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GAMES THAT INVOLVE SET THEORY OR TOPOLOGY

MARION SCHEEPERS

Game theory is the mathematical study of the intuitive notions of competition, conflict and strategy. We shall call the parties involved in the conflict *players*. Each exchange of hostilities during the conflict is said to be an *inning*, and a particular player's actions during an inning is said to be that player's *move* for that inning. The entire conflict is said to be a *play of the game*. A game is described by explicit rules which:

* prescribe the allowable actions of each player, and
* declare the circumstances under which a player has won or lost.

We shall consider games in which there are only two players and in which one inning per positive integer is played. These are called *infinite two–player games of length* ω. In all the games we consider, it will be the case that the outcome of every play of the game is a win for some player; we are *not* requiring that there is a favored player who wins every play – it is completely ok that in one play one of the players wins, while in another play, the other *players win.* What we are not allowing is that a play ends up in a *draw* where neither player wins. The study of the games we describe uses the methods of infinitary mathematics.

The two players will be named ONE and TWO; in every game we describe player ONE will in each inning be the first to act, and TWO will be the player who responds to ONE's actions. We shall let O_n denote ONE's move in the n–th inning and we shall let T_n denote TWO's move. Intuitively speaking a *strategy* for ONE is a plan which ONE has committed to before the game began, and which instructs ONE on what moves to make based on the history of the game so far. Mathematically speaking a strategy for ONE is a function F with domain the set of finite sequences of the form $(T_1, \cdots, T_n)$, where the T_i are legal moves for TWO, and with range the set of possible moves for ONE. For F a strategy of ONE, a sequence

$$O_1, T_1, \cdots, O_n, T_n, \cdots$$

which satisfies the equations

$$O_1 = F(\emptyset) \tag{1}$$

and for each positive integer n

$$O_{n+1} = F(T_1, \cdots T_n) \tag{2}$$

is said to be a *play of the game according to F*, or simply an F–play. A strategy F for ONE is a *winning strategy for ONE* if every F–play of the game is won by ONE. The notion of a strategy for TWO and of a winning strategy for TWO is defined analogously. Observe that the notion of a strategy as just described requires complete memory of all the actions of the opponent during all the preceding innings; consequently, these sorts of strategies are called *perfect memory strategies.* A game is said to be *determined* if one of the players has a winning perfect memory strategy in the game. If neither of the players has a winning memory strategy, the game is said to be *undetermined.*

It is often of interest to know, for determined games, whether the player who has the perfect memory strategy really needs all this memory to secure a win – a nice illustration of the relevance of this sort of consideration appears in [60]. Several sorts of strategies depending on less memory have been considered in the literature. The main classes of strategies of this sort are as follows:

1. *Coding strategies:* a coding strategy is a strategy which depends on at most the two most recent moves made in the game: For player TWO this means that a strategy F is a coding strategy if it is of the form $T_1 = F(O_1)$, and $T_{n+1} = F(O_{n+1}, T_n)$ for each n. For player ONE F is a coding strategy if $O_1 = F(\emptyset)$ and for each n, $O_{n+1} = F(O_n, T_n)$.
2. *n–tactics:* an n–tactic is a strategy which depends on at most the n most recent moves of the opponent: For player TWO this means that a strategy F is an n–tactic if it is of the form $T_j = F(O_1, \cdots, O_j)$ for $j \leq n$, and $T_{k+n} = F(O_{k+1}, \cdots, O_{k+n})$ for each positive integer k. The definition for player ONE is analogous.
3. *Markov n–tactics:* a Markov n–tactic is a strategy which depends on at most the n most recent moves of the opponent *and* the number of the inning in progress: For player TWO this means that a strategy F is an n–tactic if it is of the form $T_j = F(O_1, \cdots, O_j; j)$ for $j \leq n$, and $T_{k+n} = F(O_{k+1}, \cdots, O_{k+n}; k+n)$ for each positive integer k. The definition for player ONE is analogous.

Games of the sort we described can be classified according to several general criteria:

- What is the number of players? We discuss two–player games.
- What is the length of the game? We discuss games of *length* ω; that is to say, the players play an inning per positive integer.
- On how many boards is the game played? We first discuss one board games, and then we venture into the interesting area of multiple boards games.

A finer classification which takes into account the kind of rules defining the games in question is also possible. A quick glance at the table of contents below will show a few classes that are identified by this criterium.

By hindsight the serious study of infinitely long mathematical games started in the 1920's with three papers [156] by Sierpinski in 1924, [67] by Hurewicz in 1925 and [6] by Banach and Kuratowski in 1929. According to Ulam, Mazur started formulating infinitely long games and asking explicit questions about them around 1928. One such game appears as Problem 43 in **The Scottish Book** [104]. Since these early days this discipline of mathematics has grown into a recognized subject with mathematics subject classification number 90 D44 (1991 classification); the title of this classification is *Games that involve set theory or topology*.

The main purpose of this paper is to present the reader with a concise introduction to this subdiscipline of mathematics via a list of open problems. Though many theorems are stated here, none of them are proven; for proofs the reader could consult the published literature cited in the bibliography, write to the mathematicians responsible for the unpublished cited results, or wait for the textbook [152] on the subject. A second equally important purpose of this paper is to save from extinction important developments from the 1970's regarding multiboard game versions of the Gale – Stewart game and the Banach – Mazur game.

For comfortable reading of this paper it would be best if the reader has a working knowledge of cardinal numbers, general topology and Boolean algebra. The text [74] would be a sufficient general reference to look up definitions of technical terms regarding set theory (such as cardinal numbers, ideals, forcing, large cardinals, consistency– and independence–results) and Boolean algebra, while any unfamiliar term regarding topology is almost sure to be found in [96].

Telgársky's survey article [166] on the subject of infinite games contains an excellent bibliography; the bibliography we present at the end of this article, though it has many items in common with that of [166], is not a replacement but rather a supplement. For the benefit of the reader I also included a few textbooks which may be useful in pursuing some of the finer points not elaborated in the article. For example, I did not give an explanation of the important notion of a *consistency result*, or of an *equiconsistency result*: this notion lies at the center of forcing theory and inner model theory, two subdisciplines in set theory. Though [74] discusses these matters sufficiently, alternate expositions may be helpful. My personal favorite texts regarding forcing theory are [14], [23] and [132]. I also in passing mentioned that there is a deep connection between the theory of games and the theory of definable subsets of the real line, also known as descriptive set theory. Here the forthcoming text [85] or the classical text [112] would be useful; the introduction to the paper [103] contains an illuminating and concise discussion on these matters. Also, [74] would serve the reader well on these matters. The notions of *measurable cardinals* and *supercompact cardinals* are also mentioned in the article. These are examples of so-called *large cardinals*; again [74] is a good general purpose reference on these matters. Alternates would be the text [36] and the forthcoming text [82].

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1. THE ZIG-ZAG ARGUMENT

In upper division mathematics courses or early graduate level courses we usually learn facts like: the union of countably many

* first category sets is a set of the first category,
* countable sets is a countable set,
* measure zero sets is a set of measure zero.

Examples of this phenomenon abound in mathematics. The proofs are all in one way or another a variation of Cantor's *zig-zag argument* which shows that the set of rational numbers is countable. Most mathematicians would consider this argument among one of the main tools of the trade, and among the supreme examples of mathematical beauty. The argument is well worth our attention. Bibliographic references for this section include [7], [90] and [136] through [142].

To begin, we review one of the proofs that the union of countably many first category sets is a first category set. At the outset we are given a sequence $O_1, O_2, \cdots, O_n, \cdots$ of first category subsets of a topological space. Being of the first category, each can be written as an union of an increasing sequence of nowhere dense sets. Let $N_1^i \subseteq N_2^i \subseteq \cdots \subseteq N_k^i \subseteq \cdots$ be such an increasing sequence of nowhere dense sets whose union is O_i. Then we reassemble the union $\bigcup_{i=1}^{\infty} O_i$ as an union of countably many nowhere dense sets as follows: We define T_i to be the union $\bigcup_{j,k \leq i} N_k^j$. This is an union of finitely many nowhere dense sets, and so is still nowhere dense. Moreover, $\bigcup_{i=1}^{\infty} T_i = \bigcup_{j=1}^{\infty} O_j$.

When looking carefully at this proof, we see that the success of the method does not really require knowing the entire sequence of O_j–s at the outset: When we define T_i, we only used knowledge about $O_1, \cdots, O_i$. This suggests that we may even be given the terms of the sequence of first category sets one–by–one, with no knowledge of which ones we will be confronted with in the future, and yet our method will work. We may think of this as an infinitely long game between two players, ONE and TWO. In the n–th inning player ONE, who has complete freedom of choice, selects a first category set O_n, and then player TWO responds with a nowhere dense set T_n. In this manner the players build a *play*

$$(O_1, T_1, \cdots, O_n, T_n, \cdots).$$

TWO is declared the winner of this play if $\bigcup_{n=1}^{\infty} O_n \subseteq \bigcup_{n=1}^{\infty} T_n$.

The proof we have given above shows that TWO has a winning strategy in this game, i. e., there is a function F such that if TWO plays $F(O_1) = T_1, \cdots, F(O_1, \cdots, O_n) = T_n$, then TWO wins the resulting play: the function F in our argument above is given by $T_n = F(O_1, \cdots, O_n) = \bigcup_{i,j \leq n} N_j^i$. This strategy for TWO uses as information *all preceding* moves of the opponent, i. e., it is a *perfect memory* strategy. Does TWO really need all this memory to secure a win? It is this question which will occupy us for this part of the article.

We shall call the above game, as well as the variations of it which we introduce below "meager – nowhere dense games". Though we are using topological terminology to describe our games, these can equivalently be described in terms of the combinatorial concept of a free ideal. The collection J of nowhere dense subsets of a T_1–space X with no isolated points has the following properties:

1. $A, B \in J \Rightarrow A \cup B \in J$,
2. $A \in J$ and $B \subseteq A \Rightarrow B \in J$,
3. $X \notin J$ and
4. $\cup J = X$.

Any collection J of subsets of a set X is said to be *a free ideal on* X if it has these four properties. Given such a free ideal, there is a corresponding T_1–topology τ on X such that J is the collection of nowhere dense subsets of the space X. Put $\tau = \{X \backslash Y : Y \in J\} \cup \{\emptyset\}$. In what follows we shall freely interchange the combinatorial terminology of free ideals and the topological terminology of nowhere dense sets. For the remainder of this section we shall assume that we have a free ideal J on an infinite set X.

There are some matters of notation that will be important throughout this section: The symbol $\langle J \rangle$ denotes the collection of sets which can be represented as a union of countably many sets from J, and is said to be the *σ–completion* of J. For Y a subset of X, the symbol $J\lceil_Y$ denotes the set $\{T \in J : T \subseteq Y\}$. The symbol $\subset$ denotes "...is a *proper* subset of...".

Then the game we described above, called *the random game on J*, is denoted RG(J), and is played as follows: In the n–th inning ONE chooses a set O_n from $\langle J \rangle$, and TWO responds with a set $T_n \in J$. TWO is the winner of a play

$$(O_1, T_1, \cdots, O_n, T_n \cdots)$$

if $\bigcup_{n=1}^{\infty} O_n \subseteq \bigcup_{n=1}^{\infty} T_n$; otherwise, ONE is the winner.

Coding strategies

A strategy F of TWO of the form $T_1 = F(O_1)$ and for all n $T_{n+1} = F(T_n, O_{n+1})$ is said to be a *coding strategy*. There is a nice combinatorial characterization of those free ideals J for which TWO has a winning coding strategy in the game RG(J). A subset $\mathcal{A}$ of $\langle J \rangle$ is said to be *cofinal* if there is for each $Y \in \langle J \rangle$ an A in $\mathcal{A}$ such that Y is a subset of A. The *cofinality* of $\langle J \rangle$, denoted $\mathrm{cof}(\langle J \rangle)$, is defined to be the minimum of the set of cardinalities of cofinal subsets of $\langle J \rangle$.

Theorem 1. *The following statements are equivalent:*

1. *TWO has a winning coding strategy in the game* RG(J).
2. *The cofinality of $\langle J \rangle$ is no larger than the cardinality of J.*

Suppose now that we introduce more rules: From now on ONE is required to play so that for all n, $O_n \subseteq O_{n+1}$. This version of the game is denoted WMG(J), and is called *the weakly monotonic game on J*. Since the rules are stricter on ONE, it seems reasonable to expect that is may now be easier for TWO to win with coding strategies. There is a partial result in this direction. To state it, we need the following familiar terminology. The Generalized Continuum Hypothesis is the assertion that for every infinite cardinal number κ, $2^{\kappa} = \kappa^{+}$.

