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TIKHONOV REGULARIZATION FOR A CLASS OF ILL-POSED VARIATIONAL INEQUALITIES[1]

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Abstract. *The aim of this paper is to investigate a problem of variational inequalities with a constraint in the form of ill-posed nonlinear operator equation involving monotone operators under monotone perturbations. A result on convergence rate for the Tikhonov regularization is established. Then, this result will be considered in combination with finite-dimensional approximations of the space. For illustration, an example in the theory of linear integral equations in L_p-space is given.*

1. INTRODUCTION

Many problems arising in mathematical physics and mechanics (see e.g. [4, 5, 7, 20, 23]) have been formulated in the following abstract form: find an element $x_0 \in S_0$ such that

$$\varphi(x_0) = \min_{x \in S_0} \varphi(x), \tag{1}$$

where φ is a weakly lower semicontinuous and properly convex functional on the real reflexive Banach space X with the norm denoted by $\|\cdot\|$, and S_0 is a convex and closed subset of X. If we denote by $A(x)$ the subdifferential of the functional φ at the point x, then this problem is equivalent to the variational inequality (see [23])

$$\langle A(x_0),\ x - x_0 \rangle \geq 0, \forall x \in S_0,\ x_0 \in S_0, \tag{2}$$

where the symbol $\langle x^*,\ x \rangle$ denotes the value of the linear continuous functional $x^* \in X^*$ at the point $x \in X$ and X^* is the adjoint space of X. For the sake of simplicity, the norm of X^* will be also denoted by the symbol $\|\cdot\|$.

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Up to now, there have been a lot of works dealing not only with theoretical aspects but also numerical methods and applications of this problem (see [3, 9, 10, 11, 14, 18 – 20, 25, 26]). Moreover, some results have been generalized for locally Lipschitzean functional φ (see [6]) and recently for invex functional (see [13]).

In [15, 22, 23] one considered an interesting case, when φ is convex, and S_0 is the set of all solutions of the other operator equation

$$F(x) = f_0, \; f_0 \in X^*, \tag{3}$$

where F is a monotone operator from X into X^*. It is well known (see [1]) that, without additional conditions on the structure of F such as strong or uniform monotonicity, equation (3) is, in general, an ill–posed problem (see [24]). By this we mean that solutions of (3) do not depend continuously on the data F, f_0. Therefore, the problem (2) in this case is ill–posed, too. To solve it we have to use stable methods. A widely used method of this kind is the Tikhonov regularization in the form of operator equation

$$F_h(x) + \alpha A(x) = f_\delta, \tag{4}$$

where F_h and f_δ are respectively the approximations for F and f_0,

$$\|F(x) - F_h(x)\| \leq h\|x\|, \; \forall x \in X, \; \|f_\delta - f_0\| < \delta, \; h, \delta \to 0,$$

and α is the parameter of regularization. If F_h are monotone and hemicontinuous, and A is uniform monotone, then the existence and uniqueness of the solution of (4) were proved in [5, 22, 23]. There one has also showed that if δ/α and $h/\alpha \to 0$ this solution converges to x_0 and it can be approximated by solutions of the sequence of finite-dimensional problems (see [21])

$$F_{hn}(x) + \alpha A_n(x) = f_{\delta n}, \tag{5}$$

where $F_{hn} = P_n^* F_h P_n$, $A_n = P_n^* A P_n$, $f_{\delta n} = P_n^* f_\delta$ and P_n denotes the linear projection from X onto its subspace X_n satisfying the condition

$$X_n \subset X_{n+1}, \; P_n x \to x, \; n \to +\infty, \; \forall x \in X,$$

and P_n^* is the adjoint of P_n, $\|P_n\| \leq c$, where c is a positive constant.

Until now, it is still open the question whether $\lim_{\substack{\alpha, h, \delta \to 0 \\ n \to +\infty}} x^\delta_{h\alpha n} =$ x_0, where $\{x^\delta_{h\alpha n}\}$, is the squence of solutions of (5) and also the question about its convergence rate, as well as about the convergence rate for $\{x^{\delta}_{h\alpha}\}$, the solution of (4).

