,1)$.
2) If $f(x) = f(y)$ then $f(x) = f(\alpha x + (1-\alpha)y) = f(y)$, for all real number α.

So, if f is an almost-convex and quasi-concave function it must satisfy (1) and (2). Moreover, since f is almost-convex every its local minimizer must also be its global minimizer. Our algorithm is based on the following facts.

Theorem 1. *If f is an almost-convex and quasi-concave function, satisfying $f(x) \leq f(y)$ for all $x \neq y$, then $f(x) \leq f(x + \alpha(y-x))$ for all $\alpha \geq 0$.*

Proof. If $f(x) = f(y)$, for some $x \neq y$ then the assertion of Theorem 1 is an immediate consequence of property 2). Let us consider the case when $f(x) \neq f(y)$. From the quasi-concavity of $f(x)$ it follows that

$$f(x) = \min\{f(x), f(y)\} \leq f(\alpha y + (1-\alpha)x) = f(x + \alpha(y-x)), \forall\alpha \in [0, 1].$$

For $\alpha > 1$, that is $\frac{1}{\alpha} \in (0,1)$, y may be represented as

$$y = (1 - \frac{1}{\alpha})x + \frac{1}{\alpha}[x + \alpha(y-x)].$$

From the almost-convexity of $f(x)$ it implies

$$f(y) < \max\{f(x), f(x+\alpha(y-x))\}.$$

Hence,

$$f(x) < f(y) < f(x+\alpha(y-x)).$$

It is obvious from Theorem 1 that if f is an almost-convex and quasi-concave function and if for some $x, y \in R^n, f(x) \leq f(y)$ then x is a minimum point of f on the ray $x + \alpha(y-x), \alpha \geq 0$.

Theorem 2. *Let f be a continuous, almost-convex and quasi-concave function and z an arbitrary point in R^n. If $f(y) \geq f(x)$ and $f(x+z) \geq f(x)$, then:*

$$f(y+\alpha z) \geq f(y) \geq f(y-\alpha z), \forall \alpha \geq 0.$$

Proof. If $f(x+z) = f(x)$ then it is clear from the assumption of the theorem that $f(x) = f(x-\gamma z)$, for all $\gamma \in R^1$.

Assume that $f(x+z) > f(x)$. It is obvious that

$$f(x+z) > f(x) \geq f(x-\gamma z), \forall \gamma \geq 0.$$

So, in general, we always have

$$f(x) \geq f(x-\gamma z), \forall \gamma \geq 0. \tag{1}$$

Let

$$y(\gamma) = \frac{1}{1+\gamma} x + \frac{\gamma}{1+\gamma} y, \gamma \geq 0.$$

From the almost-convexity and quasi-concavity of $f(x)$ it follows

$$f(x) \leq f(y(\gamma)) \leq f(y), \forall \gamma \geq 0. \tag{2}$$

Combining (1) and (2) gives

$$f(x-\gamma z) \leq f(y(\gamma)), \forall \gamma \geq 0. \tag{3}$$

The point $y(\gamma)$ can be rewritten as

$$y(\gamma) = \frac{1}{1+\gamma}(x-\gamma z) + \frac{\gamma}{1+\gamma}(y+z).$$

From (3) and the quasi-concavity of f we can easily deduce that

$$f(x-\gamma z) \leq f(y(\gamma)) \leq f(y+z), \forall \gamma \geq 0 .$$

Therefore,

$$f(y(\gamma) \leq f(y+z), \forall \gamma \geq 0 .$$

Since f is continuous we have

$$\lim_{\gamma \to +\infty} f(y(\gamma)) = f(y) .$$

So

$$f(y) \leq f(y+z) . \tag{4}$$

From Theorem 1 and (4) we obtain

$$f(y) \leq f(y+\alpha z), \ \forall \alpha \geq 0 .$$

Since y may also be represented as

$$y = \frac{1}{2}(y-\alpha z) + \frac{1}{2}(y+\alpha z) ,$$

we get, from the assumptions of f, that

$$f(x-\alpha z) \leq f(y) \leq f(y+\alpha z), \forall \alpha \geq 0 .$$

The theorem is proved.