Theorem 2. *Assume that the Generalized Continuum Hypothesis holds. Then for every free ideal J, TWO has a winning coding strategy in the game* WMG(J).

It is not clear that the assumption about cardinal arithmetic is necessary. Indeed, in all specific examples of free ideals considered so far it was possible to prove outright that TWO has a winning coding strategy. It has also been proven that if J is any free ideal on any set of cardinality less than $\aleph_{\omega_1}$, then TWO has a winning coding strategy in the game WMG(J). This evidence suggests:

Conjecture 1 (Coding Strategy Conjecture). *For every free ideal J, TWO has a winning coding strategy in the game* WMG(J).

The simplest unresolved instance of this conjecture is:

Problem 1. *Let X be a set of cardinality $\aleph_{\omega_1}$. Is it true that for any free ideal J on X, TWO has a winning coding strategy in the game* WMG(J)?

Remainder strategies

Next we consider a different sort of strategy for TWO in the games WMG(J). Does TWO have a winning strategy which depends on knowing only the set of points from ONE's most recent choice, which have not yet been covered by TWO? A strategy of TWO which depends on knowing only this is a so–called *remainder strategy.* Formally: A strategy F for TWO is a remainder strategy if $T_1 = F(O_1)$ and for each n, $T_{n+1} = F(O_{n+1} \setminus (\bigcup_{j \leq n} T_j))$.

One can prove that if TWO has a winning remainder strategy in the game WMG(J), then TWO has a winning coding strategy in WMG(J); the converse is not true. For let J be the free ideal consisting of the finite subsets of the real line. Then TWO has a winning coding strategy in the game WMG(J) (indeed, in the game RG(J)), but as a consequence of a theorem of W. Just, TWO does not have a winning remainder strategy in the game WMG(J). It is not yet clearly understood when TWO has a winning remainder strategy in the game WMG(J). If J is for example the ideal of nowhere dense subsets of the real line, then TWO has a winning remainder strategy in WMG(J).

To give another concrete example, let κ and λ be infinite cardinal numbers such that λ is the union of countably many sets, each of cardinality less than λ (we say that λ *has countable cofinality*), and such that κ is not less than λ. Let $J_{\kappa,\lambda}$ be the ideal of subsets of κ which are of cardinality less than λ. Then $\langle J_{\kappa,\lambda}\rangle$ is the collection of subsets of κ which are of cardinality less than or equal to λ. It is known that if the cofinality of $\langle J_{\kappa,\lambda}\rangle$ is no larger than $\lambda^{<\lambda}$, then TWO has a winning remainder strategy in the game WMG($J_{\kappa,\lambda}$). A theorem of F. Galvin implies that if the cofinality of $\langle J_{\kappa,\lambda}\rangle$ is larger than 2^{λ}, then TWO does not have a winning remainder strategy in WMG($J_{\kappa,\lambda}$). This leaves the following open problem:

Problem 2. *Is it true that whenever the cofinality of $\langle J_{\kappa,\lambda}\rangle$ is no larger than 2^{λ}, then TWO has a winning remainder strategy in the game* WMG($J_{\kappa,\lambda}$)?

k–tactics

The third type of limited memory strategy which we consider requires further restrictions on player ONE. The game denoted MG(J) and called the *monotonic game on J* is played like the game WMG(J), except that ONE must now obey the rule that for each n O_n is a proper subsets of O_{n+1}.

Fix a positive integer k. A strategy of TWO which requires knowledge of only the at most k most recent moves of ONE is said to be a k–tactic for TWO. Formally, a strategy F for TWO is a k–tactic if: for $j \leq$

k, $T_j = F(O_1, \cdots, O_j)$, while for all n, $T_{n+k} = F(O_{n+1}, \cdots, O_{n+k})$. A 1–tactic will be called a tactic. We now ask ourselves if there is ever a k such that TWO has a winning k–tactic in $\mathrm{MG}(J)$.

It is known for which J TWO has a winning tactic in $\mathrm{MG}(J)$:

Theorem 3. *The following statements are equivalent:*

1. *TWO has a winning tactic in* $\mathrm{MG}(J)$.
2. $\langle J \rangle = J$.

Also, there is a nice combinatorial description for those free ideals J such that TWO has a winning 2–tactic in $\mathrm{MG}(J)$.

Theorem 4. *The following statements are equivalent:*

1. *TWO has a winning 2–tactic in* $\mathrm{MG}(J)$.
2. *There is a set S and a well-ordering $\prec$ for S and for each A in $\langle J \rangle$ there is a function f_A from A into S such that for all A and B in $\langle J \rangle$, if A is a proper subset of B, then the set $\{x \in A : f_A(x) \preceq f_B(x)\}$ is in J.*

As before, let κ and λ be cardinal numbers such that κ is at least as large as λ and such that λ has countable cofinality. It is not yet well understood under what circumstances TWO has a winning 2–tactic in $\mathrm{MG}(J_{\kappa,\lambda})$. The following theorems are known:

Theorem 5 (Koszmider)

1. *If κ is equal to $\aleph_n$ for some finite n, then TWO has a winning 2–tactic in* $\mathrm{MG}(J_{\kappa,\aleph_0})$.
2. *It is consistent that for every κ, TWO has a winning 2–tactic in* $\mathrm{MG}(J_{\kappa,\aleph_0})$.

The game $\mathrm{MG}(J_{\kappa,\aleph_0})$ is also known as the *countable–finite game on κ*. Thus, the following conjecture is consistent – it is not known if it is outright provable:

Conjecture 2 (Countable–finite conjecture). *For every cardinal number κ TWO has a winning 2–tactic in the game* $\mathrm{MG}(J_{\kappa,\aleph_0})$.

The simplest open instance of the Countable–finite conjecture is:

Problem 3. *Does TWO have a winning 2–tactic in the game* $\mathrm{MG}(J_{\aleph_\omega,\aleph_0})$?

For λ an uncountable cardinal number of countable confinality the following things are known:

Theorem 6. *Let λ be an uncountable cardinal number of countable cofinality.*

1. *It is consistent that TWO has a winning 2-tactic in* $\mathrm{MG}(J_{\lambda^+,\lambda})$.
2. *It is consistent that for all k TWO does not have a winning k-tactic in* $\mathrm{MG}(J_{\lambda^+,\lambda})$.
3. *The following statements are equivalent:*
   (a) *TWO has a winning k-tactic in* $\mathrm{MG}(J_{\lambda^+,\lambda})$.
   (b) *For each finite n, TWO has a winning k-tactic in* $\mathrm{MG}(J_{\lambda^{+n},\lambda})$.

Thus, for uncountable cardinal numbers λ of countable cofinality, it is independent of the usual axioms of mathematics whether TWO ever has a winning k–tactic for some k in the game $\mathrm{MG}(J_{\kappa,\lambda})$. Another point raised by Theorem 6 is whether $k = 2$ is exactly the breakpoint. We know:

Theorem 7. *Let λ be any cardinal number of countable cofinality. Then the following statements are equivalent.*

1. *TWO has a winning k-tactic in the game* $\mathrm{MG}(J_{\kappa,\lambda})$.
2. *TWO has a winning 3-tactic in the game* $\mathrm{MG}(J_{\kappa,\lambda})$.

This raises the following problem (which is open even for the countable–finite game):

Problem 4. *Is it ever possible that TWO has a winning 3-tactic in the game* $\mathrm{MG}(J_{\kappa,\lambda})$, *but does not have a winning 2-tactic?*

The third item in Theorem 6 suggests the following conjecture:

Conjecture 3 (λ, $<\lambda$–Conjecture). *Let λ be a cardinal number of countable cofinality. If TWO has a winning k-tactic in the game* $\mathrm{MG}(J_{\lambda^+,\lambda})$, *then for all κ TWO has a winning k-tactic in the game* $\mathrm{MG}(J_{\kappa,\lambda})$.

Notice that the truth of this conjecture implies the truth of the countable–finite conjecture.

Now the reader may start wondering if there ever are free ideals J such that TWO does not have a winning 2–tactic in $\mathrm{MG}(J)$, but does have a winning 3–tactic. This brings us to our next example. Let J_R denote the ideal of nowhere dense subsets of the real line. For this example we know the following:

Theorem 8

1. *TWO does not have a winning 2–tactic in the game* $\mathrm{MG}(J_R)$.
2. *The following statements are equivalent:*
   (a) *There is a k such that TWO has a winning k–tactic in* $\mathrm{MG}(J_R)$.
   (b) *TWO has a winning 3–tactic in* $\mathrm{MG}(J_R)$.
3. *It is consistent that TWO has a winning 3–tactic in* $\mathrm{MG}(J_R)$.
4. *It is consistent that TWO does not have a winning 3–tactic in* $\mathrm{MG}(J_R)$.

Thus it is independent of the usual axioms of mathematics whether TWO has a winning k–tactic in $\mathrm{MG}(J_R)$.

No example is known for which TWO may have a winning 4–tactic, but not a winning 3–tactic in $\mathrm{MG}(J)$. Evidence suggests the following conjecture:

Conjecture 4 (Three Tactic Conjecture). *For every free ideal* J, *if TWO has a winning k–tactic in* $\mathrm{MG}(J)$, *then TWO has a winning 3–tactic in* $\mathrm{MG}(J)$.

Not even the consistency of the three tactic conjectures is known.

Markov k–tactics

Fix a positive integer k. A function F is a *Markov k–tactic* for TWO in the game $\mathrm{WMG}(J)$ if for $i \leq k$ $T_i = F(O_1, \cdots, O_i, i)$, and for all m, $T_{m+k} = F(O_{m+1}, \cdots, O_{m+k}, m+k)$. Thus, in addition to knowing the k most recent moves of ONE, TWO now also knows the number of the inning in progress. The additional knowledge of the inning number can be an advantage to TWO, as the next theorem illustrates.

Theorem 9. *Let* λ *be a cardinal number of countable cofinality.*

1. *If* κ *is larger than* λ, *then TWO does not have a winning Markov–tactic in* $\mathrm{WMG}(J_{\kappa,\lambda})$.
2. *For all* κ *less than or equal to* $\lambda^{+\omega}$, *TWO has a winning Markov 2–tactic in* $\mathrm{WMG}(J_{\kappa,\lambda})$.

It is not known if the information in this theorem is optimal. Indeed:

Conjecture 5 (Markov 2–Tactic Conjecture). *If* λ *is a cardinal number of countable cofinality, then for every infinite cardinal number* κ, *TWO has a winning Markov 2–tactic in the game* $\mathrm{WMG}(J_{\kappa,\lambda})$.

It is also not clear what the situation concerning Markov k–tactics is for the ideal J_R. It is known that TWO does not have a winning Markov 1–tactic in WMG(J_R). It is consistent that TWO has a winning Markov 2–tactic in WMG(J_R). It is not known if this is simply a theorem:

Problem 5. *Does TWO have a winning Markov 2-tactic in* WMG(J_R)?

2. THE DIAGONAL ARGUMENT

Mathematics students learn early on in their upper division classes that

* the set of real numbers is uncountable,
* the set of all subsets of an infinite set (also called the powerset of that infinite set) cannot be partitioned into sets, in number equal to the number of elements of the underlying set, and each containing fewer sets than the powerset of the given set,
* there are statements in Peano arithmetic which are true of the set of natural numbers, but not provable from the Peano axioms of arithmetic,
* the collection of Borel subsets of the real line forms a proper hierarchy,
* the collection of projective subsets of the real line forms a proper hierarchy.

These theorems, as well as multitude of others not mentioned here, employ in one way or another in their proofs Cantor's famous *diagonal argument.* Many mathematicians would consider the diagonal argument as another of the main tools of the mathematical trade, and among the prime examples of mathematical beauty.

We now consider one (of many) games which is directly related to the diagonal argument. By hindsight, *diagonal games* have been studied for a long time. In Hurewicz's study of Menger's property and of characterizations of the F_σ– and G_δ–sets among the Borel sets ([67], [68] and [69]) he proves theorems which, in our terminology, are theorems asserting the existence or non–existence of winning strategies of one or the other of two players in a very thinly disguised game. Grigorieff's study [59] of certain ultrafilters on the set of positive integers also proceeds in the spirit of the cited Hurewicz papers, with no mention of games. Galvin, who independently formulated and studied games concerning ultrafilters, later learned that some of his game–theoretic theorems are exact analogues of Grigorieff's theorems. Cantor's diagonal argument which proves that the real line is uncountable, is similarly a thinly disguised theorem about the existence of a winning strategy for the appropriate player in a certain game.