Obviously, the answer for these questions depend on the relation between h, α and n. In this paper, applying the ideas of Engl and Groetsch in [8, 12] we shall answer these questions, under the assumption that A is an uniformly monotone operator, i. e.

$$\langle A(x) - A(y),\ x - y\rangle \geq m_A\|x - y\|^s,\ 1 < s < +\infty, \tag{6}$$

where m_A is a positive constant. Without loss of generality, we shall assume $m_A = 1$.

Below the symbols $\to$ and $\rightharpoonup$ denote the strong and weak convergence for any sequence, respectively.

2. MAIN RESULTS

First we prove a result about convergence rate for the sequence $\{x^{\delta}_{h\alpha}\}$

Theorem 1. *Suppose that the following conditions hold:*

(i) F_h *is Fréchet differentiable in some neighbourhood of* S_0 $s-1$*-times if* $s = [s]$*; the integer part of* s*, and* $[s]$*-times if* $s \neq [s]$*;*

(ii) *The derivative* $F_h^{(k)}$ *satisfies the condition*

$$\|F_h^{(k)}(x) - F_h^{(k)}(y)\| \leq L\|x - y\|,\ \forall x \in S_0,\ y \in \mathcal{U}_0,\ L > 0,$$

where $\mathcal{U}_0$ *is a neighbourhood of* S_0 *and* $k = s-1$ *if* $s = [s]$*, and* $k = [s]$ *if* $s \neq [s]$*. Moreover, if* $[s] \geq 3$*,* $F_h^{(2)}(x_0) = \cdots = F_h^{(k)}(x_0) = 0$*;*

(iii) *There exist elements* z_h *of* X *such that*

$$F_h'^{*}(x_0)z_h = A(x_0)$$

and $L\|z_h\| \leq s!$ *if* $s = [s]$*.*

Then, if α *is chosen such that* $\alpha \sim (h+\delta)^{\theta}$*,* $0 < \theta < 1$*, we have*

$$\|x^{\delta}_{h\alpha} - x_0\| = O((h+\delta)^{\rho}),\ \rho = \min\{(1-\theta)/(s-1),\ \theta/s\}.$$

Proof. From (3) and (4) it follows

$$\alpha\langle A(x^\delta_{h\alpha}) - A(x_0),\, x^\delta_{h\alpha} - x_0\rangle \le \langle f_\delta - f_0,\, x^\delta_{h\alpha} - x_0\rangle + \\ + \langle F(x_0) - F_h(x_0),\, x^\delta_{h\alpha} - x_0\rangle + \alpha\langle A(x_0),\, x_0 - x^\delta_{h\alpha}\rangle.$$

Combining this inequality, (6) and condition (iii) gives

$$\alpha\|x^\delta_{h\alpha} - x_0\|^s \le (\delta + h\|x_0\|)\|x^\delta_{h\alpha} - x_0\| \\ + \alpha\langle z_h,\, F'_h(x_0)(x_0 - x^\delta_{h\alpha})\rangle. \quad (7)$$

In the case $s = [s]$, since

$$F'_h(x_0)(x_0 - x^\delta_{h\alpha}) = F_h(x_0) - F_h(x^\delta_{h\alpha}) + r^\delta_{h\alpha}$$

with

$$\|r^\delta_{h\alpha}\| \le \frac{L}{s!}\|x^\delta_{h\alpha} - x_0\|^s$$

(see [26]) from (7) it follows that

$$\alpha\|x^\delta_{h\alpha} - x_0\|^s \le (\delta + h\|x_0\|)\|x^\delta_{h\alpha} - x_0\| + \alpha\frac{L}{s!}\|z_h\|\|x^\delta_{h\alpha} - x_0\|^s \\ + \alpha\langle z_h,\, F_h(x_0) - F_h(x^\delta_{h\alpha})\rangle.$$