The following theorems specify a sufficient condition for a function f to be almost-convex and quasi-concave.

Theorem 3. *If $f : R^1 \to R^1$ is an almost-convex and quasi-concave function in u, and $g : R^n \to R^1$ is a linear function in x, then $f(g(.))$ is an almost-convex and quasi-concave function in x.*

Proof. For every pair $x, y \in R^n (x \neq y)$, and $\forall \alpha \in [0,1]$, we always have

$$f(g(\alpha x + (1-\alpha)y)) = f(\alpha g(x) + (1-\alpha) g(y)) . \tag{5}$$

Denoting $u_x = g(x)$ and $u_y = g(y)$, we can rewrite (5) as

$$f(\alpha g(x) + (1-\alpha)g(y)) = f(\alpha u_x + (1-\alpha)u_y) . \quad (6)$$

From the almost-convex and quasi-concave property of $f(u)$ we have:

$$\min\{f(u_x), f(u_y)\} \leq f(\alpha u_x + (1-\alpha)u_y) \\ \leq \max\{f(u_x), f(u_y)\}, \ \forall \alpha \in [0, 1] . \quad (7)$$

A combination of (5), (6) and (7) implies

$$\min\{f(g(x)), f(g(y))\} \leq f(g(\alpha x + (1-\alpha)y)) \leq \max\{f(g(x)), f(g(y))\} .$$

This means that $f(g(.))$ is quasi-convex and quasi-concave in x.

Since f is almost-convex it implies

$$f(\alpha u_x + (1-\alpha)u_y) < \max\{f(u_x), f(u_y)\} ,$$

for all $u_x, u_y \in R^1$, $f(u_x) \neq f(u_y)$ and $\forall \alpha \in (0, 1)$

$$\Longleftrightarrow f(g(\alpha x + (1-\alpha)y)) < \max\{f(g(x)), f(g(y))\} ,$$

for all $x, y \in R^n$, $f(g(x)) \neq f(g(y))$ and $\forall \alpha \in (0,1)$. That is $f(g(.))$ is also an almost-convex function in x. The proof is complete.

Theorem 4. *If*

1) $f : R^1 \to R^1$ is a fully monotone function in u.

2) $g : R^n \to R^1$ is a linear function in x.

Then $f(g(.))$ is an almost-convex and quasi-concave function in x.

Proof. For all $x, y \in R^n (x \neq y)$ and $\alpha \in [0,1]$, we have

$$f(g(\alpha x + (1-\alpha)y)) = f(\alpha g(x) + (1-\alpha)g(y)) = f(\alpha u_x + (1-\alpha)u_y) ,$$

where $u_x = g(x)$ and $u_y = g(y)$. Moreover, since $\alpha u_x + (1-\alpha)u_y \in [u_x, u_y]$ and $f(u)$ is monotone, it implies that

$$\min\{f(u_x), f(u_y)\} \leq f(\alpha u_x + (1-\alpha)u_y) \leq \max\{f(u_x), f(u_y)\} \\ \Leftrightarrow \min\{f(g(x)), f(g(y))\} \leq f(g(\alpha x + (1-\alpha)y)) \\ \leq \max\{f(g(x)), f(g(y))\} .$$

This means that $f(g(.))$ is quasi-convex and quasi-concave in x.

On the other hand, from the fully monotonicity of f we deduce

$$\min\{f(u_x), f(u_y)\} < f(\alpha u_x + (1-\alpha)u_y) < \max\{f(u_x), f(u_y)\}, \forall \alpha \in (0,1).$$

So, $f(g(.))$ is also almost-convex in x. This completes the proof of Theorem 4.