The point intended here is again that the game–theoretic attitude to the study of certain mathematical objects is a natural one, and has been practised since the early days of set theory.

According to Hurewicz a topological space (X, τ) has the Hurewicz property if there is for every sequence $(\mathcal{U}_n : n = 1, 2, 3, \cdots)$ of open covers of X a sequence $(\mathcal{V}_n : n = 1, 2, 3, \cdots)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and such that each element of X is an element of all but finitely many of the sets $\cup\mathcal{V}_n$. It is clear that if X is a union of countably many of its compact subsets, then it has the Hurewicz property.

We shall call the following game the *Hurewicz game* on the space (X, τ): In the n–th inning player ONE chooses an open cover $\mathcal{O}_n$ of the space; TWO responds by choosing a finite subset $\mathcal{T}_n$ of $\mathcal{O}_n$. The players play and inning for each positive integer. TWO is the winner of the play

$$\mathcal{O}_1, \mathcal{T}_1, \cdots, \mathcal{O}_n, \mathcal{T}_n, \cdots$$

if each element is in all but finitely many of the sets $\cup\mathcal{T}_n$; otherwise, ONE wins.

The Hurewicz game can be used to characterize those topological spaces which have the Hurewicz property.

Theorem 10. *For a topological space (X, τ), the following are equivalent:*

1. *(X, τ) has the Hurewicz property.*
2. *ONE does not have a winning strategy in the Hurewicz game on X.*

Also the subsets of the real line for which TWO has a winning strategy in the Hurewicz game can be characterized very nicely:

Theorem 11. *For a subset X of the real line, the following are equivalent:*

1. *X is an union of countably many compact subsets of the real line.*
2. *TWO has a winning strategy in the Hurewicz game on X.*

In light of these two theorems, an old conjecture of Hurewicz[1] ([68], p. 200) can be reformulated as follows:

Conjecture 6 (Hurewicz). *The Hurewicz game is determined for all sets of real numbers.*

It could happen that there are subsets of the real line which are not representable as a union of countably many of its compact subsets, and yet the set has the Hurewicz property. Such sets can be constructed with the aid of the Continuum Hypothesis – see [45]. Thus, the Continuum Hypothesis implies that Hurewicz's conjecture is false. It is still an open problem if one can disprove Hurewicz's conjecture without resorting to additional hypotheses such as the Continuum Hypothesis.

3. THE CUT-AND-CHOOSE GAMES

Many a parent has found that one way out of dividing candy fairly between two children is to have the one divide the candy, and to have the other choose. Cut and choose games are to some degree a mathematical formulation of this idea.

The earliest traces of an infinite cut and choose game appears in the 1929 paper [6] of Banach and Kuratowski. Galvin described many cut and choose games in his unpublished manuscript [50]. Ulam also considered some cut and choose games – see for example pp. 346 – 347 of his paper [169].

Some of these cut and choose games were studied in the context of Boolean algebras by Jech [71, 72], Foreman [43], Kamburelis [81], Veličković [171], Vojtas [174] and Zapletal [183].

Besides their intrinsic interest the importance of the cut–and–choose games we describe here lies in the fact that they proved to be useful in giving some enlightening information regarding important open problems in connection with multiboard games discussed below.

The Bannach – Kuratowski game

The games described here are motivated by a theorem of Banach

[1] "Es enstehts nun die Vermutung dass durch die (warscheinlich schärfere) Eigenschaft E^{**} die halbkommpakten Mengen F_σ allgemein charakterisiert sind".

and Kuratowski, published in 1929. In honor of their pioneering work we shall call these games Banach – Kuratowski games. The original setting for these games is as follows. Let S be an infinite set and let μ be a cardinal number. Then define the game $\mathrm{BK}(S,\mu)$ as follows: In the n–th inning ONE partitions S into countably many disjoint pieces – let $\mathcal{O}_n$ denote the set of these pieces; TWO responds by choosing a finite subset $\mathcal{T}_n$ of $\mathcal{O}_n$. The players play an inning per positive integer. TWO wins a play

$$\mathcal{O}_1, \mathcal{T}_1, \cdots, \mathcal{O}_n, \mathcal{T}_n, \cdots$$

if the cardinality of the intersection of the sets $\cup\mathcal{T}_n$ is at least μ. Let $\mathcal{N}$ be the collection of subsets of the real line which have Lebesgue measure zero. Then $\mathrm{add}(\mathcal{N})$ denotes the smallest cardinal number such that there is a family of that many measure zero sets whose union is not a measure zero set. Here is the motivation for Banach and Kuratowski's work on this game:

Theorem 12 (Banach – Kuratowski). *If ONE has a winning strategy in the game* $\mathrm{BK}(\mathbf{R}, \mathrm{add}(\mathcal{N}))$, *then there are subsets* $S_1, S_2, \cdots, S_n, \cdots$ *of the real line such that the Lebesgue measure cannot be extended to an* $\mathrm{add}(\mathcal{N})$ *– complete measure on the* σ*–field generated by*

$$\mathcal{N} \cup \{S_n : n = 1, 2, 3, \cdots\}.$$

Now we look at some of the modern developments regarding Banach – Kuratowski games. Let $\mathcal{B}$ be a Boolean algebra, and let κ be an infinite cardinal number.

Also, let a be a non–zero element of $\mathcal{B}$. Then the game $\mathrm{BK}(\mathbf{B},\mathbf{a},\kappa)$ is played as follows: The players play one inning per positive integer; in the n–th inning ONE chooses a partition $\mathcal{O}_n$ of a of cardinality at most κ and TWO responds by choosing a finite subset $\mathcal{T}_n$ of $\mathcal{O}_n$. TWO wins a play

$$\mathcal{O}_1, \mathcal{T}_1, \cdots, \mathcal{O}_n, \mathcal{T}_n, \cdots$$

if there is a nonzero element $\mathbf{w}$ of $\mathbf{B}$ such that for every n, $\mathbf{w} \leq \bigvee \mathcal{T}_n$.

This Boolean – algebraic version of the Banach – Kuratowski game was formulated by Thomas Jech in his paper [71]. In this paper Jech shows that if ONE does not have a winning strategy in this game, then the Boolean algebra in question has certain distributivity properties. Let $\mathcal{B}$ be the σ–algebra of Borel subsets of the unit interval and let $\mathcal{N}$ be the collection of Lebesgue measure zero subsets of the unit interval. Then, let $\mathbf{A}$ be the factor Boolean algebra obtained by the construction

$\mathcal{B}/\mathcal{N}$. For any non-zero element a of **A** and for every infinite cardinal number κ, TWO has a winning strategy in the game BK(**A**,a,κ). Here is an unresolved conjecture regarding **A**:

Conjecture 7 (Gray's Conjecture). *Let* **B** *be a complete Boolean algebra having the following properties:*

1. *Any partition of a nonzero element of* **B** *is countable,*
2. **B** *is generated by countably many elements, and*
3. *TWO has a winning strategy in the game* BK(**B**,a,κ) *for all infinite* κ.

Then **B** *is isomorphic to* **A**.

A game of Galvin

Let S be an infinite set. Consider the following game where in the n–th inning, ONE chooses a partition $\mathcal{O}_n$ of S into two disjoint pieces. Then TWO responds by selecting $T_n \in \mathcal{O}_n$. The players play an inning for each positive integer. TWO wins a play

$$\mathcal{O}_1, T_1, \cdots, \mathcal{O}_n, T_n, \cdots$$

if for each n T_n has more than one element, and if the intersection of all the T_n's is non–empty. Let the symbol Gal(S) denote this game. In unpublished work Galvin proved the following theorem:

Theorem 13 (Galvin). *Let* S *an infinite set.*

1. *ONE wins the game* Gal(S) *if, and only if,* S *is a countable set.*
2. *If* S *is an uncountable set of cardinality no larger than the cardinality of the real line, then neither player has a winning strategy in the game* Gal(S).

The second item in this theorem was also later independently discovered by S. Hechler.

Problem 6 (Galvin). *Is it true that TWO has a winning strategy in the game* Gal(S) *whenever* S *is a set of cardinality* $(2^{\aleph_0})^+$?

Jech initiated the study of the Boolean algebraic version of this game. Let **B** be a complete Boolean algebra, and let **a** be a non–zero element of **B**. Then the game Gal(**B**,**a**) is played as follows: In the n–th inning ONE first partitions **a** into two disjoint pieces, i.e., ONE selects elements $\mathbf{O}_0^n$ and $\mathbf{O}_1^n$ of **B** such that $\mathbf{O}_0^n \wedge \mathbf{O}_1^n = 0$ and $\mathbf{O}_0^n \vee \mathbf{O}_1^n = \mathbf{a}$; TWO responds by selecting an element $\mathbf{T}_n$ from $\{\mathbf{O}_0^n, \mathbf{O}_1^n\}$. For convenience, let the symbol $\mathcal{O}_n$ denote the set $\{\mathbf{O}_0^n, \mathbf{O}_1^n\}$. TWO wins the play

$$\mathcal{O}_1, \mathbf{T}_1, \cdots, \mathcal{O}_n, \mathbf{T}_n, \cdots$$

if $\bigwedge_{n=1}^{\infty} \mathbf{T}_n > 0$; otherwise, ONE wins.

Jech showed that this game can be used to describe distributivity properties of Boolean algebras. Let κ and λ be cardinal numbers. Then a Boolean algebra is said to be (κ, λ)-distributive if for every doubly indexed family $(\mathbf{a}_{\alpha,\beta} : \alpha < \kappa, \beta < \lambda)$ of elements of the Boolean algebra, the equation

$$\bigwedge_{\alpha<\kappa} \Big(\bigvee_{\beta<\lambda} \mathbf{a}_{\alpha,\beta} \Big) = \bigvee_{f \in {}^{\kappa}\lambda} \Big(\bigwedge_{\alpha<\kappa} \mathbf{a}_{\alpha,f(\alpha)} \Big)$$

holds.

Theorem 14 (Jech). *For a complete Boolean algebra* $\mathbf{B}$*, the following statements are equivalent:*

1. $\mathbf{B}$ *is an* $(\aleph_0, 2)$ *-distributive Boolean algebra.*
2. *For each* $\mathbf{a}$ *in* $\mathbf{B}$*, ONE does not have a winning strategy in the game* Gal(B,a).

The Galvin – Ulam game

Let S be an infinite set and let $\lambda \geq 2$ be a cardinal number. Let GU(S, λ) denote the Galvin – Ulam game, which is played as follows: Two players, ONE and TWO, play one inning per positive integer. In the first inning ONE partitions S into at most λ pairwise disjoint nonempty pieces; TWO responds by selecting one of these pieces, call it S_1, and then TWO partitions S_1 into at most λ pairwise disjoint nonempty pieces. This concludes the first inning.

At the beginning of the second inning, ONE selects one of the pieces of TWO's partition of S_1, call it S_2, and partitions S_2 into at most λ pairwise disjoint nonempty pieces; TWO responds by selecting one of these pieces, call it S_3 and by partitioning it into at most λ pairwise disjoint pieces, and so on.

In this manner the two players construct a sequence

$$S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots \supseteq S_n \supseteq \ldots$$

ONE is the winner of this play if $\bigcap_{n=1}^{\infty} S_n \neq \emptyset$. Otherwise, TWO wins.

Ulam's original game was GU(S,2), and Ulam asked if TWO has a winning strategy in the game GU(ω_1,2).

Theorem 15 (Galvin). *Let S be an infinite set.*

1. *TWO has a winning strategy in* GU(S,2) *if, and only if, S is a countable set.*
2. *If ONE has a winning strategy in the game* GU(S,2), *then the cardinality of S is larger than that of the real line.*
3. *ONE has a winning strategy in* GU(S,2) *if, and only if, ONE has a winning strategy in the game* GU($S,\aleph_0$).

The second item in this theorem was also discovered independently by S. Hechler.

