VIETNAM JOURNAL OF MATHEMATICS
Volume 23, Number 2, 1995

A Short Communication

ON BEST MULTIVARIATE TRIGONOMETRIC APPROXIMATIONS[1]

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1. A characterization of the smoothness properties of multivariate periodic functions which govern a preassigned speed of the best approximation by trigonometric polynomials with frequencies from so called hyperbolic crosses was given in [3], [4]. The smoothness of functions is characterized in terms of some "moduli of smoothness" which are defined by the help of their convolutions with certain distributions on the d-dimensional torus $\boldsymbol{T}^d$. In an earlier manuscript [1], its authors gave such a characterization for the best approximation by trigonometric polynomials with frequencies from the regular hyperbolic cross by introducing new moduli of smoothness of functions which are defined by means of the convolutions of their higher-order mixed differences with the symmetric multivariate B-splines.

The present note continues the investigation in [3], [4]. We will give a different characterization of the smoothness properties of functions for the best hyperbolic cross approximation considered in [3], [4]. This characterization will be defined in terms of higher-order mixed differences of their higher-order mixed integrals.

2. For a finite subset A of $\boldsymbol{R}^d_+ := \{x \in \boldsymbol{R}^d : x_j \geq 0\}$ and $t > 0$, the set

$$\Gamma_t(A) := \{k \in \boldsymbol{Z}^d : \prod_{j \in J(\alpha)} |k_j|^{\alpha_j} < t, \ \alpha \in A\}$$

is called a hyperbolic cross, where $J(\alpha) := \{j : 1 \leq j \leq d, \ \alpha_j > 0\}$. We are interested in the best $L_p(\boldsymbol{T}^d)$-approximation, $1 < p < \infty$, of functions by elements from $\mathcal{P}_t^A$ which is defined as $L_p(\boldsymbol{T}^d)$-closure of

[1]Supported by Contract 1.5.5 of the National Program for Fundamental Researches in Natural Sciences.

the span of the harmonics $e^{i\langle k,\cdot\rangle}$, $k \in \Gamma_t(A)$. The most interesting and important case is that when $\mathcal{P}_t^A$ consists of real trigonometric polynomials, i.e. $\Gamma_t(A)$ is a finite set. The hyperbolic cross $\Gamma_t(A)$ is a finite set for each $t > 0$ if and only if

$$a\varepsilon^j \in A\,,\ j = 1, \ldots, d\,;\ \text{for some}\ a > 0\,,$$

where $\varepsilon^1 = (1, 0, \ldots, 0)$, $\varepsilon^2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\varepsilon^d = (0, \ldots, 0, 1)$ are the basis vectors in $\boldsymbol{R}^d$. However, we would like to emphasize that the results of the present note will be asserted for arbitrary finite subset A of $\boldsymbol{R}_+^d$, without requirement on finiteness of $\Gamma_t(A)$.

For any $t > 0$ we let

$$E_t^A(f)_p := \inf_{g \in \mathcal{P}_t^A} \|f - g\|_{L_p(\boldsymbol{T}^d)}$$

denote the best $L_p(\boldsymbol{T}^d)$-approximation of $f \in L_p(\boldsymbol{T}^d)$ by elements from $\mathcal{P}_t^A$. We are interested in characterization of the smoothness properties of f which guarantee a preassigned degree of $E_t^A(f)_p$. We let Φ denote the set of all continuous functions φ on $[0,1]$ such that $\varphi(x) > 0$ for $x > 0$, and $\varphi(0) = 0$ and φ is nondecreasing on $[0, \tau]$ with some $\tau \in [0, 1]$. The degrees of $E_t^A(f)_p$, which are treated in our note, are functions $\varphi(1/t)$ for $\varphi \in \Phi$. Thus, for example, the function t^{-1} is the degree of $E_t^A(f)_p$ on the class of functions f with $L_p(\boldsymbol{T}^d)$-bounded mixed derivatives in the Weil sense of order α for all $\alpha \in A$, while the function $t^{-1} \log^\nu t$ with some nonnegative integer $\nu < d$ is the degree of $E_t^B(f)_p$ on the class of all functions f such that

$$\|\Delta_h^r f\|_{L_p(T^d)} \leq \prod_{j \in J(\alpha)} |h_j|^{\alpha_j}\ (r \in \boldsymbol{Z}_+^d\,,\ r > \alpha)$$

for all $\alpha \in A$, where Δ_h^r is the mixed difference operator (see a definition below), B is a certain finite subset of $\boldsymbol{Z}_+^d$, constructed from A (cf. e.g, [2], [5]).