There are several classes of almost-convex and quasi-concave functions. Here we note some:

$$y = \langle a,\, x\rangle,\; y = \frac{\langle a,\, x\rangle + b}{\langle c, x\rangle + d},\; y = e^{\langle a, x\rangle},$$

$$y = \sqrt{\langle a,\, x\rangle}(\langle a,\, x\rangle \geq 0),\; y = \ln\langle a,\, x\rangle(\langle a,\, x\rangle > 0), \ldots$$

3. ALMOST-CONVEX AND QUASI-CONCAVE PROGRAMS OVER LINEAR CONSTRAINT SETS

3.1. Problem setting

In this section we will be concerned with the following programming problem:

$$(P) \qquad \begin{cases} f(x) \to \min, \\ \langle a^j,\, x\rangle + b_j \leq 0,\; j = 1, 2, \ldots, m. \end{cases} \tag{8}$$

Where f is a continuous, almost-convex and quasi-concave function (not necessary differentiable) in R^n, $a^j \in R^n, m \geq n$. Let assume that (8) is a convex polyhedron. We show that the linear structure of the constraint set of problem (P) may be exploited to develop an efficient finite algorithm for solving it, based on the outer-approximation scheme (see [3], [4], [6], [7],...). By its linear nature, this algorithm is somehow similar to the dual simplex method. At each iteration we construct some linear outer cone with the vertice being the minimizer for f over the given cone. This cone and the associated objective function value will be updated in subsequent steps until a solution to problem (P) is found (or it will reveals that (P) has empty feasible solution set).

3.2. Theoretical background

Definition 4. A polyhedral cone M with the vertice x^M is called a min-cone for f if

$$f(x^M) \leq f(x), \forall x \in M.$$

If, in addition, f is the objective function of problem (P), then M is called a min-cone of problem (P).

Let M be defined by

$$M := \{x \in R^n : \langle a^i, x\rangle + b_i \leq 0, \ i \in I\},$$

where $I \subset \{1, 2, ..., m\}, |\ I\ | = n$ and a^i are linearly independent. M is then a simplicial cone with the vertice x^M being the solution to the following system:

$$\langle a^i, x\rangle + b_i = 0, \ \forall i \in I, \tag{9}$$

and has exactly n edges z^i, $i \in I$, defined by

$$\begin{cases} \langle a^r, z^i\rangle = 0, & \forall r \in I, r \neq i, \\ \langle a^i, z^i\rangle = -1 & . \end{cases} \tag{10}$$

Now, assume that M is a min-cone for (P), and consider the set

$$J^+(x^M) := \{j \in \{1, ..., m\} : \langle a^j, x^M\rangle \ + b_j > 0\}.$$

It is clear that if $J^+(x^M) = \emptyset$ then x^M must be a feasible point for problem (P) and therefore it must also be a solution to (P). So, we assume that $J^+(x^M) \neq \emptyset$. For each $s \in J^+(x^M)$, we define the following sets:

$$I^s := \{i \in I : \ \langle a^s, z^i\rangle \neq 0\},$$
$$I^0 := \{i \in I : \ \langle a^s, z^i\rangle = 0\}.$$

For every $i \in I^s$, the line $x = x^M + t.z^i$ will intersect with the hyperplane $\langle a^s, x\rangle + b_s = 0$ at the point

$$x^i = x^M + \alpha_i.z^i, \tag{11}$$

where

$$\alpha_i = -\frac{\langle a^s, x^M\rangle + b_s}{\langle a^s, z^i\rangle}. \tag{12}$$

Let denote by I^s_+ the set

$$I^s_+ := \{i \in I^s : \alpha_i > 0\}.$$

Theorem 5. *If $I^s_+ = \emptyset$, then the feasible set of the problem (P) is empty.*

Proof. We proceed by contradiction. Assume that x^0 is a feasible point of (8). It's clear that x^0 must belong to the cone M and may be represented as

$$x^0 = x^M + \sum_{i\in I} \alpha_i . z^i,\ \alpha_i \geq 0,\ \forall i \in I. \tag{13}$$

Since $I^s_+ = \emptyset$ it implies that

$$\langle a^s, z^i\rangle\ > 0,\ \forall i \in I^s \Longrightarrow \langle a^s, z^i\rangle\ \geq 0,\ \forall i \in I. \tag{14}$$

Substituting (13) into the left hand side of $s-th$ constraint from (8) gives

$$\begin{aligned}\langle a^s,\, x^0\rangle + b_s &= \langle a^s,\, x^M + \sum_{i\in I} \alpha_i . z^i\rangle + b_s\\ &= \langle a^s, x^M\rangle + b_s + \sum_{i\in I} \alpha_i \langle a^s, z^i\rangle.\end{aligned} \tag{15}$$

From (13), (14) and the fact that $s \in J^+(x^M)$, we can deduce that the right hand side of (15) is positive. It means that $\langle a^s,\, x^0\rangle + b_s > 0$.