The connection between Galvin's game Gal(S), and the Galvin–Ulam game GU(S,2) is as follows:

Theorem 16 (Galvin). *For an infinite set S, the following are equivalent:*

1. *TWO has a winning strategy in the game* Gal(S).
2. *Either ONE has a winning strategy in the game* GU(S,2) *or else the cardinality of S is at least as large as the first uncountable measurable cardinal number.*

It is also known that the assertion that there is an infinite set S such that ONE has a winning strategy in the game GU(S,2) far transcends the traditional axioms of set theory in consistency strength. This is because of the following result of Solovay and Gray:

Theorem 17 (C. Gray and R. M. Solovay). *If it is consistent that there is an infinite set such that ONE has a winning strategy in the game* GU(S,2), *then it is also consistent that there is a measurable cardinal number.*

M. Magidor then completed the "consistency picture" with the result:

Theorem 18 (M. Magidor). *If it is consistent that there is a measurable cardinal number, then it is consistent that there is a set S such that ONE has a winning strategy in the game* GU(S,2).

One might wonder now if the assertion that the cardinality of S is measurable implies that ONE has a winning strategy in the game GU(S,2) or if the fact that ONE has a winning strategy in the game GU(S,2) implies that the cardinality of S is measurable. Solovay showed that the first implication is not provable, and Magidor has shown that the second one is not provable. R. Laver improved the results of Magidor as follows:

Theorem 19 (R. Laver). *If it is consistent that there is a measurable cardinal, then it is consistent that ONE has a winning strategy in the game* $\mathrm{GU}(\omega_2,2)$.

Laver moreover proved:

Theorem 20 (R. Laver). *If it is consistent that there is a proper class of supercompact cardinals, then it is consistent that for every uncountable regular cardinal number* κ, *ONE has a winning strategy in the game* $\mathrm{GU}(\kappa^+,\kappa)$.

It is known if such a strong large cardinal hypothesis is necessary in Theorem 20

Problem 7 (Laver). *If it is consistent that for every regular uncountable cardinal number* κ *ONE has a winning strategy in the game* $\mathrm{GU}(\kappa^+,\kappa)$, *is it then consistent that there is a proper class of supercompact cardinal numbers?*

4. THE DESCENDING CHAIN ARGUMENT

We now turn our attention to games which require that the players construct a descending chain of elements of a partially ordered set. There are much quoted and much used games of this sort.

One example was invented by Mazur in the late 1920's. It appears as Problem 43 of The Scottish Book [104]. The first important result about this game was proved by Banach. The game is now widely known as the Banach – Mazur game. The game was later generalized to its present form by Oxtoby. Several authors also studied various generalizations of the Banach – Mazur game. The Banach – Mazur game has also been generalized to the context of Boolean algebras.

Another important example, the Banach – Galvin game, also known as the precipitous ideal game, was invented during the 1930's by Banach (Problem 67 of The Scottish Book) and generalized to its present form in the 1970's by Galvin. It is a much used game in connection with the study of precipitous ideals.

A third example, the Sierpinski game, was invented by Sierpinski in the 1920's [156] and was later explicitly formulated by Telgársky. A fourth example is now known as Michael's game; it was invented by Ernest Micheal during his study of completeness properties of metric spaces. A fifth example is due to Choquet [18], and is a sort of Banach–Mazur game with side conditions.

There are many other games of this sort in the literature, but the five we look at here should give the reader some indication of the sorts of concerns that are important in these games.

The Banach – Mazur game for topological spaces

Let (X,τ) be a topological space. The Banach–Mazur game on (X,τ), denoted $\mathrm{BM}(X,\tau)$, is played as follows: In the n–th inning, ONE selects a nonempty open subset O_n of X; TWO responds by selecting a nonempty open subset T_n of X. The players must further obey the rule that for each n, $O_n \supseteq T_n \supseteq O_{n+1}$. Player ONE wins a play $(O_1, T_1, \ldots, O_n, T_n, \ldots)$ if $\bigcap_{n=1}^{\infty} O_n = \emptyset$; otherwise, TWO wins.

Mazur originally formulated this game for the case when X is a subset of the real line and τ is the topology which X inherits from the real line. In the 1950's Oxtoby extended the definition to its present form [123]. The first interesting theorem regarding this game is due to Banach, and was later extended to the more general setting by Oxtoby:

Theorem 21 (Banach – Oxtoby). *For a topological space (X,τ), the following are equivalent:*

1. *ONE has a winning strategy in the game* $\mathrm{BM}(X,\tau)$.
2. *X has a nonempty open subset which is of the first category in X.*

It was later observed that it follows almost immediately from the Banach – Oxtoby theorem that if ONE has a winning strategy in the game $\mathrm{BM}(X,\tau)$, then ONE has a winning 1–tactic in this game. In the late 1970's W. G. Fleissner and K. Kunen asked if it is also true that if TWO has a winning strategy in the game $\mathrm{BM}(X,\tau)$, then TWO has a winning 1–tactic [42]. In the early 1980's G. Debs showed that the answer was "no" [29, 30]. Debs gave three examples of spaces for which TWO had a winning strategy in the Banach – Mazur game, but did not have a winning 1–tactic. In all three examples it turned out that TWO had a winning 2–tactic. Two examples were treated in the cited paper of Debs, and the third one was treated in [7]. Shortly afterwards R. Telgársky made the following conjecture [166]:

Conjecture 8 (Telgársky's Conjecture). *For every positive integer k there exists a topological space such that TWO does not have a winning k-tactic, but does have a winning $k+1$-tactic in the Banach – Mazur game on the space.*

The truth of Telgársky's conjecture would imply that there exists a topological space such that TWO has a winning strategy in the Banach – Mazur game on it, but there is no k such that TWO has a winning k–tactic. Thus, as a first attack on this conjecture one might ask:

Problem 8. *Is there a space such that TWO has a winning strategy in the Banach – Mazur game on the space, but there is no k such that TWO has a winning k-tactic in the Banach – Mazur game?*

One may also ask if the situation is any better for TWO if instead of asking for a winning k–tactic we ask for a winning Markov k–tactic. It turns out that the answer is "no": the case $k = 1$ of the following theorem is due to Galvin and Telgársky:

Theorem 22 (Bartoszynski – Just – Scheepers). *Let (X, τ) be a topological space and let k be a positive integer. Then the following statements are equivalent:*

1. *TWO has a winning k-tactic in* $\mathrm{BM}(X, \tau)$.
2. *TWO has a winning Markov k-tactic in $BM(X, \tau)$.*

The notion of a coding strategy for TWO also makes sense in the context of Banach – Mazur games. Here the situation is completely understood, due to the following theorem:

Theorem 23 (Galvin – Telgársky). *For a topological space (X, τ), the following are equivalent:*

1. *TWO has a winning strategy in* $\mathrm{BM}(X, \tau)$.
2. *TWO has a winning coding strategy in* $\mathrm{BM}(X, \tau)$.

The Banach – Mazur game has been used to characterize certain topological aspects of function spaces. We give two examples of this phenomenon:

Let X be a completely regular space and let $C^*(X)$ denote the set of bounded continuous real–valued functions on X. Endow $C^*(X)$ with the supremum norm: $\|f\| = \sup\{|f(x)| : x \in X\}$. It is well known that $\|\cdot\|$ is a complete norm on $C^*(X)$. If X is compact then $C^*(X) = C(X)$, and for every $f \in C^*(X)$ we have

$$\|f\| = \max\{|f(x)| : x \in X\}.$$

For which spaces X is it true that "most" elements of $C^*(X)$ satisfy this equation? The following beautiful theorem gives a description of some of these spaces [87]:

Theorem 24 (Kenderov – Revalski). *For completely regular spaces (X,τ) the following statements are equivalent:*

1. $\{f \in C^*(X) : \|f\| = \max\{|f(x)| : x \in X\}\}$ *is a dense G_δ subset of* $(C^*(X), \|\cdot\|)$.
2. *Player TWO has a winning strategy in* $\mathrm{BM}(X,\tau)$.

Again let X be a completely regular Hausdorff space. Let $C(X)$ be the set of all continuous real–valued functions on X. Now endow $C(X)$ with the *compact-open topology*, τ_{co}: This topology is generated by the subbase whose elements are of the form $\{f \in C(X) : f[K] \subseteq U\}$, where K ranges over the compact subsets of X while U ranges over the open subsets of the real line. For which spaces X is it true that TWO has a winning strategy in the game $\mathrm{BM}(C(X), \tau_{co})$? Also this question has a very satisfying answer [98], relating the Banach – Mazur game to another old and well–studied concept in metrization theory:

Theorem 25 (D. K. Ma). *For completely regular Hausdorff spaces (X,τ), the following are equivalent:*

1. *TWO has a winning strategy in* $\mathrm{BM}(C(X), \tau_{co})$.
2. (X,τ) *is paracompact.*

The Banach – Mazur game for Boolean algebras

Let $\mathbf{B}$ be a complete Boolean algebra. Then the Banach – Mazur game on $\mathbf{B}$, denoted $\mathrm{BM}(\mathbf{B})$ was introduced by Jech [71], and is defined as follows: TWO players, ONE and TWO, alternately choose non–zero elements of $\mathbf{B}$. In the n–th inning ONE selects an element $\mathbf{O}_n$ of $\mathbf{B}$, and TWO responds with an element $\mathbf{T}_n$ of $\mathbf{B}$. The players must obey the rule that for each n $\mathbf{O}_{n+1} \leq \mathbf{T}_n \leq \mathbf{O}_n$. TWO wins a play

$$\mathbf{O}_1, \mathbf{T}_1, \ldots, \mathbf{O}_n, \mathbf{T}_n, \ldots$$

$\bigwedge_{n=1}^{\infty} \mathbf{T}_n > 0$; otherwise, ONE wins.

Jech showed that also this game is related to distributivity laws for Boolean algebras. If the Boolean algebra is for each cardinal number λ (κ, λ)–distributive, then it is said to be (κ, ∞)–distributive.

Theorem 26 (Jech). *For a complete Boolean algebra $\mathbf{B}$, the following statements are equivalent:*

1. $\mathbf{B}$ *is* $(\aleph_0, \infty)$*-distributive.*
2. *ONE does not have a winning strategy in the game* BM($\mathbf{B}$).

As in the case of the Banach – Mazur game for topological spaces, also for Boolean algebras the conditions for the existence of a winning strategy for TWO is less well understood. Jech found a natural condition under which TWO has a winning strategy in the game BM($\mathbf{B}$). A subset D of a Boolean algebra $\mathbf{B}$ is said to be *dense in* $\mathbf{B}$ if, for every non–zero element $\mathbf{b}$ of $\mathbf{B}$, there is a non–zero element $\mathbf{d}$ of D such that $\mathbf{d} \leq \mathbf{b}$. A subset D is said to be $\aleph$–closed if for every sequence $(\mathbf{d}_n : n = 1, 2, 3, \ldots)$ from D such that for each n $0 < \mathbf{d}_{n+1} \leq \mathbf{d}_n$, there is a non–zero $\mathbf{d}$ in D such that for each n $\mathbf{d} \leq \mathbf{d}_n$.

Theorem 27 (Jech). *Let* $\mathbf{B}$ *be a complete Boolean algebra. If* $\mathbf{B}$ *has a dense subset which is* $\aleph_0$*–closed, then TWO has a winning strategy in the game* BM($\mathbf{B}$).

The question whether the converse of this theorem is true has been open for a long time:

Problem 9 (Jech). *Is it true that if* $\mathbf{B}$ *is a complete Boolean algebra for which TWO has a winning strategy in the game* BM($\mathbf{B}$), *then* $\mathbf{B}$ *has a dense subset which is* $\aleph_0$*–closed?*

Considerable progress has been made on this problem. In particular, Foreman proved that if $\mathbf{B}$ is a Boolean algebra such that TWO has a winning strategy in the game BM($\mathbf{B}$), and if $\mathbf{B}$ has a dense subset of cardinality at most $\aleph_1$, then $\mathbf{B}$ has a dense subset which is $\aleph_0$–closed. This was later improved by Velickovic to:

Theorem 28 (Velickovic). *If* $\mathbf{B}$ *is a complete Boolean algebra which has a dense subset of cardinality at most* $2^{\aleph_0}$*, and if TWO has a winning strategy in the game* BM($\mathbf{B}$)*. then* $\mathbf{B}$ *has a dense subset which is* $\aleph_0$*–closed.*

Jech also describes a connection between the games Gal($\mathbf{B}$,a) and the game BM($\mathbf{B}$):

Theorem 29 (Jech). *Let* $\mathbf{B}$ *be a complete Boolean algebra. If player TWO has a winning strategy in the game* BM($\mathbf{B}$), *then for every non-zero* $\mathbf{a}$ *in* $\mathbf{B}$, *TWO has a winning strategy in the game* Gal($\mathbf{B}$,a).