If $\varphi \in \Phi$ and $0 < q \leq \infty$, we let $\mathcal{E}_{p,q}^{A,\varphi}$ denote the space of all functions $f \in L_p(\boldsymbol{T}^d)$ such that the quasinorm

$$|f|_{\mathcal{E}_{p,q}^{A,\varphi}} := \begin{cases} \left(\sum_{n=0}^{\infty} \{E_{2^n}^A(f)_p / \varphi(2^{-n})\}^q \right)^{1/q}, & q < \infty \\ \sup_{0 \leq n < \infty} \{E_{2^n}^A(f)_p / \varphi(2^{-n})\}\,, & q = \infty \end{cases}$$

is finite.

3. For $h \in \boldsymbol{R}$, the univariate difference operators Δ_h^r, $r = 0, 1$, are defined by

$$\Delta_h^0 f := f \,; \quad \Delta_h^1 f := f(\cdot + h/2) - f(\cdot - h/2)\,.$$

For positive integer r, the r–th difference operator Δ_h^r is defined inductively by $\Delta_h^r := \Delta_h^1 \circ \Delta_h^{r-1}$. If $r \in \boldsymbol{Z}_+^d := \{k \in \boldsymbol{Z}^d : k_j \geq 0\}$, then for $h \in \boldsymbol{R}^d$, the mixed difference operator Δ_h^r is defined by

$$\Delta_h^r f := \prod_{j=1}^{d} \Delta_{h_j}^{r_j} f$$

for functions f on $\boldsymbol{T}^d$, where the operator $\Delta_{h_j}^{r_j}$ is applied to the variable x_j. If $r \in \boldsymbol{Z}_+^d$ and $h \in \boldsymbol{R}^d$ then the mixed integral operator I_h^r can be defined for integrable functions f on $\boldsymbol{T}^d$ in the same way, starting with the univariate operators

$$I_h^0 f := f \,; \quad (I_h^1 f)(x) := \frac{1}{h}\int_0^x f(t)dt\,.$$

For a triple $\gamma = (\alpha, r, \beta) \in \boldsymbol{R}_+^d \times \boldsymbol{Z}_+^d \times \boldsymbol{Z}_+^d$ and $t \geq 0$, we define the operator

$$D_t^\gamma f := \int\limits_{V(\alpha,t)} \Delta_h^r I_h^\beta f \prod_{j \in J(\alpha)} \frac{dh_j}{h_j}$$

for integrable functions f on $\boldsymbol{T}^d$, where

$$V(\alpha,t) := \Big\{(h_j)_{j \in J(\alpha)} : h_j > 0,\ t \leq \prod_{j \in J(\alpha)} h_j^{\alpha_j} \leq 2^{\xi_0} t\Big\}$$

with $\xi_0 = 1 + \sum\limits_{j \in J(\alpha)} \alpha_j$. For a finite subset G of $\boldsymbol{R}_+^d \times \boldsymbol{Z}_+^d \times \boldsymbol{Z}_+^d$, we define the modulus of smoothness

$$\Omega^G(f,t)_p := \sum_{\gamma \in G} \|D_t^\gamma f\|_p$$

for functions $f \in L_p(\boldsymbol{T}^d)$. If $\varphi \in \Phi$ and $0 < q \leq \infty$, let $B_{p,q}^{G,\varphi}$ denote the space of all functions $f \in L_p(\boldsymbol{T}^d)$ such that the quasinorm

$$|f|_{B_{p,q}^{G,\varphi}} := \begin{cases} \left(\sum\limits_{n=0}^{\infty} \{\Omega^G(f,2^{-n})_p \,/\, \varphi(2^{-n})\}^q\right)^{1/q}, & q < \infty \\ \sup\limits_{0 \leq n < \infty} \{\Omega^G(f,2^{-n})_p \,/\, \varphi(2^{-n})\}\,, & q = \infty \end{cases}$$

is finite.