It contradicts the assumption imposed on x^0. The theorem is proved.

So, if (8) has a feasible point then the set I^s_+ must be non-empty. For $r \in I^s_+$, let us consider the set of all point x satisfying following inequalities

$$\begin{cases} \langle a^i, x\rangle + b_i \leq 0, \forall i \in I, i \neq r,\\ \langle a^s, x\rangle + b_s \leq 0.\end{cases}$$

This set determines a new cone (denoted by $M(r,\, s)$) with the vertice

$$x^{M(r,s)} = x^r = x^M + \alpha_r . z^r, \tag{16}$$

where α_r comes from (12).

The vertice x^r satisfies (9) with the new index set $I(r, s) = (I \cup \{s\}) \setminus \{r\}$. Thus, the new index set $I(r, s)$ may be obtained from the old index set I by just replacing one index (index s) by a new one (index r). The direction vectors z^i, $i \in I(r, s)$, of the new cone $M(r, s)$ can be determined by solving (10) with the new index set $I(r, s)$, or they may be obtained using following simple formulas which are direct consequences of (10) and (11):

$$z^i = \begin{cases} z^i & \text{if } i \in I^0, \\ x^i - x^r & \text{if } i \in I^s_+, i \neq r, \\ x^r - x^i & \text{if } i \in \{I \setminus \{I^s_+ \cup I^0\}\}, \\ z^r & \text{if } i = s. \end{cases} \tag{17}$$

Denote

$$V^{\min} := \{v \in I^s_+ : f(x^v) = \min_{i \in I^s_+}\{f(x^i)\}\}.$$

Theorem 6. *For each $r \in V^{\min}$, the cone $M(r, s)$ is the min-cone of problem (P).*

Proof. Since M is a min-cone of the problem (P) we have

$$f(x^M) \leq f(x^M + \alpha_i . z^i), \ \forall i \in I.$$

In particular, $f(x^M) \leq f(x^M + \alpha_r . z^r) = f(x^r)$. By applying Theorem 2 with $x = x^M$, $y = x^r$, $z = \alpha_i . z^i$, $i \in I^0$ and $i = s$ we obtain

$$f(x^r) \leq f(x^r + \alpha_i . z^i), \ \forall i \in I^0, \ i = s. \tag{18}$$

On the other hand, since $r \in V^{\min}$, we have

$$f(x^r) \leq f(x^i), \ \forall i \in I^s_+.$$

Therefore, from Theorem 1 we get

$$f(x^r) \leq f(x^i + \alpha(x^i - x^r)), \ \forall \alpha \geq 0, \forall i \in I^s_+, i \neq r. \tag{19}$$

Taking into acount that $\alpha_i < 0$ for every $i \in \{I \setminus (I^s_+ \cup I^0)\}$, we deduce from Theorem 2 that

$$f(x^i) = f(x^M + \alpha_i . z^i) \leq f(x^M), \ \forall i \in \{I \setminus (I^s_+ \cup I^0)\}.$$

But $f(x^M) \le f(x^r) \Rightarrow f(x^i) \le f(x^r)$, for all $i \in \{I \setminus (I_+^s \cup I^0)\}$.

Therefore, from Theorem 1, we have

$$f(x^i) \le f(x^i + \alpha(x^r - x^i)),\ \forall \alpha \ge 0, \forall i \in \{I \setminus (I_+^s \cup I^0)\}.$$

Applying Theorem 2 again gives

$$f(x^r) \le f(x^r + \alpha(x^r - x^i)),\ \forall \alpha \ge 0,\ \forall i \in \{I \setminus (I_+^s \cup I^0)\}. \tag{20}$$

A combination of (17), (18), (19) and (20) show that f is non-decreasing along the directions z^i for all $i \in I(r, s)$. It means that $M(r, s)$ is indeed a min-cone of the problem (P). The proof is complete.