This raises the obvious question whether the converse of the statement in the theorem is also true.

Problem 10 (Jech). *Is it true of each complete Boolean algebra* $\mathbf{B}$ *that if for every non-zero* $\mathbf{a}$ *in* $\mathbf{B}$ *TWO has a winning strategy in the game* $\mathrm{Gal}(\mathbf{B},\mathbf{a})$, *then TWO has a winning strategy in the game* $\mathrm{BM}(\mathbf{B})$?

Zapletal made considerable progress on this problem. To state his result concisely, we need to introduce another concept. A subset A of a Boolean algebra $\mathbf{B}$ is said to be an *antichain* if it consists of more than one element, none of its elements is the zero element, and for every two elements $\mathbf{a}$ and $\mathbf{b}$ from A, $\mathbf{a} \wedge \mathbf{b} = 0$.

Theorem 30 (Zapletal). *If* $\mathbf{B}$ *is a complete Boolean algebra such that no antichain of* $\mathbf{B}$ *is of cardinality larger than* $2^{\aleph_0}$ *and if for every non-zero element* $\mathbf{a}$ *of* $\mathbf{B}$ *TWO has a winning strategy in the game* $\mathrm{Gal}(\mathbf{B},\mathbf{a})$, *then TWO has a winning strategy in the game* $\mathrm{BM}(\mathbf{B})$.

The Banach – Galvin game

In Problem 67 of **The Scottish Book** S. Banach proposed the following infinite game: An uncountable set S is given. In the n–th inning player ONE chooses a subset O_n of S which has the same cardinality as S, and TWO responds with a set T_n which has the same cardinality as S. The players must further obey the rule that for each n we have $O_{n+1} \subseteq T_n \subseteq O_n$. Player ONE wins a play of the game if $\bigcap_{n=1}^{\infty} T_n$ is empty, and TWO wins if this intersection is nonempty. One can show that ONE has a winning strategy in this game – see for example [52].

Galvin later generalized the game as follows: Again let S be an uncountable set, and let J be a free ideal on S. Players ONE and TWO now play so that for each n we have O_n and T_n not elements of J, and $O_{n+1} \subseteq T_n \subseteq O_n$. As before, ONE wins a play of this game if the intersection $\bigcap_{n=1}^{\infty} T_n$ is empty, and TWO wins otherwise. To indicate the involvement of both Banach and Galvin in the formulation of this game, we shall use the symbol $\mathrm{BG}(J)$ to denote the game, and we shall call it the *Banach – Galvin game.*

To see that this game is a generalization of Banach's game, observe that the collection J of subsets of S of cardinality less than S is a free ideal on S. The existence of a winning strategy for either player now depends heavily on the properties of the ideal J.

Independently of these game–theoretic considerations, T. Jech and K. Prikry introduced the notion of a *precipitous ideal* [77]. This notion was motivated by measure theoretic concerns as well as concerns about cardinal arithmetic.

The definition of a precipitous ideal was originally given in terms of mathematical notions related to the theory of models for set theory. As often happens with game–theoretic notions, it was later discovered (by T. Jech) that an elegant combinatorial description of precipitous which requires no knowledge of metamathematics can be given, namely: An ideal J on an infinite set S is a ***precipitous ideal*** if, and only if, player ONE has no winning strategy in the game $\mathrm{BG}(J)$. We shall take this (very often used) characterization of precipitous ideas as our definition. Because of its direct connection with precipitous ideals, the Banach–Galvin game is often called the ***precipitous ideal game***.

The Banach–Galvin game was studied in the joint paper [52]. In order to give a description of the results from this paper, we introduce the following concepts: Let κ be a regular cardinal number. A subset C of κ is ***closed*** if for every subset U of C which his an upper bound in C, the supremum of U is in C; C is said to be an unbounded subset of κ if there is for each α in κ and element β in C such that α is less than β. Then set Club_κ denotes the collection of closed, unbounded subsets of κ. The collection Stat_κ is the family of those subsets of κ which have nonempty intersection with each element of Club_κ. A typical element of Stat_κ is the set of elements of κ which have countable cofinality. Elements of Stat_κ are said to be ***stationary*** subsets of κ. All other subsets of κ are said to be ***nonstationary.*** For convenience we set $\mathrm{NST}_\kappa = \{U \subset \kappa : U \text{ is nonstationary}\}$. A little bit of thought shows that NST_κ is a free ideal on κ. By a well–known theorem of Fodor, if S is a stationary subset of κ and $f : S \to \kappa$ is a function such that for each α in S $f(\alpha) < \alpha$, there is a subset T of S which is a stationary subset of κ and on which f is constant.

Set theorists call an ideal J on κ ***normal*** if: For every set S not in J and for every function $f : S \to \kappa$ such that for each $\alpha \in S$ we have $f(\alpha) < \alpha$, there is a subset T of S such that T is not in J and f is constant on T. An ideal J on κ is also said to be ***κ–complete*** if for every subset S of J, if S has cardinality less than κ, then $\cup S$ is an element of J. A standard exercise which also uses Fodor's theorem shows that the ideal NST_κ is a κ–complete normal ideal on κ. It can be shown that NST_κ is a subset of every κ–complete normal ideal on κ.

In [52] the authors prove among other things:

Theorem 31 (Galvin, Jech, Magidor). *Let S be an infinite set and let J be a free ideal on S.*

1. *If the cardinality of S is no larger than that of the real line, then TWO does not have a winning strategy in the game* $\mathrm{BG}(J)$.
2. *If J is a κ–complete normal ideal on the regular initial ordinal κ, and if $\{\alpha \in \kappa : cof(\alpha) = \omega\}$ is not an element of J, then TWO does not have a winning strategy in the game* $\mathrm{BG}(J)$.

Thus, if J is a precipitous ideal on a regular initial ordinal κ and if TWO has a winning strategy in $\mathrm{BG}(J)$, then κ is larger than $2^{\aleph}_0$; if moreover J is normal and κ–complete, then $\{\alpha < \kappa : cof(\alpha) = \omega\}$ is in J.

It is ever possible to have a precipitous ideal? It turns out that it is consistent that there is a precipitous ideal if, and only if, it is consistent that there is a measurable cardinal number [75]. Going a little further one many inquire if there could ever be a precipitous ideal for which TWO has a winning strategy in the Banach – Galvin game. It turns out that it is consistent that there is a precipitous ideal if, and only if, it is consistent that there is a precipitous ideal for which TWO has a winning strategy in the Banach – Galvin game. In [52] the authors give an example, due to R. Laver, of precipitous ideals for which TWO has a winning strategy in the Banach – Galvin game:

Theorem 32 (R. Laver). *If it is consistent that there exists a measurable cardinal number, then it is consistent that there is a precipitous ideal J on ω_2 such that TWO has a winning 1-tactic in the game* $\mathrm{BG}(J)$.

The properties of this example suggest the following generalization of the Banach – Galvin game: Let besides the free ideal J also a family $I \subseteq J$ with the property that if X is in I and if Y is a subset of X then Y is in I, be given. Then the game which is denoted by $\mathrm{BG}(J, I)$ is defined so that a sequence

$$O_1, T_1, \ldots, O_n, T_n, \ldots$$

is a play if for each n we have O_n not in J and $O_{n+1} \subseteq T_n \subseteq O_n$. Such a play is won by ONE if $\bigcap_{n=1}^{\infty} T_n \in I$; otherwise, TWO wins.

Indeed, the example given in Theorem 32 is such that TWO has a winning 1–tactic in the game $\mathrm{BG}(J, J)$. It is not clear that this is particular to the example – in particular:

Problem 11. *Is it true that if J is a precipitous ideal on an infinite set of uncountable regular cardinality such that TWO has a winning 1-tactic in* $\mathrm{BG}(J)$, *then TWO has a winning strategy in* $\mathrm{BG}(J, J)$?

Consider a precipitous ideal J for which TWO has a winning strategy in the game $\mathrm{BG}(J)$; by a theorem of Galvin and Telgársky [55] it follows that TWO has a winning coding strategy. It is not clear how much memory of only ONE's moves TWO would need to insure winning $\mathrm{BG}(J)$.

Problem 12. *If it is consistent that there is a precipitous ideal, is it then consistent that there is a precipitous ideal J such that TWO does not have a winning 1-tactic in the game* $\mathrm{BG}(J)$, *but does have a winning 2-tactic?*

Several examples of obtaining precipitous ideals from appropriately large cardinal numbers are nowadays available in the literature; the paper [76] is an early source for such examples.

J. E. Baumgartner noted a connection between the Banach – Galvin game and the Galvin – Ulam game:

Theorem 33 (Baumgartner). *If J is a free ideal on* κ *such that TWO has a winning strategy in the game* $\mathrm{BG}(J)$, *then ONE has a winning strategy in the game* $\mathrm{GU}(\kappa, \aleph_0)$.

This also connects the Banach – Galvin game with the upcoming discussion of multiboard games.

Making further modifications, Jech introduced two more games for ideals; the ideals which have certain properties with respect to these games are called *pseudo-precipitous* and *weakly precipitous*. These games do not belong to the class of descending chain games, and so will not be introduced right here.

Sierpinski's game

The main problem in the early days of set theory was the status of the Continuum Hypothesis: are there any uncountable sets of real numbers of cardinality less than that of the set of real numbers? Cantor proved early on that for closed sets of real numbers, the answer is "no". Then later Hausdorff and, independently, Alexandroff showed that for all Borel sets the answer is also "no". In 1924 Sierpinski [156] gave a new proof of the Alexandroff – Hausdorff theorem. Sierpinski's proof is one of the early examples of a game – theoretic argument. Telgársky formulated the game that implicitly occurred in Sierpinski's proof. The game is defined as follows:

Let (X,τ) be an uncountable topological space and Y be an uncountable subset of X. Then Sierp(X,Y) denotes the Sierpinski game, which is played as follows: In the n-th inning ONE first selects an uncountable subset O_n of Y, and TWO then responds with an uncountable subset T_n of O_n. Player ONE must further obey the rule that for each n, $O_{n+1} \subseteq T_n$. Player TWO wins a play

$$O_1, T_1, \ldots, O_n, T_n, \ldots$$

of Sierp(X,Y) if $\bigcap_{n=1}^{\infty} \overline{T_n} \subseteq Y$; otherwise, ONE wins. Kubicki proved that if Y is an analytic subset of the real line, then TWO has a winning strategy in the game Sierp$(\mathbf{R},Y)$. This implies the classical theorem of Lusin that every uncountable analytic set has the same cardinality as the real line. The following question was formulated by Kubicki, and still unsolved:

Problem 13 (Kubicki). *Is it true that if Y is in the σ-algebra generated by the analytic subsets of the real line, then the game* Sierp(X,Y) *is determined?*

Michael's game

Let (X,τ) be a topological space. In his study [109] of completely metrizable space E. Michael defines the following game, denoted EMG(X,τ): In the n-th inning ONE chooses a non-empty subset O_n of X, and TWO responds by choosing a non-empty subset T_n of O_n. The players must further obey the rules that for each n,

1. $O_{n+1} \subseteq T_n$, and
2. T_n is open in the relative topology of S_n.

Player TWO wins a play

$$O_1, T_1, \ldots, O_n, T_n, \ldots$$

if the set $\bigcap_{n=1}^{\infty} \overline{T_n}$ is nonempty; otherwise, ONE wins. Michael then shows that a metric space (X,d) has a complete metrization if, and only if, TWO has a winning strategy in the game EMG(X,d). Michael further finds a characterization of those spaces for which TWO has a winning strategy in this game. He also shows that for metric spaces TWO has a winning strategy if, and only if, TWO has a winning 1–tactic. The following problem from Michael's paper appears to be still unsolved:

Problem 14 (E. Michael). *Is it true for any topological space (X,τ) that if TWO has a winning strategy in the game* $\mathrm{EMG}(X,\tau)$ *then TWO has a winning 1-tactic?*

Choquet's game

Like the game of Christensen which is treated later, Choquet's game (introduced in [18]) is a descending chain game with side conditions. This game, denoted $\mathrm{Ch}(X,\tau)$, is played as follows: In the n–th inning ONE chooses a pair (O_n, x_n) where O_n is a nonempty open subset of the space (X,τ), and TWO responds with a nonempty open subset $T_n \subseteq O_n$. ONE must obey the rule that for each n, O_{n+1} is a subset of T_n, and TWO must obey the rule that for each n, x_n is an element of T_n. TWO wins a play

$$(O_1, x_1), T_1, \ldots, (O_n, x_n), T_n, \ldots$$

if $\bigcap_{n=1}^{\infty} T_n$ is nonempty; otherwise, ONE wins.

