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ON ALMOST SURE CONVERGENCE OF TWO-PARAMETER RANDOM PROCESSES[1]

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Abstract. *The aim of this note is to give some criteria of almost sure convergence of two-parameter random processes.*

1. INTRODUCTION

Convergence of two-parameter martingales and amarts have considered by Cairoli [5], Cairoli-Walsh [6] and some others. Further, some types of convergence of discrete parameter random processes in Polish spaces were studied by Billingsley [4], Szynal-Zieba [8] etc. The main aim of this paper is to prove some criteria of almost sure convergence of two-parameter random processes in Polish spaces.

2. DEFINITIONS AND BASIC FACTS

Throughout this note, let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $I = \{t = (i,j) : i, j \in N\}$. Then I is a directed set with the usual partial order given by: $t = (i,j) \leq t' = (i',j')$ iff $i \leq i'$ and $j \leq j'$. Further, assume that we are given an increasing sequence $(\mathcal{A}_t,\ t \in I)$ of complete sub-σ-fields of $\mathcal{A}$ with $\mathcal{A} = \bigvee_{n\geq 1} \mathcal{A}_{\overline{n}}$, where $\overline{n} = (n,n)$, $n \in N$.

A function $\tau : \Omega \to I$ is said to be a bounded 1-topping time, write $\tau \in T^1$, iff τ is finitely-valued and the set $\{\tau = (i,j)\} \in \mathcal{A}^1_i$ for every $(i,j) \in I$, where $\mathcal{A}^1_i = \bigvee_{j\geq 1} \mathcal{A}_{ij}$ for any $i \in N$. Thus T^1 is also a direct set with the partial order defined by $\tau \leq \tau'$ iff $\tau(\omega) \leq \tau'(\omega)$ almost surely $(a.s)$.

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Let $L^0(E, \mathcal{A})$ stand for the space of all $\mathcal{A}$–measurable random elements defined on Ω, taking values in a Polish space (E, ρ). Then, a sequence $(X_t,\ t \in I)$ in $L^0(E, \mathcal{A})$ is said to be adapted to $(\mathcal{A}_t,\ t \in I)$ if $X_t \in L^0(E, \mathcal{A}_t)$ for every $t \in I$. Next, given an sequence $(X_t,\ t \in I)$ adapted to $(\mathcal{A}_t)$ and $\tau \in T^1$, we define $X_\tau : \Omega \to E$ and $\mathcal{A}_\tau \subset \mathcal{A}$ by

$$X_\tau(\omega) = X_{\tau(\omega)}(\omega)$$

and
$$\mathcal{A}_\tau = \{A \in \mathcal{A} : A \cap [\tau = (i,j)] \in \mathcal{A}_i^1,\ (i,j) \in I\}.$$

As in the discrete case, $(\mathcal{A}_\tau,\ \tau \in T^1)$ is an increasing family of complete sub–σ–fields of $\mathcal{A}$ and $X_\tau \in L^0(E, \mathcal{A}_\tau)$ for all $\tau \in T^1$.

Now we recall some definitions.

Definition 1. A sequence $(X_n,\ n \in N)$ in $L^0(E, \mathcal{A})$ is said to converge in law to some X in $L^0(E, \mathcal{A})$, write $X_n \overset{D}{\to} X$ as $n \in N$, if the sequence $(P_{X_n},\ n \in N)$ of the probability distributions of X_n, $n \in N$ converges weakly to the probability distribution P_X of X (see [4]).

Definition 2. A sequence $(X_t,\ t \in I)$ in $L^0(E, \mathcal{A})$ is said to converge *a.s.* to some X in $L^0(E, \mathcal{A})$, write $X_t \overset{a.s.}{\to} X$ as $t \in T$, if

$$P\big[\lim_{n\to\infty} \sup_{t \geq \overline{n}} \rho(X_t, X) = 0\big] = 1,$$

where ρ is metric in the Polish space E.

3. MAIN RESULTS

The following lemma is immediate from Definition 2.