In several optimization problems such a simplicial min-cone may be easily identified from the problem setting. If it is not the case, we can proceed as follows. Since the constraint set of (P) is a polyhedron, we cover it by a sufficiently large simplex and choose a vertice of that simplex where f reaches its minimum. The chosen vertice and the edges of the simplex, adjacent to it, determine then a min-cone of (P).

In the next subsection we will develop a simple algorithm, based on Theorems 5 and 6, for solving problem (P).

3.3. ALGORITHM 1.

Initialization. We start with an initial min-cone M_0 of the given problem (P). Its vertice x^0 and the direction vectors z^{i0} are determined respectively by (9) and (10).

Iteration $k (k = 1, 2, ...)$. Assume that a min-cone of (P), denoted by M_k, has been found. Its index set, vertice and direction vectors are respectively I_k, x^k, and z^{ik}.

Calculate $J^+(x^k)$.

a) If $J^+(x^k) = \emptyset$ then stop, x^k is a solution to (P).

b) If $J^+(x^k) \ne \emptyset$ then compute $s_k = \min\{j : j \in J^+(x^k)\}$ and determine $I_+^{s_k}$.

b.1) If $I_+^{s_k} = \emptyset$ then stop, (8) has no feasible point.

b.2) If $I_+^{s_k} \ne \emptyset$ then intersection points of the hyperplane $\langle a^{s_k}, x\rangle + b_{s_k} = 0$ with all edges of M_k will be calculated using (11):

$$x^{ik} = x^k + \alpha_i^k . z^{ik}.$$

Next, calculate

$$V_k^{\min} := \{v \in I_+^{s_k} : f(x^{vk}) = \min_{i \in I_+^{s_k}} \{f(x^{ik})\}\}$$

and

$$r_k = \min\{v : v \in V_k^{\min}\}\,.$$

Then we construct $M_{k+1} = M_k(r_k, s_k)$; $I_{k+1} = I_k(r_k, s_k) = (I_k \cup \{s_k\}) \setminus \{r_k\}$; $x^{k+1} = x^{M_k(r_k,s_k)} = x^{r_k}$; (from (16)), and new direction vectors $z^{i,k+1}$ (using (17)).
Return to iteration k with $k \leftarrow k+1$.

Notes.

1) From Theorem 6 it is easy to see that the newly constructed cone M_{k+1} is still a min-cone of the problem (P).

2) The choice of $r_k = \min\{v : v \in V_k^{\min}\}$ and $s_k = \min\{j : j \in J^+(x^k)\}$ prevents our algorithm from cycling. So, it can solve problem (P) in a finite time.

Theorem 7. *Algorithm 1 terminates after a finite number of iterations, either indicating that the feasible set (8) is empty or yielding an optimal solution to problem* (P).

Proof. We will show that the selection rule for r_k (the index going out of index set I_k) and s_k (the index entering into I_k) prevents our algorithm from cycling and, therefore, guarantees its finiteness.

Indeed, let assume the contrary that some cycling occurs. Since the number of different n-elements subsets I of $\{1,2,...,m\}$ could not excess C_m^n, some of indexes must be entered into and taken out of I_k infinitely many times. Let denote by V the set of all such indexes. We notice that if some $j \notin V$ then either $j \in I_k$ for all k or j never enter I_k.

Let denote by

$$p = \max\{j : j \in V\}. \tag{21}$$

Assume that at iteration k_1 index p is chosen to enter I_{k_1}. From (21) and selection rule for entering index we have

$$\begin{cases} \langle a^p, x^{k_1}\rangle + b_p > 0\,, \\ \langle a^j, x^{k_1}\rangle + b_j \le 0\,,\ \forall j < p\,. \end{cases} \tag{22}$$

For $p \in V$, it will be taken out of index set I_k some iteration later, say k_2.

