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Short Communication

SOME REMARKS ON THE STABILITY OF THE CHARACTERIZATION OF THE COMPOSED RANDOM VARIABLES

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Let ξ be a random variable (r. v.) with the characteristic function $\varphi(t)$ and η be a r. v. with the generating function $a(z)$. It is known [2] that the composed r. v. of the two random variables ξ, η denoted by $\langle \xi, \eta \rangle$ is a r. v. with the characteristic function $\psi(t) = a(\varphi(t))$. In [1] we have dealt with the class N consisting of the characteristic functions of composed r. v. 's and proved that it is a proper extension of the class L of characteristic function of infinitely divisible laws. However, there are still many properties of the class N which have not yet considered.

In this paper we shall investigate a subclass N_ε of N possessing the stability in the following sense: the small changes in the distribution functions of the components ξ and η only lead to the small changes in the distribution function of the composed r. v. $\langle \xi, \eta \rangle$. We will give some remarks on stability conditions for this class.

1. Suppose that $\psi_1(t)$ and $\psi_2(t)$ are two characteristic functions of the class N_ε with the same generating function:

$$\psi_1(t) = a(\varphi_1(t)), \quad \psi_2(t) = a(\varphi_2(t)), \tag{1}$$

where $a(z)$ satisfies the following conditions:

$$|a(z_1) - a(z_2)| \le K \, |z_2 - z_1| \tag{2}$$

for all complex numbers z_1, $z_2, |z_1| \le 1$, $|z_2| \le 1$, and K is a constant.

If for any sufficiently small ε $(0 < \varepsilon < 1)$, we can choose a number $T = T(\varepsilon)$ $(T(\varepsilon) \to +\infty$ when $\varepsilon \to 0)$ so that

$$\max_{|t| \le T(\varepsilon)} \left| \varphi_1(t) - \varphi_2(t) \right| \le \varepsilon \tag{3}$$

then we shall have the following estimation

$$\lambda(\Psi_1;\Psi_2) \leq \max\left\{K\varepsilon;\ \frac{1}{T(\varepsilon)}\right\}, \tag{4}$$

where $\Psi_1(x)$ and $\Psi_2(x)$ are two distribution functions which has the corresponding characteristic functions $\psi_1(t)$, $\psi_2(t)$ and the metric λ is defined by

$$\lambda(\Psi_1;\Psi_2) = \min_{T>0}\max\left\{\max_{|t|\leq T}\left|\psi_1(t)-\psi_2(t)\right|;\ \frac{1}{T}\right\} \tag{5}$$

Indeed, from (1), (2) and (3), we have, for $|t| \leq T(\varepsilon)$,

$$|\psi_1(t)-\psi_2(t)| = |a(\varphi_1(t)) - a(\varphi_2(t))| \leq K\varepsilon\,,$$

and hence (4) follows, by the definition of λ.

2. If ν is the r. v. having the Poisson law with the parameter $\lambda > 0$ and $\varphi_1(t)$ is the characteristic function of the r. v. ξ having ε-exponential law (i. e. $\exists T(\varepsilon)$, $T(\varepsilon) \to +\infty$ when $\varepsilon \to 0$, $\forall t$, $|t| \leq T$, $\left|\varphi_1(t) - \frac{1}{1-it\theta}\right| \leq \varepsilon$) then the composed r. v. of ξ and ν has the distribution function $\Psi_1(x)$ which satisfies the following estimation

$$\lambda(\Psi_1;\Psi_1^{\lambda\theta}) \leq \max\left\{e^{4\lambda}\varepsilon;\ \frac{1}{T(\varepsilon)}\right\}, \tag{6}$$

where $\Psi_1^{\lambda\theta}(x)$ is the distribution function with the characteristic function