Lemma 1. *A sequence $(X_t,\ t \in I)$ converges a.s. to X if and only if for every $\varepsilon > 0$ there exists some $n(\varepsilon) \in N$ such that for all $n \geq n(\varepsilon)$*

$$P\big[\sup_{t \geq \overline{n}} \rho(X_t, X) \geq \varepsilon\big] \leq \varepsilon.$$

Theorem 1. *A sequence $(X_t,\ t \in I)$ converges a.s. to X iff for every (τ_n) in T^1 with $\tau_n \geq \overline{n}$, $n \in N$, $X_{\tau_n} \overset{a.s.}{\to} X$ as $n \in N$.*

Proof. Assume that $X_t \overset{a.s.}{\to} X$ as $t \in T$ and (τ_n) is a sequence in T^1 with $\tau_n \geq \overline{n}$, $n \in N$. Then for every $n \in N$,

$$\sup_{k\geq n}\rho(X_{\tau_k},X)\leq \sup_{t\geq \overline{n}}(X_t,X),\ a.s.$$

Thus, by Definition 2,

$$1\geq P\left[\lim_{n\to\infty}\sup_{k\geq n}\rho(X_{\tau_k},X)=0\right]\geq P\left[\lim_{n\to\infty}\sup_{k\geq \overline{n}}\rho(X_t,X)=0\right]=1\,.$$

It means that $X_{\tau_n}\xrightarrow{a.s.}X$ as $n\in N$.

Now suppose that $X_t \overset{a.s.}{\nrightarrow} X$ as $t\in T$. Define $s_n=\sup_{t\geq\overline{n}}\rho(X_t,X)$, $n\in N$. Then the sequence $(s_n,\ n\in N)$ is decreasing. By virtue of Lemma 1, there exists $\varepsilon>0$ such that for all $n\in N$

$$P\left[s_n\geq 8\varepsilon\right]\geq 8\varepsilon\,. \tag{1}$$

Further, since $X\in L^0(E,\mathcal{A})$ there exists a sequence Y_n, $n\in N$ such that $Y_n\in L^0(E,\mathcal{A}_n)$, $n\in N$ and $(Y_n,\ n\in N)$ converges in probability to X, write $Y_n\xrightarrow{P}X$ as $n\to\infty$. Then we can find some $n\varepsilon\in N$ such that

$$P\left[\rho(Y_n,X)\geq\varepsilon\right]\leq\varepsilon,\ n\geq n(\varepsilon)\,. \tag{2}$$

But for every $n\in N$

$$\lim_{m\to\infty}\sup_{\overline{n}\leq t\leq\overline{m}}\rho(X_t,X)=s_n\ (a.s.)\,,$$

then by (1), for every $n\in N$ there exists some $m_n>n$ such that

$$P\left[\sup_{\overline{n}\leq t\leq\overline{m}_n}\rho(X_t,X)\geq 4\varepsilon\right]\geq 4\varepsilon\,. \tag{3}$$

Now define $\tau_n:\Omega\to I$ by

$$\tau_n(\omega)=\overline{m}_n\ \text{for}\ \omega\in\left[\sup_{\overline{n}\leq t\leq\overline{m}_n}\rho(X_t,Y_n)<2\varepsilon\right]$$

and $\tau_n(\omega)=(i,j)$ for $\omega\in\left[\sup_{\overline{n}\leq t\leq\overline{m}_n}\rho(X_t,Y_n)\geq 2\varepsilon\right]$, where $i=\inf\{s:n\leq s\leq m_n,\ \omega\in\bigcup_{n\leq j\leq m_n}[\rho(X_{s,j},Y_n)\geq 2\varepsilon]\}$ and $j=\inf\{\ell:n\leq\ell\leq m_n,\ \omega\in[\rho(X_{i,\ell},Y_n)\geq 2\varepsilon]\}$.