We denote by s_{k_2} the index that replaces p, and by I_{k_2} the index set obtained at this iteration.

We note that $s_{k_2} \in V$ and $p \in I_{k_2}$.

For every $i \in I_{k_2}$ and $i \notin V$ we have

$$\langle a^i, x^{k_1} \rangle + b_i = 0, \tag{23}$$

$$\langle a^{s_{k_2}}, z^{ik_2} \rangle = 0. \tag{24}$$

For those $i \in I_{k_2} \cap V$ we can get from (22) that

$$\langle a^i, x^{k_1} \rangle + b_i \le 0,$$

$$\langle a^p, x^{k_1} \rangle + b_p > 0.$$

Thus, for all $i \in I_{k_2}$ we have

$$\langle a^i, x^{k_1} \rangle + b_i \le 0, \forall i \in I_{k_2} (i \ne p), \tag{25}$$

$$\langle a^p, x^{k_1} \rangle + b_p > 0. \tag{26}$$

On the other hand, for p is leaving the index set I_{k_2} at iteration k_2, we must have

$$\alpha_i^{k_2} = -\frac{\langle a^{s_{k_2}}, x^{k_2} \rangle + b_{s_{k_2}}}{\langle a^{s_{k_2}}, z^{ik_2} \rangle} \le 0$$

for all $i \in I_{k_2}$ and $i \in V$. It implies

$$\langle a^{s_{k_2}}, z^{ik_2} \rangle > 0. \tag{27}$$

For i in I_{k_2} and not belonging to V, we get from (24) that

$$\langle a^{s_{k_2}}, z^{ik_2} \rangle = 0. \tag{28}$$

Combining (27) and (28) gives

$$\langle a^{s_{k_2}}, z^{ik_2} \rangle \ge 0, \forall i \in I_{k_2} (i \ne p). \tag{29}$$

Now, let us consider the following linear system of inequalities in y

$$\begin{cases} \langle a^i, y \rangle + b_i \le 0, \ \forall i \in I_{k_2} (i \ne p), \\ \langle a^p, y \rangle + b_p = 0. \end{cases} \tag{30}$$

It is easy to see that for every solution y of (30) we always have

$$\langle a^{s_{k_2}}, y\rangle + b_{s_{k_2}} > 0\,. \tag{31}$$

Indeed, in this case y may be written as

$$y = x^{k_2} + \sum_{i\in I_{k_2}(i\neq p)} \beta_i . z^{ik_2},\ \beta_i \geq 0\,.$$

From (29), it implies that

$$\begin{aligned}\langle a^{s_{k_2}}, y\rangle + b_{s_{k_2}} &= \langle a^{s_{k_2}}, x^{k_2}\rangle + b_{s_{k_2}} \\ &+ \sum_{i\in I_{k_2}(i\neq p)} \beta_i . \langle a^{s_{k_2}}, z^{ik_2}\rangle > 0\,.\end{aligned}$$

So, (31) holds.

Since p is the index to be eliminated from I_{k_2} at iteration k_2 we have

$$\langle a^i, x^{k_2}\rangle + b_i = 0,\ \forall i \in I_{k_2}(i\neq p)\,, \tag{32}$$

$$\langle a^{s_{k_2}}, x^{k_2}\rangle + b_{s_{k_2}} = 0\,, \tag{33}$$

$$\langle a^p, x^{k_2}\rangle + b_p < 0\,. \tag{34}$$

By using (25), (26) and (32), (34) we can find

$$y^* = \lambda_* . x^{k_2} + (1-\lambda_*) . x^{k_1},\ \lambda_* \in [0,1]\,, \tag{35}$$

so that

$$\begin{cases} \langle a^i, y^*\rangle + b_i \leq 0,\ \forall i \in I_{k_2}(i\neq p)\,, \\ \langle a^p, y^*\rangle + b_p = 0\,. \end{cases}$$

From (30) and (31) we have

$$\langle a^{s_{k_2}}, y^*\rangle + b_{s_{k_2}} > 0\,. \tag{36}$$

Together with (33) and (35) it implies that

$$\langle a^{s_{k_2}}, x^{k_1}\rangle + b_{s_{k_2}} > 0\,.$$

On the other hand, since $s_{k_2} < p$, we can deduce from (22) that

$$\langle a^{s_{k_2}}, x^{k_1}\rangle + b_{s_{k_2}} \le 0 .$$

This contradiction proves that the cycling can not occur and algorithm 1 is finite.