We can estimate the last term in the following way

$$\langle z_h,\, F_h(x_0) - F_h(x^\delta_{h\alpha})\rangle = \langle z_h,\, F_h(x_0) - F(x_0) + f_0 - f_\delta + \alpha A(x^\delta_{h\alpha})\rangle \\ \le \|z_h\|(\delta + h\|x_0\| + \alpha\|A(x^\delta_{h\alpha})\|),$$

that is equivalent to $\sim \delta + h + \alpha$ because $x^\delta_{h\alpha} \to x_0$ and the hemi-continuous monotone operator A is locally bounded (see [25]). Thus, the inequality (7) can be written in the form

$$\left(1 - \frac{L}{s!}\|z_h\|\right)\|x^\delta_{h\alpha} - x_0\|^s \le \frac{h\|x_0\| + \delta}{\alpha}\|x^\delta_{h\alpha} - x_0\| + O(h + \delta + \alpha). \quad (8)$$

Using the implication in [17]:

$$a,\ b>0,\ p>q>0 \text{ and } t^p \le at^q + b \Rightarrow t^p = O(a^{p/(p-q)} + b)$$

we get

$$\|x^\delta_{h\alpha} - x_0\| = O((h+\delta)^\rho).$$

If $s \neq [s]$, the left-hand side of the inequality (8) will be replaced by

$$\Big(1 - \frac{L}{([s]+1)!}\|z\|\,\|x^\delta_{h\alpha} - x_0\|^{[s]+1-s}\Big)\|x^\delta_{h\alpha} - x_0\|^s.$$

As $[s]+1-s>0$ and $\|x^\delta_{h\alpha} - x_0\| \to 0$, the requirement $L\|z_h\| \le s!$ is not necessary. Q.E.D.

The following theorem is the answer for the question about the convergence of $\{x^\delta_{h\alpha n}\}$.

Theorem 2. *If the conditions* (i) *and* (ii) *of Theorem 1 hold. Moreover, suppose additionally that*

(iv) *There exists* $\alpha = \alpha(n) \to 0$ *such that*

$$(\tilde{\gamma}_{hn}(x) + \|(I-P_n)x\|^{[s]})\alpha^{-1} \to 0,\ \forall x \in S_0,$$

as $h,\ \delta \to 0$ *and* $n \to +\infty$, *where* $\tilde{\gamma}_{hn}(x)$ *is defined by*

$$\tilde{\gamma}_{hn}(x) = \|F'_h(x)(I-P_n)x\|.$$

Then the sequence $\{x^\delta_{h\alpha n}\}$ *converges to* x_0.

Proof. From (5) it follows

$$\begin{aligned} &F_{hn}(x^\delta_{h\alpha n}) - F_{hn}(x_n) + \alpha(A_n(x^\delta_{h\alpha n}) - A_n(x_n)) \\ &\quad = f_{\delta n} - F_{hn}(x_n) - \alpha A_n(x_n),\ x_n = P_n x,\ \forall x \in S_0. \end{aligned}$$

Multiplying the two parts of this equality by $x^\delta_{h\alpha n} - x_n$, and using the projective property of P_n : $P^*_n = P^*_n P^*_n$ and the monotone property of F_{hn} and A, we obtain

$$\begin{aligned} \alpha\langle A(x^\delta_{h\alpha n}) - A(x_n),\ x^\delta_{h\alpha n} - x_n\rangle &\le (\delta + h\|x\|)\|x^\delta_{h\alpha n} - x_n\| \\ &+ \langle F_h(x) - F_h(x_n),\ x^\delta_{h\alpha n} - x_n\rangle + \alpha\langle A(x_n),\ x_n - x^\delta_{h\alpha n}\rangle. \end{aligned} \tag{9}$$

In the case $s=[s]$ we can write

$$F_h(x_n) = F_h(x) + F'_h(x)(x_n - x) + r_{hn}$$

with

$$\|r_{hn}\| \leq \frac{L}{s!}\|(I - P_n)x\|^s.$$

Therefore, from (6) and (9) we get

$$\begin{aligned} &\alpha\|x^\delta_{h\alpha n} - x_n\|^s \leq \\ &\leq \Big(\delta + h\|x\| + \frac{L}{s!}\|(I - P_n)x\|^s + \|F'_h(x)(I - P_n)x\|\Big)\|x^\delta_{h\alpha n} - x_n\| \\ &\quad + \alpha\langle A(x_n),\ x_n - x^\delta_{h\alpha n}\rangle. \end{aligned} \tag{10}$$

Because of $s > 1$, this inequality gives us the boundedness of the sequence $\{x^\delta_{h\alpha n}\}$. Without loss of generality, suppose that $x^\delta_{h\alpha n} \rightharpoonup x_1 \in X$, as $n \to +\infty$ and $h,\ \alpha,\ \delta \to 0$. We shall prove that $x_1 \in S_0$.