It is easy to check that $\tau_n \in T^1$ with $\tau_n \geq \overline{n}$, $n \in N$. For every $n \in N$ we have

$$\Big[\sup_{\overline{n}\leq t\leq \overline{m}_n} \rho(X_t, Y_n) \geq 2\varepsilon\Big] = [\rho(X_{\tau_n}, Y_n) \geq 2\varepsilon].$$

On the other hand

$$\begin{aligned} &P\Big[\sup_{\overline{n}\leq t\leq \overline{m}_n} \rho(X_t, X_n) \geq 4\varepsilon\Big] \leq \\ &\leq P\Big[\sup_{\overline{n}\leq t\leq \overline{m}_n} \rho(X_t, Y_n) \geq 2\varepsilon\Big] + P[\rho(Y_n, X) \geq 2\varepsilon] \leq \\ &\leq P[\rho(X_{\tau_n}, Y_n) \geq 2\varepsilon] + P[\rho(Y_n, X) \geq \varepsilon],\ m \in N. \end{aligned}$$

From (2) and (3) we obtain

$$P[\rho(X_{\tau_n}, Y_n) \geq 2\varepsilon] \geq 3\varepsilon,\ n \in N. \tag{4}$$

But, since $P[\rho(X_{\tau_n}, Y_n) \geq 2\varepsilon] \leq P[\rho(X_{\tau_n}, X) \geq \varepsilon] + P[\rho(Y_n, X) \geq \varepsilon]$ so by (2) and (4) we get

$$P[\rho(X_{\tau_n}, X) \geq \varepsilon] \geq 2\varepsilon,\ n \in N.$$

It follows that $X_{\tau_n} \overset{a.s.}{\not\to} X$ as $n \in N$, a contradiction. The proof of Theorem 1 is completed.

Before giving a criterion of almost sure convergence of $(X_t,\ t \in I)$ in terms of the convergence in law of $(X_\tau,\ \tau \in T^1)$ we need the following lemma which is a two parameter version of a result of Austin – Edgar – Ionescu Tulcea [2, p. 18].

Lemma 2. *Let $(X_t,\ t \in I)$ and X be in $L^0(E, \mathcal{A})$. Then there exists a sequence (τ_n) in T^1 with $\tau_n \geq \overline{n}$, $n \in N$ such that the sequence $(X_{\tau_n},\ n \in N)$ converges a.s. to X, write $X_{\tau_n} \overset{a.s.}{\to} X$ as $n \in N$, iff X is cluster point of $(X_t,\ t \in I)$ a.s. i. e.*

$$P\Big[\inf_{t\geq \overline{n}} \rho(X_t, X) = 0\Big] = 1,\ n \in N. \tag{5}$$

Proof. The part "iff" is obvious.

To prove the part "if" we assume that X is an element of $L^0(E, \mathcal{A})$. Then there exists a sequence $(Y_n,\ n \in N)$ adapted to $\mathcal{A}_{\overline{n}}$, $n \in N$, such that $Y_n \xrightarrow{P} X$ as $n \in N$. Therefore there exists also an increasing sub-sequence (n_k) such that for every $k \in N$

$$P\left[\rho(Y_{n_k}, X) \geq 2^{-(k+1)}\right] \leq 2^{-(k+1)}, \tag{6}$$

and hence the sequence Y_{n_k}, $k \in N$ converges *a.s.* to X.

Now let $(X_t, t \in I)$ be a sequence in $L^0(E, \mathcal{A})$. Then for every $k \in N$, the sequence $\left(\inf_{\overline{n}_k \leq t \overline{m}} (X_t, X),\ m \geq n_k\right)$ decreases to $\inf_{t \geq \overline{n}_k} \rho(X_t, X)$, *a.s.*

By (5) for each $k \in N$ there exists some $m_k > n_k$ such that

$$P\left[\inf_{\overline{n}_k \leq t \leq \overline{m}_k} \rho(X_t, X) \geq 2^{-(k+1)}\right] \leq 2^{-(k+1)}.$$

Then by (6) we get

$$\begin{aligned} P\left[\inf_{\overline{n}_k \leq t \leq \overline{m}_k} \rho(X_t, Y_{n_k}) \geq 2^{-k}\right] &\leq P\left[\inf_{\overline{n}_k \leq t \leq \overline{m}_k} \rho(X_t, X) \geq 2^{-(k+1)}\right] \\ &\quad + P\left[\rho(Y_{n_k}, X) \geq 2^{-(k+1)}\right] \\ &\leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}. \end{aligned} \tag{7}$$