We note here some interesting features of the above algorithm: It may be used to solve (P) without knowing weather its feasible set is empty or not; or it is quite convenient for solving problems with re-optimization required. When applying to linear programs, our algorithm behaves exactly like the well known dual simplex algorithm.

4. ALMOST-CONVEX AND QUASI-CONCAVE PROGRAMS OVER COMPACT CONVEX CONSTRAINT SETS

In this section we will be concerned with the following problem

$$(Q) \qquad \min\{f(x) : x \in D\},$$

where f is as in previous section and D is a compact convex set in R^n, defined by

$$D := \{x \in R^n : g_i(x) \le 0,\ i = 1, 2, ..., m\}, \tag{37}$$

$g_i (i = 1, ..., m)$ are convex functions in R^n.

For solving (Q) we can use algorithm 1 and the outer-approximation scheme (see [3], [4], [6], [7],...). Combining these two techniques lead to following.

ALGORITHM 2.

Initialization. Cover D by following simplex

$$D_0 := \{x \in R^n : h_j(x) = \langle a^j, x\rangle + b_j \le 0,\ j = 1, 2, ..., n+1\}.$$

Set $x^0 = \text{argmin}\ \{f(x) : x \in V(D_0)\}$, where $V(D_0)$ denotes the set of all vertices of D_0.

Iteration k (k=1,2,...). Solve the subproblem

$$\min\{f(x) : x \in D_k\}$$

using algorithm 1 with an initial feasible point x^{k-1} being the solution of the subproblem encountered at previous iteration $k-1$. Let denote by x^k the obtained solution.

If x^k satisfies $g_i(x^k) \le 0$ for all $i = 1, 2, ...m$ then stop; x^k is the solution to problem (Q).

Else: Select some index i_k such that $g_{i_k}(x^k) > 0$. Let $a^{i_k}(x^k)$ be a subgradient vector of $g_{i_k}(x)$ at x^k, we define D_{k+1} as follows

$$D_{k+1} = D_k \cap \{x : h_{n+k+1}(x) = \langle a^{i_k}(x^k), x - x^k\rangle + g_{i_k}(x^k) \le 0\}.$$

Then return to iteration k with $k \leftarrow k+1$.

Algorithm 2 may either be finite or infinite. In the first case, like algorithm 1, it provides us with an exact solution of (Q). If it is infinite then the following result may be established using Theorem 7 and results given in [4].

Theorem 8. *If algorithm 2 is infinite then every accumulation point of the sequence $\{x^k\}$ is an optimal solution to problem (Q).*

5. COMPUTATIONAL EXPERIMENT

Let function f from R^2 into R, be given as follows

$$f(x) = \begin{cases} 3(x_1 - x_2) + 2sin(x_1 - x_2) + 1 & \text{with } x_1 - x_2 < 0, \\ 2(x_1 - x_2)^{1/2} + sin(x_1 - x_2)^{1/2} + 1 & \text{with } 0 \le x_1 - x_2 \le 1, \\ 2(x_1 - x_2) + sin(x_1 - x_2) + 1 & \text{with } x_1 - x_2 > 1. \end{cases}$$

We consider the problem

$$\begin{cases} f(x) \to \min \\ 3x_1 + 4x_2 - 12 \le 0 \\ -4x_1 + x_2 + 2 \le 0 \\ -x_1 + 4x_2 - 2 \le 0 \\ -x_1 - x_2 + 2 \le 0 \\ -x_1 \le 0 \\ -x_2 \le 0. \end{cases}$$

First, we cover the constraint set of above problem by the following simplex

$$x_1 \geq 0, x_2 \geq 0, 3x_1 + 4x_2 - 12 \leq 0,$$

with vertices (0,0), (0,3), (4,0). We choose point (0,3) as the vertice of the initial min-cone for f. It is determined by the constraints

$$-x_1 \leq 0; 3x_1 + 4x_2 - 12 \leq 0.$$

Initialization. Set $x^0 = (0,3)$, $I_0 = \{1,5\}$ and direction vectors z^{10} and z^{50} of the edges of M_0 are obtained from (10) as: $z^{10} = (0,-3); z^{50} = (4,-3)$.