4. We say that the function $\varphi \in \Phi$ satisfies Condition (BS) if

$$\int_0^t \varphi(x)\frac{dx}{x} \le C\,\varphi(t)$$

and Condition (Z_θ), $\theta > 0$, if

$$\int_t^1 \varphi(x)\,x^{-\theta}\frac{dx}{x} \le C'\varphi(t)t^{-\theta}$$

and we say that the subset G of $\boldsymbol{R}^d_+ \times \boldsymbol{Z}^d_+ \times \boldsymbol{Z}^d_+$ satisfies Condition (R), if $J(\alpha) = J(r) = J(\beta) \neq \infty$ and $1 < \beta_j < r_j$, $j \in J(\alpha)$, for each $\gamma = (\alpha, r, \beta) \in G$. If G satisfies Condition (R), we define

$$\rho(G) := \min\{\rho(r-\beta,\alpha) : (\alpha, r, \beta) \in G\},$$
$$\nu(G) := \max\{\nu(r-\beta,\alpha) : (\alpha, r, \beta) \in G,\ \rho(r-\beta,\alpha) = \rho(G)\},$$

where $\rho(x,y) := \min\{x_j/y_j : j \in J(y)\}$ for $x, y \in \boldsymbol{R}^d_+$, and $\nu(x,y)$ denotes the number of $j \in J(y)$ such that $x_j/y_j = \rho(x,y)$. We let $p^* = \min(p,2)$ for $1 < p < \infty$.

Theorem 1. *Let $1 < p < \infty$, $0 < q \le \infty$ and let A be a finite subset of $\boldsymbol{R}^d_+$. Then for any $\theta > 0$ and any natural number $\nu \le \max\{\operatorname{card} J(\alpha) : \alpha \in A\}$, we can construct a finite subset G of $\boldsymbol{R}^d_+ \times 2\boldsymbol{Z}^d_+ \times \boldsymbol{Z}^d_+$, such that*

(i) $A = \{\alpha : (\alpha, r, \beta) \in G\}$

(ii) *G satisfies Condition (R)*

(iii) $\rho = \rho(G) \ge \theta$

(iv) $\nu(G) = \nu$

Moreover, if G is such a set and $f \in L_p(\boldsymbol{T}^d)$, then there hold the direct inequality

$$E^A_{2^n}(f)_p \le C\Big(\sum_{m=n+1}^{\infty} \{\Omega^G(f, 2^{-m})_p\}^{p^*}\Big)^{1/p^*} \tag{1}$$

for any nonnegative integer n, and the inverse inequality

$$\Omega^G(f, 2^{-n})_p \le C'\Big(\sum_{m=0}^{n} \{2^{-\rho(m-n)}(n-m)^{\nu-1}E^A_{2^n}(f)_p\}^{p^*}\Big)^{1/p^*} \tag{2}$$

for any natural n, with C and C' depending only on p and G.

Theorem 2. *Under the assumptions of Theorem 1, let $\varphi \in \Phi$ and φ satisfy Conditions (BS) and (Z_θ). Then for any finite subset G of $\boldsymbol{R}^d_+ \times 2\boldsymbol{Z}^d_+ \times \boldsymbol{Z}^d_+$, satisfying Conditions* (i) – (iii) *in Theorem 1, we have*

$$\mathcal{E}^{A,\varphi}_{p,q} = B^{G,\varphi}_{p,q}$$

and, moreover for functions $f \in \mathcal{E}^{A,\varphi}_{p,q}$, the following quasinorm equivalence holds

$$|f|_{\mathcal{E}^{A,\varphi}_{p,q}} \asymp |f|_{B^{G,\varphi}_{p,q}} .$$

If in Theorem 1 we take $\nu = 1$, then the inverse inequality (2) becomes

$$\Omega^G(f, 2^{-n})_p \leq C'\Big(\sum_{m=0}^{n} \{2^{-\rho(n-m)} E^A_{2^n}(f)_p\}^{p^*}\Big)^{1/p^*}. \tag{3}$$

Some inequalities weaker than (1), (3) , were obtained in [1] for the best hyperbolic cross approximations and moduli of smoothness considered by its authors. The inequalities (1), (3) with replacing $\Omega^G(f,t)_p$ by the moduli of smoothness introduced in [3], [4] were formulated and proved in these papers. Theorem 2 shows that if p, q, φ, A given, different sets G satisfying the conditions of Theorem 2, determine the same space $B^{G,\varphi}_{p,q}$.