Now we define $\tau_k : \Omega \to I$ by

$$\tau_k(\omega) = \overline{m}_k \text{ for } \omega \in \left[\inf_{\overline{n}_k \leq t \leq \overline{m}_k} \rho(X_t, Y_{n_k}) < 2^{-k}\right]$$

and $\tau_k(\omega) = (i, j)$ for other ω, where

$$i = \inf\left\{x : n_k \leq s \leq m_k,\ \omega \in \bigcup_{n_k \leq j \leq m_k} \left[\rho(X_{s,j}, Y_{n_k}) \geq 2^{-k}\right]\right\}$$

and $j = \inf\left\{\ell : n_k \leq \ell \leq m_k,\ \omega \in \left[\rho(X_{i,\ell}, Y_{n_k}) \geq 2^{-k}\right]\right\}$. Then $\tau_k \in T^1$ with $\tau_k \geq \overline{n}_k \geq \overline{k}$ and

$$\left[\inf_{\overline{n}_k \leq t \leq \overline{m}_k} \rho(X_t, Y_{n_k}) \geq 2^{-k}\right] = \left[\inf_{\overline{n}_k \leq t \leq \overline{m}_k} \rho(X_{\tau_k}, Y_{n_k}) \geq 2^{-k}\right],\ k \in N.$$

This combining with (7) implies that

$$P\left[\rho(X_{\tau_k}, Y_{n_k}) \geq 2^{-k}\right] \leq 2^{-k},\ k \in N,$$

and thus the sequence $\rho(X_{\tau_k}, Y_{n_k})$, $k \in N$ converges *a.s.* to zero.

Moreover, by (6) the sequence $(X_{\tau_k},\ k \in N)$ also converges *a.s.* to X completing the proof of Lemma 2.

The following theorem gives a criterion of the almost sure convergence.

Theorem 2. *For a sequence $(X_t,\ t \in I)$ and X in $L^0(E, \mathcal{A})$, the following conditions are equivalent*

(i) $X_t \xrightarrow{a.s.} X$ *as* $t \in I$.

(ii) $X_\tau \xrightarrow{P} X$ *as* $\tau \in T^1$.

(iii) $X_\tau \xrightarrow{D} X$ *and* X *is a cluster point of* $(X_t,\ t \in I)$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $X_t \xrightarrow{a.s.} X$ as $t \in I$. Then by Theorem 1, $X_{\tau_n} \xrightarrow{a.s.} X$ and hence $X_{\tau_n} \xrightarrow{P} X$ as $n \in N$ for every sequence (τ_n) in T^1 with $\tau_n \geq \bar{n}$, $n \in N$. But the convergence in probability is metrizable, so $X_\tau \xrightarrow{P} X$ as $\tau \in T^1$. The implication (ii) $\Rightarrow$ (iii) is obvious. It remains to prove that (iii) $\Rightarrow$ (i). Let X be a cluster point of $(X_t,\ t \in I)$. Then by Lemma 2, there exists a sequence (σ_n) in T^1 such that $\sigma_n \geq \bar{n}$, $n \in N$ and

$$X_{\sigma_n} \xrightarrow{a.s.} X \text{ as } n \in N\,. \tag{8}$$

Now assume that $X_\tau \xrightarrow{D} X$ as $\tau \in T^1$. By Theorem 1 and Theorem 2 in [7], $X_t \xrightarrow{a.s.} X'$ as $t \in I$ for some $X' \in L^0(E, \mathcal{A})$ with $P_{X'} = P_X$. Then by Theorem 1, we have

$$X_{\tau_n} \xrightarrow{a.s.} X' \text{ as } n \in N\,, \tag{9}$$

for every sequence (τ_n) in T^1 with $\tau \geq \bar{n}$, $n \in N$. From (8) and (9) we conclude that $X' = X$, a.s. and $X_t \xrightarrow{a.s.} X$ as $t \in I$. This completes the proof.

For other results related to the above Theorem we refer to [1] and [3]. Here we present only the following corollary of Theorem 2 which can be considered as a two-parameter version of Theorem 2 in [8].

Corollary. *Let C be an element of E and $(X_t,\ t \in I)$ a sequence in $L^0(E, \mathcal{A})$. Then $X_t \xrightarrow{a.s.} C$ as $t \in I$ iff $X_\tau \xrightarrow{D} C$ as $\tau \in T^1$.*

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