So, $J^+(x^0) = \{2,3\}, s_0 = 2$. Using (12) we get

$$\alpha_1^0 = \frac{\langle a^2, x^0\rangle + b_2}{\langle a^2, z^{10}\rangle} = 5/3, \ \alpha_5^0 = \frac{\langle a^2, x^0\rangle + b_2}{\langle a^2, z^{50}\rangle} = 5/19.$$

Hence, $I_+^2 = \{1,5\}$.

Next, we use (11) to determine

$$x^{10} = x^0 + \alpha_1^0.z^{10} = (0,-2), \ x^{50} = x^0 + \alpha_5^0.z^{50} = (20/19, 42/19).$$

After substituting them into the objective function and comparing obtained results we obtain the point $x^{50} = (20/19, 42/19)$ which will be the vertice of the newly constructed min-cone M_1 with new index set $I_1 = \{1,2\}$ ($r_0 = 5$, and $s_0 = 2$). Direction vectors of M_1 are calculated using (17):

$$z^{11} = x^{10} - x^{50} = (-20/19, -80/19), \ z^{21} = z^{50} = (4,-3).$$

Now, we are in a position to start iteration 1.

Iteration 1. Putting $x^1 = x^{50}$ into the constraint system and comparing obtained results gives

$$J^+(x^1) = \{3\} \Rightarrow s_1 = 3.$$

Using (12) and (11) we get

$$\alpha_1^1 = 11/30, \alpha_2^1 = 11/8, \Rightarrow I_+^3 = \{1,2\},$$

and

$$x^{11} = (2/3, 2/3), x^{21} = (5/2, 9/8).$$

New min-cone M_2 then has the vertice at x^{11} and index set $I_2 = \{2, 3\}$ ($r_1 = 1$, and $s_1 = 3$).

$$z^{22} = x^{21} - x^{11} = (11/6, 11/24), \ z^{32} = z^{11} = (-20/19, -80/19).$$

Iteration 2. $x^2 = x^{11} = (2/3, 2/3)$; $J_2^+ = \{4\} \Rightarrow s_2 = 4$.
$\alpha_2^2 = 16/55, \alpha_3^2 = -19/55, \Rightarrow I_+^4 = \{2\}$,
$x^{22} = (6/5, 4/5), x^{32} = (34/33, 70/33)$.
Min-cone M_3 has its vertice at x^{22}; $I_3 = \{3, 4\}$ ($r_2 = 2$ and $s_2 = 4$),

$$z^{33} = z^{32} = (-20/19, -80/19), z^{42} = x^{22} - x^{32} = (28/165, -218/165).$$

Iteration 3. $x^3 = x^{22} = (6/5, 4/5) \Rightarrow J^+(x^3) = \emptyset$, we stop. The optimal solution is

$$x_{\text{opt}} = x^3 = (6/5, 4/5).$$

Results obtained using an IBM PC show that Algorithm 1 is quite efficient. Calculating time of some experiments are given in the following table.

The calculating time table

Dimention of problem (n)	Number of constraints (m)	The time take by computer 286 (RAM 2 MB)	The time take by computer 486 (RAM 4 MB)
17	35	0' 54" 48	0' 02" 91
20	32	0' 19" 45	0' 00" 99
20	38	0' 33" 23	0' 01" 75
20	41	0' 32" 60	0' 05" 06
25	51	0" 49" 32	0' 06" 16
25	46	0' 42" 67	0' 01" 63
30	50	0' 47" 61	0' 02" 12
30	56	1' 02" 56	0' 07" 03
30	58	2' 45" 00	0' 09" 34
30	61	3' 07" 35	0' 14" 42

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