First, since $F_n := P_n^* F P_n$ is monotone, we have

$$\langle F_n(x_n) - F_n(x^\delta_{h\alpha n}),\ x_n - x^\delta_{h\alpha n}\rangle \ \geq\ 0,\ x_n = P_n x,\ \forall x \in X.$$

Since $P_n^* P_n^* = P_n^*$, this inequality can be written in the form

$$\langle F(x_n) - F_n(x^\delta_{h\alpha n}),\ x_n - x^\delta_{h\alpha n}\rangle \ \geq\ 0.$$

Together with (5), it gives

$$\begin{aligned} &\langle F(x_n) - f_{\delta n} + \alpha A_n(x^\delta_{h\alpha n}),\ x_n - x^\delta_{h\alpha n}\rangle \\ &+ \langle F_{hn}(x^\delta_{h\alpha n}) - F_n(x^\delta_{h\alpha n}),\ x_n - x^\delta_{h\alpha n}\rangle \ \geq\ 0, \end{aligned}$$

that is equivalent to

$$\begin{aligned} &\langle F(x_n) - f_\delta,\ x_n - x^\delta_{h\alpha n}\rangle + \alpha\langle A(x_n),\ x_n - x^\delta_{h\alpha n}\rangle \\ &+ \langle F_h(x^\delta_{h\alpha n}) - F(x^\delta_{h\alpha n}),\ x_n - x^\delta_{h\alpha n}\rangle \ \geq\ 0. \end{aligned}$$

Thus we have

$$\begin{aligned} &h\|x^\delta_{h\alpha n}\|\,\|x_n - x^\delta_{h\alpha n}\| + \langle F(x_n) - f_\delta,\ x_n - x^\delta_{h\alpha n}\rangle \\ &+ \alpha\langle A(x_n),\ x_n - x^\delta_{h\alpha n}\rangle \ \geq\ 0. \end{aligned}$$

Letting $n \to +\infty$ and $h,\ \delta,\ \alpha \to 0$ we obtain

$$\langle F(x) - f_0,\ x - x_1\rangle \ \geq\ 0,\ \forall x \in X.$$

By virtue of Minty's lemma, $x_1 \in S_0$. Replacing x_n in (10) by $x_{1n} = P_n x_1$ and letting $n \to +\infty$ and $h,\ \delta,\ \alpha \to 0$, we can see that $x^{\delta}_{h\alpha n} \to x_1$ and

$$\langle A(x),\ x - x_1\rangle\ \geq\ 0,\ \forall x \in S_0.$$

On the other hand, this variational inequality is equivalent to

$$\langle A(x_1),\ x - x_1\rangle\ \geq\ 0, \forall x \in S_0.$$

Therefore, x_1 is a solution of the problem (1) (see [23]). Since A is uniformly monotone, the last variational inequality has an unique solution x_0. Thus, $x_1 = x_0$ and the entire sequence $\{x^{\delta}_{h\alpha n}\}$ converges to x_0.

For the case $s \neq [s]$, the term $\frac{L}{s!}\|(I-P_n)x\|^s$ in (10) will be replaced by $\frac{L}{([s]+1)!}\|(I-P_n)x\|^{[s]+1}$ and the process of proof of the theorem will be entirely repeated. Q.E.D.

It is not difficult to verify that all results in Theorems 1 and 2 are still true, if all conditions on F_h are stated in the similar way only for F, i.e. the index h in Theorems 1 and 2 for F_h, its derivatives, z_h and $\tilde{\gamma}_{hn}$ can be obmitted. Indeed, for instance, in the proof of Theorem 2 the inequalities (9) and (10) will be replaced by

$$\alpha\langle A(x^{\delta}_{h\alpha n}) - A(x_n),\ x^{\delta}_{h\alpha n} - x_n\rangle \leq\ (\delta + h\|x\|)\|x^{\delta}_{h\alpha n} - x_n\| + \langle F(x) - F(x_n),\ x^{\delta}_{h\alpha n} - x_n\rangle + \alpha\langle A(x_n),\ x_n - x^{\delta}_{h\alpha n}\rangle. \quad (9')$$