One might think that the extra demand on TWO does not cause a big difference in the outcome between the Banach – Mazuz game or Choquet's game. However, there are examples such that TWO has a winning 1–tactic in the Banach – Mazuz game, while ONE has a winning strategy in Choquet's game. As was remarked above, if TWO has a winning Markov strategy in the Banach – Mazuz game, then TWO has a winning 1–tactic. It is not known if this is true for the Choquet game. I don't know the origin of this problem; I seem to recall hearing this from Galvin – thus:

Problem 15 (Galvin). *Let (X,τ) be a topological space such that TWO has a winning Markov – strategy in the game* $\mathrm{Ch}(X,\tau)$. *Does TWO then have a winning 1-tactic?*

5. DISJOINT OCCUPANCY GAMES

The types of games considered here can be described as follows: Two player, ONE and TWO, play and infinitely long game which has an inning per positive integer. In each inning they choose subsets of a prescribed sort from a given infinite set S. They must obey the rule that when a player chooses a set, this set must be disjoint from all sets chosen so far by either player. Player ONE usually wins when the set of points chosen by ONE is large in some sense, and TWO wins otherwise. This sort of game can be conceived of as that the two players complete for territory.

These sorts of games have shown up in a variety of contexts, quite independently of each other. I shall describe some of these here.

The Lutzer – McCoy game

The game of Lutzer and McCoy made its debut in the paper [97], where the authors were studying topological properties of the space of real-valued continuous functions, endowed with the point-wise topology. In particular, let (X, τ) be a Hausdorff space with the property that for every pair of distinct points in the space there is a continuous real-valued function taking distinct values at these two points. Such spaces are said to be *completely Hausdorff* spaces. The set of all functions from X to the real line, considered as the Cartesian product of X copies of the real line, carries the Tychonoff product topology. The set $C(X)$ of continuous functions from X to the real line is a natural subspace of this product space. The topology it inherits from this product is denoted τ_p and is said to be the *topology of pointwise convergence* on $C(X)$. The question which motivated the disjoint occupancy game to be described now, was: When is $(C(X), \tau_p)$ a Baire space?

The Lutzer – McCoy game, $\mathrm{LM}(X)$, is played as follows: First, a finite subset S_0 of X is given. Then ONE starts by choosing, in the first inning, a finite set O_1 which is disjoint from S_0; TWO responds by choosing a finite set T_1 which is disjoint from $S_0 \cup O_1$. In general, in the $n+1$–st inning ONE first chooses a finite set O_{n+1} which is disjoint from $S_0 \cup O_1 \cup T_1 \cup \cdots \cup O_n \cup T_n$, and then TWO responds by choosing a finite set T_{n+1} which is disjoint from $S_0 \cup O_1 \cup T_1 \cup \cdots \cup O_n \cup T_n \cup O_{n+1}$. ONE wins the play

$$S_0, O_1, T_1, O_2, T_2, \ldots, O_n, T_n, \ldots$$

if the set $\cup_{n=1}^{\infty} O_n$ has a limit point. Otherwise, TWO wins.

They prove the following interesting theorems regarding this game: First, a bit of terminology. A topological space is *pseudonormal* if there is for every pair of disjoint closed sets, one of which is countable, a pair of disjoint open sets, each containing one of these two closed sets.

Theorem 34 (Lutzer and McCoy). *Let (X, τ) be a pseudonormal completely regular space. Then the following statements are equivalent:*

1. *TWO has a winning strategy in the game* LM(X).
2. *TWO has a winning strategy in the game* BM$(C(X), \tau_p)$.
3. *Every countable subset of X is closed.*

The situation for player ONE is less clear:

Theorem 35 (Lutzer and McCoy). *Let (X, τ) be a completely Hausdorff space.*

1. *If $(C(X), \tau_p)$ is a Baire space, then ONE does not have a winning strategy in the game* LM(X).
2. *If X is the topological sum of Hausdorff spaces, each having at most one nonisolated point, and if ONE has no winning strategy in* LM(X), *then $(C(X, \tau_p)$ is a Baire space.*

Now 2 of Theorem 35 is a weak converse of 1, and one may ask if the full converse is in fact true. Indeed, Lutzer and McCoy ask on p. 158 of [97].

Problem 16 (Lutzer and McCoy). *Is it true for normal Hausdorff spaces (X, τ) that $(C(X), \tau_p)$ is a Baire space if, and only if, ONE does not have a winning strategy in the game* LM(X)?

They also state a conjecture about this problem:

Conjecture 9 (Lutzer and McCoy's Conjecture). *There exists a normal Hausdorff space (X, τ) such that $(C(X), \tau_p)$ is not a Baire space, and yet ONE does not have a winning strategy in the game* LM(X).

On page 151 of [97] the authors also ask the following (slightly reformulated) interesting question:

Problem 17 (Lutzer and McCoy). *Let (X, τ) be a normal Hausdorff space. Is it true that if ONE does not have a winning strategy in the game* LM(X), *then there is a countable subset Y of X such that ONE has not winning strategy in the game* LM(Y)?

An ultrafilter game of Galvin

F. Galvin, S. Hechler and R. McKenzie studied a variety of infinite games related to ultrafilter on the set of positive integers. A set $\mathcal{U}$ of subsets of the positive integers is a filter if: the empty set is not an element of $\mathcal{U}$, if A and B are elements of $\mathcal{U}$, then $A \cap B$ is an element of $\mathcal{U}$, and if A is an element of $\mathcal{U}$ and $A \subseteq B$, then B is an element of $\mathcal{U}$.

If a filter has the property that for every set of positive integers either that set of or its complement is in the filter, then the filter is said to be an *ultrafilter*. An ultrafilter is said to be *free* if the complement of every finite set of positive integers is an element of it. Then ultrafilter $\mathcal{U}$ is said to be a *Q-point* if for every partition of the set of positive integers into pairwise disjoint nonempty finite sets, there is a set in $\mathcal{U}$ which has at most one element in common with each of the finite sets in the partition.

One can show that the Continuum Hypothesis implies that there are Q-points. K. Kunen has shown that to prove the existence of a Q-point ultrafilter requires some hypothesis beyond the usual axioms of set theory.

Let r and s be positive integers and let $\mathcal{U}$ be a free ultrafilter on the set of positive integers. Then the game $\mathrm{GHM}^{r,s}(\mathcal{U})$ is played as follows: In the n-th inning ONE first chooses a subset of $\mathbb{N}\setminus(O_1\cup\cdots\cup O_{n-1}\cup T_1\cup\cdots\cup T_{n-1})$ which contains no more than r points, and then TWO chooses a subset T_n of $\mathbb{N}\setminus(O_1\cup\cdots\cup O_n\cup T_1\cup\cdots\cup T_{n-1})$ which contains no more than s points. Player ONE wins a play

$$O_1, T_1, \ldots, O_n, T_n, \ldots$$

if $\cup_{n=1}^{\infty} O_n$ is an element of $\mathcal{U}$; otherwise, TWO wins.

Theorem 36 (Galvin, Hechler and McKenzie). *Let r and s be positive integers and let $\mathcal{U}$ be a free ultrafilter on the set of positive integers.*

1. *ONE has a winning strategy in $\mathrm{GHM}^{r,s}(\mathcal{U})$ if, and only if, TWO has a winning strategy in the game $\mathrm{GHM}^{s,r}(\mathcal{U})$.*
2. *Neither player has a winning strategy in the game $\mathrm{GHM}^{r,r}(\mathcal{U})$.*
3. *If ONE has a winning strategy in $\mathrm{GHM}^{r,s}(\mathcal{U})$, then $\mathcal{U}$ is not a Q-point ultrafilter, and $r > s$.*
4. *There exists a free ultrafilter $\mathcal{U}$ such that for each r, ONE has a winning strategy in the game $\mathrm{GHM}^{2r,r}(\mathcal{U})$.*

It is not known if the information in 4 of this theorem is optimal. In particular, the following is an old unsolved problem of Galvin:

Problem 18 (Galvin). *Is there a free ultrafilter $\mathcal{U}$ on the positive integers such that ONE has a winning strategy in the game $\mathrm{GHM}^{3,2}(\mathcal{U})$?*

6. PURSUIT - EVASION GAMES

We now consider an example of a pursuit and evasion. There are several of these in the literature. Strictly speaking, the problem that we state in connection with the example we discuss here belongs under the topic of multiboard games.

Telgársky's pursuit–evasion game

A topological space (X,τ) is given. Every one–element subset of the space is assumed to be a closed set. A collection $\mathbf{K}$ of closed subsets of X is also given. Then Telgársky's game, denoted $\mathrm{TG}(\mathbf{K},(X,\tau))$, is played as follows: In the n–th inning ONE first chooses a set O_n from $\mathbf{K}$; TWO responds with a closed subset T_n of X. The players must further obey the rules that for each n we have

$$O_{n+1} \cup T_{n+1} \subseteq T_n \text{ and } O_n \cap T_n = \emptyset$$

ONE wins a play $O_1, T_1, \ldots, O_n, T_n, \ldots$ if $\bigcap_{n=1}^{\infty} T_n$ is empty; otherwise, TWO wins.

Intuitively one may regard this game as that ONE is a pursuer (predator) and TWO attempts to evade ONE (TWO is the prey). In every inning ONE invades some of TWO's hiding grounds, and TWO then withdraws to a smaller region. ONE wins if eventually TWO has no place to hide; TWO wins otherwise. This game has been a quite successful tool in analysing a wide variety of topological concepts. One of the reasons for the versatility of this game is the freedom one has in choosing the class of spaces represented by the parameters $\mathbf{K}$. In particular, it has been used to prove quite interesting theorems in dimension theory. We shall not attempt here to outdo the excellent survey of this game given by Yajima in [182].

Instead, we illustrate some of the uses of this game for a particular choice of the parameter $\mathbf{K}$. Consider the following old problem of Tamano:

Problem 19 (Tamano). *Give a characterization of the class of paracompact spaces with the property that for any one of these spaces, its product with every paracompact space is paracompact.*

A partial answer was given by Telgársky: Let $\mathbf{K}$ denote the class of Hausdorff topological spaces which have discrete closed covers by compact sets.

Theorem 37 (Telgársky). *If (X,τ) is a paracompact space and if ONE has a winning strategy in the game* $\mathrm{TG}(\mathbf{K},(X,\tau))$, *then the product of* (X,τ) *with any Hausdorff paracompact space is again a paracompact space.*

Yajima used the game to also prove analogous theorems for two other classes of spaces: the subparacompact spaces, and the metacompact spaces:

Theorem 38 (Yajima). *Let (X,τ) be a regular Hausdorff space such that ONE has a winning strategy in the game* $\mathrm{TG}(\mathbf{K},(X,\tau))$.

1. *If (X,τ) is subparacompact, then its product with any subparacompact space is subparacompact.*
2. *If (X,τ) is metacompact and a P-space, then its product with any metacompact space is metacompact.*

These sorts of results inspire the following natural question, which is important for the development of a multiboard theory of this game (see below):

Problem 20 (Telgársky). *Let (X,τ) and (Y,σ) be completely regular spaces. Is it true that if ONE has a winning strategy in each of the games* $\mathrm{TG}(\mathbf{K}(X,\tau))$ *and* $\mathrm{TG}(\mathbf{K},(Y,\sigma))$, *then ONE has a winning strategy in the game* $\mathrm{TG}(\mathbf{K},(X\times Y,\tau\times\sigma))$?

7. SIMULTANEOUS GAMES

What we have described so far can be viewed as follows: We have a game of some sort which is played between players ONE and TWO, and they play an inning per positive integer. This game is played on a "board" of some sort; the board could for example be a topological space, and moves by the players may be some topological objects. There are three possibilities: Either ONE has a winning strategy, or else TWO has a winning strategy, or else neither player has a winning strategy.