$$\psi_1^{\lambda\theta}(t) = e^{\lambda(\frac{1}{1-i\theta t}-1)}\,.$$

3. If ν is r. v. having the Poisson law with the parameter $\lambda > 0$ and ξ has the ε-Normal distribution function (i.e. its the characteristic function $\varphi_2(t)$ satisfies the estimation: $\left|\varphi_2(t) - e^{-t^2/2}\right| \leq \varepsilon$, $\forall t : |t| \leq T(\varepsilon)$, $T(\varepsilon) \to +\infty$ when $\varepsilon \to 0$). Then the composed r. v. of ν and ξ has the distribution function $\Psi_2(x)$ which satisfies the following estimation:

$$\lambda(\Psi_2;\Psi_2^{0,1,\lambda}) \leq \max\left\{e^{4\lambda}\varepsilon;\ \frac{1}{T(\varepsilon)}\right\}, \tag{7}$$

where $\Psi_2^{0,1,\lambda}(x)$ is the distribution with the characteristic function

$$\psi_2^{0,1,\lambda}(t) = e^{\lambda(e^{-t^2/2}-1)}.$$

4. If ν is r.v. having the binomial distribution function with the parameters p, q and ξ has the ε-exponential distribution, then the composed r.v. of ν and ξ has the distribution function $\Psi_3(x)$ which satisfies the following estimation:

$$\lambda\big(\Psi_3;\, \Psi_3^{p,\theta}\big) \le \max\Big\{np(1+2p)^{n-1}\varepsilon;\, \frac{1}{T(\varepsilon)}\Big\}, \tag{8}$$

where $\Psi_3^{p,\theta}$ is the distribution with the characteristic function

$$\psi_3^{p,\theta}(t) = \Big[1 + p\Big(\frac{1}{1-i\theta t} - 1\Big)\Big]^n.$$

5. If ν is the r.v. having the negative binomial distribution with the parameters p, q and ξ has the characteristic $\varphi_4(t)$ which is ε-exponential then the composed r.v. of ν and ξ has the distribution function $\Psi_4(x)$ satisfying the following estimation

$$\lambda\big(\Psi_4;\, \Psi_4^{p,q,\theta}\big) \le \max\Big\{\frac{p}{q}\varepsilon;\, \frac{1}{T(\varepsilon)}\Big\}, \tag{9}$$

where $\Psi_4^{p,q,\theta}(x)$ is the distribution function with the characteristic function:

$$\psi_4^{p,q,\theta}(t) = \frac{p - i\alpha t}{p - i\theta t} \quad (\alpha = \theta q).$$

The above Remarks 2, 3, 4, 5 are immediate from Remark 1 since the corresponding generating functions clearly satisfy the condition (2). Indeed, for instance, to show Remark 5, let $a_4(z)$ be the generating function of the negative-binomial law, i.e.:

$$a_4(z) = p\,[1 - qz]^{-1}.$$

For any the complex numbers z_1, z_2 satisfying $|z_1| \le 1$, $|z_2| \le 1$, we have the following estimation:

$$\big|a_4(z_1) - a_4(z_2)\big| = \Big|\frac{p}{1-qz_1} - \frac{p}{1-qz_2}\Big| \le \frac{pq\,|z_1 - z_2|}{|1-qz_1|\cdot|1-qz_2|}. \tag{10}$$

Notice that

$$\begin{aligned} |1-qz_1| &\geq \big|1-q|z_1|\big| \geq 1-q \ \text{ for all } \ |z_1| \leq 1\,, \\ |1-qz_2| &\geq \big|1-q|z_2|\big| \geq 1-q \ \text{ for all } \ |z_2| \leq 1\,. \end{aligned} \tag{11}$$

From (10) and (11) it follows that

$$\big|a_4(z_1) - a_4(z_2)\big| \leq \frac{pq|z_1 - z_2|}{(1-q)^2}\,.$$

Thus $a_4(z)$ satisfies the condition (2) with the constant $K = \dfrac{p}{q}$.

REFERENCES

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