$$\alpha\|x^{\delta}_{h\alpha n} - x_n\|^s \leq \leq\ (\delta + h\|x\| + \frac{L}{s!}\|(I-P_n)x\|^s + \|F'(x)(I-P_n)x\|)\|x^{\delta}_{h\alpha n} - x_n\| + \alpha\langle A(x_n), x_n - x^{\delta}_{h\alpha n}\rangle. \quad (10')$$

respectively, and the process of proof of the theorem is completely repeated. We will do this in detail in answering the question about convergence rate for the sequence $\{x^{\delta}_{h\alpha n}\}$.

Theorem 3. *Let the conditions* (i) – (iii) *of Theorem 1 hold for* F. *Moreover, let*

(v) *There exists a positive constant* d *and a neighbourhood* O_0 *of* x_0 *such that*

$$\|A(x) - A(x_0)\|\ \leq\ d\|x - x_0\|^{\tilde{s}},\ \tilde{s}\ >\ 0,\ \forall x \in O_0;$$

(vi) α *is chosen such that* $\alpha \sim (h + \delta + \gamma_n)^{1/2}$, *where*

$$\gamma_n = \max \{\|(I - P_n)x_0\|, \ \|(I^* - P_n^*)f_0\|, \ \|(I - P_n)z\|.$$

Then

$$\|x_{h\alpha n}^{\delta} - x_0\| = O((h+\delta)^{1/2s} + \gamma_n^{\bar{s}}), \ \bar{s} = \ min \{1/2s, \ \tilde{s}\}.$$

Proof. From the condition (v) of this theorem, for sufficiently large n, we have

$$\langle A(x_{0n}), \ x_n - x_{h\alpha n}^{\delta}\rangle \ \le \ d\gamma_n^{\tilde{s}} \|x_{h\alpha n}^{\delta} - x_n\| \\ + \langle A(x_0), \ x_n - x_{h\alpha n}^{\delta}\rangle, \ x_n \in X_n.$$

Thus, in the case $s = [s]$, the inequality (10′) with $x_n = x_{0n}$ has the form

$$\alpha\|x_{h\alpha n}^{\delta} - x_{0n}\|^s \le O(\delta + h\|x_0\| + \gamma_n + \gamma_n^s + \alpha d\gamma_n^{\tilde{s}})\|x_{h\alpha n}^{\delta} - x_{0n}\| \\ + \alpha\langle A(x_0), \ x_{0n} - x_{h\alpha n}^{\delta}\rangle. \tag{11}$$

It is easy to see that in this case of s

$$\langle A(x_0), \ x_{0n} - x_{h\alpha n}^{\delta}\rangle = \langle A(x_0), \ (P_n - I)x_0\rangle + \langle z, \ F'(x_0)(x_0 - x_{h\alpha n}^{\delta})\rangle \\ \le \ O(h + \gamma_n) + \langle z, \ F(x_0) - F_h(x_{h\alpha n}^{\delta})\rangle + L\|z\|\|x_{h\alpha n}^{\delta} - x_0\|^s/s!, \\ \|x_{h\alpha n}^{\delta} - x_0\|^s \ \le \ \|x_{h\alpha n}^{\delta} - x_{0n}\|^s + O(\gamma_n),$$

and

$$\langle z, \ F(x_0) - F_h(x_{h\alpha n}^{\delta})\rangle = \langle z, \ f_0 - f_\delta\rangle + \alpha\langle z, \ A(x_{h\alpha}^{\delta})\rangle \\ + \langle z, \ F_h(x_{h\alpha}^{\delta}) - F_h(x_{h\alpha n}^{\delta})\rangle \\ \le O(\delta + \alpha) + \langle z, \ F_h(x_{h\alpha}^{\delta}) - F_h(x_{h\alpha n}^{\delta})\rangle.$$