Suppose now that we place two boards, each representing the same original game, before the two players, and we require that the game be played as follows: In the first inning ONE first makes a move on each of the two boards, after which TWO responds by making a move on each of the boards, and so on. In the n-th inning ONE first makes a legal move on each of the boards, and then TWO responds by making a move on each of the two boards. Finally, ONE will be declared the winner if ONE won on at least one of the boards; TWO is declared the winner if ONE didn't win on any of the boards.

It is very important to note that the rules require that the players do not play asynchronously – ONE is not allowed to make a move on one of the boards, wait for TWO's response there, and then make a move on the second board – similar restrictions apply to TWO. This may appear at first glance to be an innocent restriction which would not affect the outcome if the game were played differently. It is a serious restriction!

It is reasonably clear that if ONE had a winning strategy in the one-board version of the game, then ONE would also have a winning strategy in the multiple board version – simply play the winning strategy on each board. The same remark applies to TWO. Thus, this simultaneous version is of interest when neither player has a winning strategy in the one – board version.

Games of this sort have been studied under a different guise for many years by general topologists – these games are especially relevant to such questions as whether a certain topological property is preserved by products of spaces having the property.

The Gale – Stewart game

Let S be a nonempty set, and let X be a set of functions from the set of positive integers to S, while ${}^{\mathbb{N}}S$ is the set of all functions from the set of positive integers to S. In the early 1950's Gale and Stewart [46] initiated the serious study of the following game, denoted $\mathrm{GS}_S(X)$: In the n-th inning ONE first chooses an element O_n of S to which TWO replies with an element T_n of S. ONE is declared the winner of the play

$$O_1, T_1, \ldots, O_n, T_n, \ldots$$

if the function f whose value for each positive integer n is such that $f(2n-1) = O_n$ and $f(2n) = T_n$, is an element of X; otherwise, TWO wins.

We shall consider the set S as a topological space which has the discrete topology, and ${}^{\mathbb{N}}S$ is then taken as the Tychonoff product of countably many copies of this space. One can for example show that there is a subset of ${}^{\mathbb{N}}2$ such that neither player has a winning strategy in the game $\mathrm{GS}_2(X)$. Gale and Stewart showed that if X happens to be an open subset or a closed subset of this topological space, then one of the players has a winning strategy in the game $\mathrm{GS}_S(X)$. After several generalizations of the Gale–Stewart theorem, D. A. Martin finally in the 1970's proved [100, 101]:

Theorem 39 (D. A. Martin). *If X is a Borel subset of ${}^{\mathbb{N}}S$, then one of the players has a winning strategy in the game $\mathrm{GS}_S(X)$.*

It was soon realized that the assertion that: *for every analytic subset X of the space ${}^{\mathbb{N}}\mathbb{N}$, some player of the game $\mathrm{GS}_{\mathbb{N}}(X)$ has a winning strategy* transcended ordinary set theory in deductive strength. One of the spectacular developments in set theory during the 1980's was that the exact requirements for having the truth of the assertion that *for every projective subset X of ${}^{\mathbb{N}}\mathbb{N}$ some player has a winning strategy in the game $\mathrm{GS}_{\mathbb{N}}(X)$* have been determined in terms of the more basic set theoretic notions of cardinals numbers. The three main architects in the finalization of this pursuit were D. A. Martin, J. Steele and W. H. Woodin. The interest in such a result is that the theory of the projective subsets of the real line (and other spaces of interest to analysts) is particularly elegant and satisfying when one has available as a working tool the assertion that for every projective subset of ${}^{\mathbb{N}}\mathbb{N}$ some player has a winning strategy in the corresponding Gale–Stewart game.

In a different direction, F. Galvin in the early 1970's started studying the multiboard- (= simultaneous-) versions of this game. For a nonzero cardinal number κ and for a subset X of ${}^{\mathbb{N}}S$, we let $\mathrm{GS}_S(X,\kappa)$ denote the following game: In the n-th inning ONE selects a function $O_n : \kappa \to S$; TWO responds by selecting a function $T_n : \kappa \to S$. Player ONE wins the game $\mathrm{GS}_S(X,\kappa)$ if there is some $\alpha < \kappa$ such that $(O_1(\alpha), T_1(\alpha), \ldots, O_n(\alpha), T_n(\alpha), \ldots)$ is an element of X; otherwise, TWO wins. Thus, $\mathrm{GS}_S(X,1)$ is the original Gale–Stewart game.

Observe that for a particular X there are the following possibilities:

* Either some player has a winning strategy in $\mathrm{GS}_S(X)$, in which case that same player will for each $\kappa > 0$ have a winning strategy in the game $\mathrm{GS}_S(X,\kappa)$.
* Or else neither player has a winning strategy in the game $\mathrm{GS}_S(X)$, in which case there is no κ such that TWO has a winning strategy in $\mathrm{GS}_S(X,\kappa)$. For this case there are two possibilities:
  - there is some κ_0 such that for all $\kappa < \kappa_0$ neither player has a winning strategy in the game $\mathrm{GS}_S(X,\kappa)$, but for all $\kappa \geq \kappa_0$

ONE has a winning strategy in the game $\mathrm{GS}_S(X,\kappa)$, or
- or else for each κ neither player has a winning strategy in the game $\mathrm{GS}_S(X,\kappa)$.

Galvin investigated these possibilities for the case when S happens to be 2 or $\mathbb{N}$. Improving on early results of McKenzie, he found examples of the following facts:

Theorem 40 (Galvin). *Let $0 < \lambda \leq \omega$ be a cardinal number.*

1. *There is a set $X \subset {}^{\mathbb{N}}\mathbb{N}$ such that for each nonzero cardinal $\kappa < 2^{\aleph_0}$ neither player has a winning strategy in the game $\mathrm{GS}_{\mathbb{N}}(X,\kappa)$.*
2. *There is a set $X_\lambda \subseteq {}^{\mathbb{N}}2$ such that for every nonzero $n < \lambda$, neither player has a winning strategy in the game $\mathrm{GS}_2(X_\lambda,n)$ but ONE has a winning strategy in the game $\mathrm{GS}_2(X_\lambda,\lambda)$.*
3. *There is a set $X \subseteq {}^{\mathbb{N}}2$ such that neither player has a winning strategy in the game $\mathrm{GS}_2(X,\kappa)$ wherever κ is a cardinal number such that either $2^\kappa \leq 2^{\aleph_0}$, or else the real line is not the union of κ nowhere dense sets of Lebesgue measure zero.*

The following questions of Galvin's regarding these matters are still unsolved:

Problem 21 (Galvin). *Is there a set $X \subset {}^{\mathbb{N}}\mathbb{N}$ such that neither player has a winning strategy in $\mathrm{GS}_{\mathbb{N}}(X,2^{\aleph_0})$?*

The reason for this question is that this is the smallest value which has not been taken care of by the examples in item 1 of Theorem 40.

Problem 22 (Galvin). *Is it true that for every subset X of ${}^{\mathbb{N}}2$, some player has a winning strategy in the game $\mathrm{GS}_2(X,(2^{\aleph_0})^+)$?*

The reason for this question is that $(2^{\aleph_0})^+$ is a cardinal number not satisfying the requirements of item 3 of Theorem 40.

Problem 23 (Galvin). *Is there a cardinal number κ such that for each subset X of ${}^{\mathbb{N}}2$, some player has a winning strategy in the game $\mathrm{GS}_2(X,\kappa)$?*

It is further not known if from this point of view there really is a distinction between ${}^{\mathbb{N}}2$ and ${}^{\mathbb{N}}\mathbb{N}$:

Problem 24 (Galvin). *Is it true that if κ is cardinal number such that for every subset X of ${}^{\mathbb{N}}2$ some player has a winning strategy in the game $\mathrm{GS}_2(X,\kappa)$, then also for every subset Y of ${}^{\mathbb{N}}\mathbb{N}$ some player has a winning strategy in the game $\mathrm{GS}_{\mathbb{N}}(Y,\kappa)$?*

Galvin conjectured the following about these multiboard versions of the Gale – Stewart games:

Conjecture 10 (Galvin's Conjecture 1). *For every nonempty set S there is a cardinal number λ such that for every subset X of ${}^{\mathbb{N}}S$, some player has a winning strategy in the game $\mathrm{GS}_S(X,\lambda)$.*

Before we discuss what is known regarding this conjecture, we first mention two special instances of it which are completely open.

Problem 25 (Galvin). *Is it true (or consistent) that for every subset X of ${}^{\mathbb{N}}2$, some player has a winning strategy in the game $\mathrm{GS}_2(X,2^{\aleph_0})$?*

Problem 26 (Galvin). *Is it true (or consistent) that for every subset X of ${}^{\mathbb{N}}\mathbb{N}$, some player has a winning strategy in the game $\mathrm{GS}_{\mathbb{N}}(X,2^{\aleph_0})$?*

We now return to the Galvin – Ulam game. It is related as follows to the multiboard version of the Gale – Stewart game:

Theorem 41 (Galvin). *Let S be a nonempty set and let κ be an infinite cardinal number. If player ONE has a winning strategy in the game $\mathrm{GU}(\kappa,|S|)$, then for every subset X of ${}^{\mathbb{N}}S$, some player has a winning strategy in the game $\mathrm{GS}_S(X,\kappa)$.*

Moreover, the Galvin – Ulam game has the following *monotonicity* property:

Lemma 42 (Galvin). *Let $\kappa \leq \kappa'$, and $\lambda \geq \lambda'$ be cardinal numbers. If ONE has a winning strategy in the game $\mathrm{GU}(\kappa,\lambda)$, then ONE has a winning strategy in the game $\mathrm{GU}(\kappa',\lambda')$.*

Thus, putting Laver's consistency result Theorem 20, Galvin's result relating the multiboard Gale – Stewart game and the Galvin – Ulam game Theorem 41, and Lemma 42 together, one obtains

Theorem 43. *If it is consistent that there is a proper class of supercompact cardinals, then it is consistent that Galvin's Conjecture 1 holds.*

If instead of Theorem 20 we use Theorem 19, then we get from the consistency of the existence of a measured cardinal the consistency of a positive answer to Problem 22.

At this stage there is no evidence to preclude the possibility that Galvin's Conjecture 1 is simply a theorem of ZFC. This appears to be what is known regarding the multiboard theory for Gale – Stewart games.

Simultaneously played Banach – Mazur games

A topological space is said to be *Baire space* if for every countable collection of dense open subsets of this space, the intersection of these sets is a dense subset of the space. If (X,τ) is a topological space such that TWO has a winning strategy in the game $\mathrm{BM}(X,\tau)$, then this space is a Baire space in a very strong sense. It is known that there are topological spaces (Y,σ) such that (Y,σ) is a Baire space, but $Y \times Y$ with the product topology is not a Baire space [42]. We discuss this phenomenon from the point of view of game theory.

Playing on two boards simultaneously

It follows from classical work of Oxtoby [124] and Cohen [21] that there exist topological spaces (X,τ) and (Y,σ) such that neither player has a winning strategy in the games $\mathrm{BM}(X,\tau)$ and $\mathrm{BM}(Y,\sigma)$ and yet ONE has a winning strategy in the game $\mathrm{BM}(X \times Y, \tau \times \sigma)$. Using a method of Krom [91] one can show that these spaces can be taken to be metrizable. Due to a result of Oxtoby [124] such that a metric space cannot have a countable dense subset (equivalently, it cannot have countable weight – the *weight* of a topological space is the minimal cardinality for a basis of the topology of the space).

Along these lines, J. Van Mill and R. Pol showed [110]:

Theorem 44 (Van Mill and Pol). *If (X,d) is any non-separable completely metrizable topological vector space, then it has two vector subspaces S_1 and S_2 such that neither player has a winning strategy in either of the games $\mathrm{BM}(S_1,d)$ and $\mathrm{BM}(S_2,d)$, but ONE has a winning strategy in the game $\mathrm{BM}(S_1 \times S_2, d \times d)$.*

When (X,d) has weight $\aleph_1$ or $\aleph_2$, one can moreover insure that X is the direct sum of the vector spaces S_1 and S_2. Using the Generalized Continuum Hypothesis one can show that for every non-separable Banach space there is a bounded linear transformation from it onto a Banach space of weight N_1 or N_2. It then follows that every non-separable Banach space is the direct sum of two vector subspaces, each a space such that neither player has a winning strategy in the Banach – Mazur game on that space, but ONE has a winning strategy in the simultaneous game. But this conclusion relies on the hypothesis that the Generalized Continuum Hypothesis holds. Thus:

Problem 27 (Van Mill and Pol). *Let $(B, \|\cdot\|)$ be a non-separable Banach space. Are there then vector subspaces S_1 and S_2 of B such* *that B is the direct sum of S_1 and S_2, neither player has a winning strategy in either of the games* $\mathrm{BM}(S_1, \|\cdot\|)$ *or* $\mathrm{BM}(S_2, \|\cdot\|)$, *and yet ONE has a winning strategy in the game* $\mathrm{BM}(S_1 \times S_2, \|\cdot\| \times \|\cdot\|)$?