5. The proofs of Theorems 1 – 2 rest on the Littlewood – Paley theorem, the Marcinkiewicz multiplier theorem, a generalization of the discrete Hardy inequalities and the following lemma. We let $\mathbf{1} = (1, 1, \dots, 1) \in \boldsymbol{R}^d$, and $\mathcal{D}(\alpha, \eta)$ denote the $L_p(\boldsymbol{T}^d)$ – closure of the span of the harmonics $e^{i\langle k, \cdot\rangle}$, $k \in Z(\alpha, \eta)$ for $\alpha \in \boldsymbol{R}^d_+$ and $\eta > 0$ where

$$Z(\alpha, \eta) := \{\cup \square_s : s \in \boldsymbol{Z}^d_+,\ s_j > 0,\ j \in J(\alpha),\ \eta - 1 \leq \langle \alpha, s\rangle < \eta\}$$

$$\square_s = \{k \in \boldsymbol{Z}^d : [2^{s_j - 1}] \leq |k_j| < 2^{s_j}\}.$$

Lemma. *Let $1 < p < \infty$, $\gamma = (\alpha, r, \beta) \in \boldsymbol{R}^d_+ \times 2\boldsymbol{Z}^d_+ \times \boldsymbol{Z}^d_+$, and let $\{\gamma\}$ satisfy Condition (R). Then for any ξ, $\eta > 0$ and any $f \in \mathcal{D}(\alpha, \eta)$, we have*

$$\|f\|_{L_p(T^d)} \leq C\,\|D^\gamma_{2^{-\xi}} f\|_{L_p(T^d)}\,,\quad \eta = \xi,$$

$$\|D^{\gamma}_{2-\xi} f\|_{L_p(\boldsymbol{T}^d)} \le C\,\|f\|_{L_p(\boldsymbol{T}^d)} \begin{cases} 2^{-\rho(\xi-\eta)}(\xi-\eta+1)^{\nu-1}, & \eta \le \xi \\ 2^{-\rho'(\eta-\xi)}(\eta-\xi+1)^{\nu'-1}, & \eta \ge \xi \end{cases}$$

with C and C' depending only on γ, p, where $\rho = \rho(r-\beta,\alpha)$, $\nu = \nu(r-\beta,\alpha)$ and $\rho' = \rho(\beta-\mathbf{1},\alpha)$, $\nu' = \nu(\beta-\mathbf{1},\alpha)$.

6. The hyperbolic cross $\Gamma_t(\{\alpha\})$, $\alpha \in \boldsymbol{R}^d_+$, can be considered as the simplest among $\Gamma_t(A)$ for finite sets A. However, it is not finite for any α, and therefore, is not of great interest for trigonometric polynomial approximation. One of the simplest finite hyperbolic crosses which is the most important is $\Gamma_t(A^*)$ where $A^* = \{\alpha^e\}_{e\in J}$ for some $\alpha \in \boldsymbol{R}^d$ with positive coordinates, $J = \{1, 2, \ldots, d\}$, α^e is the vector with $\alpha^e_j = \alpha_j$ for $j \in e$ and $\alpha^e_j = 0$ for $j \notin e$. The hyperbolic crosses $\Gamma_t(\{\alpha\})$ and $\Gamma_t(A^*)$ with $\alpha = \mathbf{1}$ were treated in [1]. If for a triple $\gamma = (\alpha, r, \beta) \in \boldsymbol{R}^d_+ \times 2\boldsymbol{Z}^d_+ \times \boldsymbol{Z}^d_+$, $\{\gamma\}$ satisfies Conditions (i) – (iv) with given θ and ν, for the case $A = \{\alpha\}$ of Theorems 1 – 2, then the set $G^* = \{(\alpha^e, r^e, \beta^e)\}_{e\subset J}$ will satisfy Conditions (i) – (iv) with the same θ and ν for the case $A = A^*$. Moreover, we have $\rho(G^*) = \rho(r-\beta,\alpha)$ and $\nu(G^*) = \nu(r-\beta,\alpha)$ in Theorem 1.

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Received November 10, 1995

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