Since

$$\langle z, \ F_h(x_{h\alpha}^{\delta}) - F_h(x_{h\alpha n}^{\delta})\rangle = \langle z, \ F_h(x_{h\alpha}^{\delta}) - F_{hn}(x_{h\alpha n}^{\delta})\rangle \\ + \langle z, \ F_{hn}(x_{h\alpha n}^{\delta}) - F_h(x_{h\alpha n}^{\delta})\rangle,$$

where

$$\begin{aligned}\langle z,\ F_{hn}(x^{\delta}_{h\alpha n}) - F_h(x^{\delta}_{h\alpha n})\rangle &= \langle (P_n - I)z, F_h(x^{\delta}_{h\alpha n})\rangle \\ &\le \gamma_n \|F_h(x^{\delta}_{h\alpha n}) - F(x^{\delta}_{h\alpha n}) + F(x^{\delta}_{h\alpha n})\| \le O(\gamma_n)\end{aligned}$$

and

$$\begin{aligned}\langle z,\ F_h(x^{\delta}_{h\alpha}) - F_{hn}(x^{\delta}_{h\alpha n})\rangle &= \langle z,\ f_\delta - \alpha A(x^{\delta}_{h\alpha}) - f_{\delta n} + \alpha A_n(x^{\delta}_{h\alpha n})\rangle \\ &\le O(\delta + \alpha) + \langle z,\ (I^* - P_n^*) f_0\rangle \le O(\gamma_n + \delta + \alpha).\end{aligned}$$

The inequality (11) can be written in the form

$$\begin{aligned}&\alpha(1 - L\|z\|/s!)\|x^{\delta}_{h\alpha n} - x_{0n}\|^s \\ &\le O(h + \delta + \gamma_n + \gamma_n^s + \alpha d\tilde{\gamma}_n^{s})\|x^{\delta}_{h\alpha n} - x_{0n}\| \\ &+ \alpha O(h + \delta + \alpha + \gamma_n).\end{aligned}$$

Then

$$\begin{aligned}\|x^{\delta}_{\alpha n} - x_{0n}\|^s \le O((h + \delta + \gamma_n)^{1/2} + \gamma_n^{s/2} + \tilde{\gamma}_n^{s})\|x^{\delta}_{\alpha n} - x_{0n}\| \\ + O((h + \delta + \alpha + \gamma_n).\end{aligned}$$

Applying again the relation in [17] to the last inequality we obtain

$$\|x^{\delta}_{\alpha n} - x_{0n}\| \le O((h + \delta)^{1/2s} + \gamma_n^{\bar{s}}).$$

Therefore

$$\|x^{\delta}_{\alpha n} - x_0\| \le O((h + \delta)^{1/2s} + \gamma_n^{\bar{s}}).$$

The case $s \neq [s]$ is proved in the similar way as in the proof of Theorem 1. Q.E.D.

3. AN EXAMPLE

Consider the following problem: find a norm-minimal solution of the linear integral equation

$$(Fx)(t) \equiv KfK^*x(t) = f_0(t),\ t \in \Omega, \tag{12}$$

for a given $f_0(x) \in L_q(\Omega)$, $1 < q < +\infty$. Here the operator K is defined by

$$(Kx)(t) = \int_\Omega k(t, s)\, x(s)\, ds, \quad t \in \Omega,$$

where the kernel function $k(t, s)$ is a measurable on $\Omega \times \Omega$ such that the null space $\mathcal{N}(K) \neq \{\mathcal{O}\}$ and Ω is a closed set in $\mathbf{R}^n$. The function $f(t)$, $t \in \mathbf{R}^1$, is a real and non-decreasing function.

Suppose

$$\int_{\Omega\times\Omega} |k(x,y)|^q < +\infty, \ |f(t)| \leq b_0 + b_1|t|^{q-1}, \ b_0, \, b_1 > 0, \ p^{-1} + q^{-1} = 1.$$