Van Mill and Pol also show:

Theorem 45 (Van Mill and Pol). *If $(G, d, +)$ is a metrizable Abelian topological group which as topological space is non-separable and topologically complete, and if G has a dense path component, then G has two subgroups G_1 and G_2 such that neither player has a winning strategy in either of the games* $\mathrm{BM}(G_1, d)$ *or* $\mathrm{BM}(G_2, d)$, *but ONE has a winning strategy in game* $\mathrm{BM}(G_1 \times G_2, d \times d)$.

They moreover show that if $(G, d, +)$ has weight $\aleph_1$, then G_1 and G_2 can be taken so that $G_1 \cap G_2$ is separable. In light of their results on completely metrizable topological vector spaces, these results are not nearly as sharp as one might expect. Their results for example do not show that one may take G to be the direct sum of G_1 and G_2. Accordingly, one could ask:

Problem 28 (Van Mill and Pol). *If $(G, d, +)$ is a pathconnected completely metrizable Abelian topological group of uncountable weight, are there then subgroups G_1 and G_2 such that G is the direct sum of the groups G_1 and G_2, neither player has a winning strategy in either of the games* $\mathrm{BM}(G_1, d)$ *or* $\mathrm{BM}(G_2, d)$, *but ONE has a winning strategy in the game* $\mathrm{BM}(G_1 \times G_2, d \times d)$?

Even the weaker version of the problem where we only demand that the weight of G is $\aleph_1$, and that each element of G is expressible as a sum of an element of G_1 and an element of G_2 is unsolved.

In [42] Fleissner and Kunen pose the following question which seems to be still unsolved:

Problem 29 (Fleissner and Kunen). *Let (X, d) be a metric space such that neither player has a winning stratey in the game* $\mathrm{BM}(X, \tau)$ *and such that there is a space (Y, σ), for which neither player has a winning strategy in the game* $\mathrm{BM}(Y, \sigma)$, *but ONE has a winning strategy in the game* $\mathrm{BM}(X \times Y, \tau \times \sigma)$. *Must there then be a metric space (Y, ρ) which has the same weight as (X, d), such that neither player has a winning strategy in the game* $\mathrm{BM}(Y, \rho)$, *but ONE has a winning strategy in the simultaneous game* $\mathrm{BM}(X \times Y, d \times \rho)$?

Now let us look at what is known when we demand that the two spaces involved in the simultaneous game are homeomorphic. As we observed earlier, there is a space (X, τ) such that neither player has a winning strategy in the game $\mathrm{BM}(X, \tau)$ and yet ONE has a winning strategy in the game $\mathrm{BM}(X \times X, \tau \times \tau)$; by Krom's results we may assume that the space is a metric space. Could one have examples of this kind where the metric space has additional mathematical structure? In particular:

Problem 30 (Van Mill and Pol). *Is there a normed vector space* $(X, \|\cdot\|)$ *such that neither player has a winning strategy in the game* $\mathrm{BM}(X, \|\cdot\|)$, *but ONE has a winning strategy in the game* $\mathrm{BM}(X \times X, \|\cdot\| \times \|\cdot\|)$?

There have been claims towards a positive solution to this problem, but the offered proofs had some shortcomings. Along the lines of this problem Valdivia [170] also asked:

Problem 31 (Valdivia). *Does every non-separable completely metrizable topological vector* (V, d) *space contain a dense vector subspace* S *such that neither player has a winning strategy in the game* $\mathrm{BM}(S, d)$, *but ONE has a winning strategy in* $\mathrm{BM}(S \times S, d \times d)$?

Playing on several boards simultaneously

Fleissner and Kunen also constructed in [41] for each cardinal number $\kappa \geq 2$ a family $((X_\alpha, \tau_\alpha) : \alpha < \kappa)$ of Baire spaces such that for each $\gamma < \kappa$ the Tychonoff product of the family of spaces $((X_\alpha, \tau_\alpha) : \alpha < \kappa$ and $\alpha \neq \gamma)$ is a Baire space, but the product of all the spaces is not a Baire space. In particular, their examples show that:

1. For every integer $n > 1$ there is a single topological space (X_n, τ_n) such that neither player has a winning strategy when the Banach–Mazur game is played simultaneously on fewer than n copies of this space, but ONE has a winning strategy in the Banach–Mazur game played simultaneously on n copies of the space.
2. There is a topological space (X, τ) such that neither player has a winning strategy if the game is played simultaneously on finitely many copies of the space, but ONE has a winning strategy in the Banach–Mazur game played simultaneously on countably many copies of the space – i.e., in $\mathrm{BM}(\prod_{n=1}^{\infty} X, \prod_{n=1}^{\infty} \tau)$.

In connection with the second fact they raise the following open problem:

Problem 32 (Fleissner and Kunen). *Let (X, d) be a metric space such that neither player has a winning strategy in the game* $\mathrm{BM}(X, d)$. *Assume that ONE has a winning strategy in the game* $\mathrm{BM}(\prod_{n=1}^{\infty} X, \prod_{n=1}^{\infty} d)$. *Does there then exists a metric space (Y, ρ) such that neither player has a winning strategy in the game* $\mathrm{BM}(Y, \rho)$ *but ONE has a winning strategy in the game* $\mathrm{BM}(X \times Y, d \times \rho)$?

An important aspect of the examples given in [42] is that the infinitary products of the spaces considered are Tychonoff– (also known as Cartesian–) products; the product theory for Baire spaces seems dramatically different for *box products*. To illustrate, let $((X_i, \tau_i) : i \in I)$ be a family of topological spaces. Then the *box topology*, denoted $\Box_{i\in I}\tau_i$ is defined to be the topology on $\prod_{i\in I} X_i$ generated by sets of the form $\prod_{i\in I} U_i$, where for each i U_i is an open subset of X_i. If I is a finite set, then the box topology coincides with the ordinary product topology, but if I is infinite then the box topology property contains the ordinary Tychonoff product topology. The connection between the two topologies as far as the existence of winning strategies in the Banach – Mazur game is as follows:

Theorem 46. *Let $((X_i, \tau_i) : i \in I)$ be a family of topological spaces. If for every countable subset J of I, ONE does not have a winning strategy in the game* $\mathrm{BM}(\prod_{j\in J} X_j, \Box_{j\in J}\tau_j)$, *then ONE does not have a winning strategy in* $\mathrm{BM}(\prod_{i\in I} X_i, \prod_{i\in I} \tau_i)$.

In light of Fleissner and Kunen's examples, one can find an uncountable family of topological spaces such that in the Tychonoff product topology ONE has no winning strategy in the Banach – Mazur game, while there is a countable subcollection of these spaces such that ONE has a winning strategy in the simultaneous version of the game on these countably many spaces.

Further interesting examples regarding these matters are given in [170]; some of Valdivia's results imply:

Theorem 47 (Valdivia). *There is a countable family $(S_n : n = 1, 2, 3, \ldots)$ of dense vector subspaces of $c_0(\omega_1)$ such that:*

1. *For each n, ONE has no winning strategy in the game* $\mathrm{BM}(\prod_{j=1}^{\infty} S_n, \Box_{j=1}^{\infty} d)$,
2. *For each n, ONE has no winning strategy in* $\mathrm{BM}(\prod_{m\neq n} S_m, \Box_{j=1}^{\infty} d)$, *but*
3. *ONE has a winning strategy in* $\mathrm{BM}(\prod_{n=1}^{\infty} S_m, \Box_{n=1}^{\infty} d)$.

The reader should consult [170] on further examples illustrating a variety of interesting phenomena regarding the multiboard theory of Banach–Mazur games.

If TWO has a winning strategy in $\mathrm{BM}(X, \tau)$, then the product of (X, τ) with any space in which ONE has no winning strategy is still a space in which ONE has no winning strategy. This unfortunately does not characterize those spaces for which TWO has a winning strategy; there are spaces (X, τ) for which neither player has a winning strategy in the game $\mathrm{BM}(X, \tau)$ and for every space (Y, σ) such that neither player has a winning strategy in the game $\mathrm{BM}(Y, \sigma)$ also neither player has a winning strategy in the game $\mathrm{BM}(X \times Y, \tau \times \sigma)$.

There is another property which might give such a characterization. Note that if TWO has a winning strategy in the game $\mathrm{BM}(X, \tau)$, all powers of (X, τ) endowed even with the *box-product* topology are still spaces in which TWO has a winning strategy, and thus in which ONE does not have a winning strategy, in the Banach–Mazur game. F. Galvin conjectured a long time ago that the weaker looking conclusion that there is no box-power of the space such that ONE has a winning strategy for the Banach–Mazur game played on that power, is in fact equivalent to the assertion that for all box powers, TWO has a winning strategy. More precisely.

Conjecture 11 (Galvin's Conjecture 2). *For every space (X, τ), if every power of it, endowed with the box-product topology, is a space for which ONE has no winning strategy in the Banach–Mazur game, then TWO has a winning strategy in the game* $\mathrm{BM}(X, \tau)$.

Galvin observed that Galvin's Conjecture 1 implies Galvin's Conjecture 2. Consequently, using the consistency results mentioned earlier in connection with the Gale–Stewart game, one sees that (modulo the consistency of the existence of large cardinal numbers) Galvin's Conjecture 2 is consistent. But there is at present no evidence to preclude the possibility that this conjecture is simply a theorem which needs no extraneous hypotheses.

A game of Christensen

Let Y be a compact Hausdorff space and let Z be a metric space. A Hausdorff space X is said to be a *Namioka space* if there is for each function $f : X \times Y \to Z$ which has the properties that

* for each $x \in X$ the function $f_x : Y \to Z$ defined so that for each $y \in Y$ $f_x(y) = f(x,y)$ is continuous and
* for each $y \in Y$ the function $f^y : X \to Z$ defined so that for each $x \in X$ $f^y(x) = f(x,y)$ is continuous,

there is a dense subset A of X such that A is an intersection of countably many open subsets of X, and f restricted to $A \times Y$ is continuous.

In [120] I. Namioka proved the important theorem that locally compact Hausdorff spaces and complete metric spaces are Namioka spaces. In an effort to present a technically easier proof of Namioka's results and to determine which spaces are Namioka spaces, J. P. R. Christensen introduced two infinite games in [19]. We are interested in one of these here:

Let (X,τ) be a Hausdorff space. Then the game JPR is played as follows: In the n-th inning ONE first selects a nonempty open subset O_n of X; TWO responds by selecting a pair (T_n, x_n) where T_n is nonempty open subset of O_n and x_n is an element of T_n. ONE must further obey the rule that for each n, O_{n+1} is a subset of T_n TWO wins a play,

$$O_1, (T_1, x_1), O_2, (T_2, x_2), \ldots, O_n, (T_n, x_n), \ldots$$

if for every subsequence of $(x_n : n = 1, 2, 3, \ldots)$, there is a point in $\cap_{n=1}^{\infty} T_n$ which is an accumulation point for the sequence; otherwise, ONE wins.

Christensen then proves that if TWO has a winning 1-tactic in the game JPR(X,τ) then (X,τ) is a Namioka space. Saint Raymond later observed in [135] that indeed, if ONE does not have a winning strategy in the game JPR(X,τ) then (X,τ) is a Namioka space. One of the important problems raised here ([19], p. 456) is:

Problem 33 (Christensen). *Is it true that if for Hausdorff spaces* (X,τ) *and* (Y,σ) *TWO has a winning 1-tactic in each the games* JPR(X,τ) *and JPR*(Y,σ), *then TWO has a winning 1-tactic in the game* JPR$(X \times Y, \tau \times \sigma)$?

Saint Raymond reported some partial progress on this problem in Theorem 8 of his paper [135]. This problem is an important first step towards developing a multiboard theory for this game.

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Department of Mathematics,
Boise State University,
Boise, Idaho 83725