Then the operator K is a linear, bounded operator from $X = L_p(\Omega)$ into $X^* = L_q(\Omega)$, and $(fx)(t) = f(x(t)) : X^* \to X$, is a monotone operator. Therefore, $F = KfK^*$ is also a monotone operator from X into X^*. If either K or f is compact, then (12) is ill–posed. We are interested in a solution $x_0(t)$ of (12) with minimal norm. In this case, the operator A is the dual mapping U^s of the spaces L_p. It satisfies all necessary conditions (see [2]): If X is a Hilbert space, then $U^s = I$, $s = 2$, $m_A = 1$, $\tilde{s} = 1$ and $d = c(R) = 1$, $R = \max\{\|x\|, \|x_0\|\}$. For the spaces of Lebesgue's type l_p, L_p, W^p_m, $p > 1$ we can construct U^s satisfying condition (6) and the condition (v) of Theorem 3 with

$$1 < p < 2 \ : \ s = 2, \ m_A = p - 1, \ c(\rho) = p2^{2p-1}e^p L^{p-1},$$

$$e = \max\{2^p, 2\rho\}, \ 1 < L < 3.18, \ \tilde{s} = p - 1;$$

$$2 < p : s = p, \ m_A = 2^{2-p}/p,$$

$$c(\rho) = 2^p \rho^{p-2}\{p[p-1+\max\{\rho, \, L\}]\}^{-1}, \ \tilde{s} = 1.$$

In particular, if $1 < p < 2$, then $s = 2$ and we only need to verify the condition (iii) of Theorem 1 and the condition (iv) of Theorem 2 for each concrete problem and concrete form of finite-dimensional approximations of $L_p(\Omega)$. We shall see this in the following case, when $\Omega = [0, 1]$, $p = 2$. Then condition (iii) of Theorem 1 is written in the form

$$x_0 = Kf_h'^*(K^*x_0)K^*z_h,$$

where f_h is the approximations of f. This equation has solution if $x_0 \in R(K)$, the range of K, and there exist $u_h(t) \in R(K^*)$ such that

$$\xi_0 = f_h'^*(y_0)u_h\,, \;\; y_0 = K^*x_0\,, \;\; K\xi_0 = x_0\,, \tag{13}$$

where $f_h'^*(y_0)$ is coercive on $L_2[0,\,1]$.

Indeed, let for example

$$f(t) = \begin{cases} e_1(t-t_0)+d, & \text{if } t-t_0 \le 0\,, \\ e_2(t-t_0)+d, & \text{if } t-t_0 > 0\,, \end{cases}$$

$$e_2 > e_1 > 0,\;\; d \in R\,.$$

Then we can approximate $f(t)$ by $f_h(t)$ in the following form

$$f_h(t) = \begin{cases} f(t)\,, & \text{if } t \notin (t_0-h,\, t_0+h]\,, \\ d + e_1(t-t_0) + p(t-t_0+h)^2 & \\ +q(t-t_0+h)^3\,, & \text{if } t \in (t_0-h,\, t_0+h]\,. \end{cases}$$

The coefficients p and q can be calculated by solving the system of two linear equations

$$f_h(t_0+h) = e_2h + d\,,$$
$$f_h'^*(t_0+h) = e_2\,.$$

It is easy to verify that for sufficiently small h the coefficients p ans q are defined uniquely and the functions $f_h(t)$ are monotone and differentiable. Moreover

$$|F_{1h}(t) - f(t)| \le |F_{1h}(t_0) - F(t_0)| \le ch,\; c > 0\,.$$

On the other hand, we have

$$f_h'(t) = \begin{cases} e_1 \text{ or } e_2\,, & \text{if } t \notin (t_0-h,\, t_0+h)\,, \\ e_1 + (e-e_1)(t-t_0+h)/4h\,, & \text{if } t \in [t_0-h,\, t_0+h]\,. \end{cases}$$

Therefore, condition (13) is satisfied, because $e_1 > 0$. Now, we approximate the Hilbert space $H = L_2[0,\,1]$ by the sequence of linear subspaces H_n, defined by

$$H_n = L\{\psi_1, \psi_2, \ldots, \psi_n\}\,,$$

$$\psi_j = \begin{cases} 1, & t \in (t_{j-1},\, t_j]\,, \\ 0, & t \notin (t_{j-1},\, t_j]\,. \end{cases}$$

It is well known that

$$\|(I - P_n)y\| = O(n^{-1}), \ \forall y \in L_2[0, 1],$$

where

$$P_n y = \sum_{j=1}^{n} y(t_j)\psi_j(t).$$

By taking $\alpha_n = O(n^{-1/2})$ and $h = \delta = O(n^{-2})$ we can see that almost all the conditions of Theorems 1 - 3 are satisfied.

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