Vietnam Journal of Mathematics
Volume 23, Number 1, 1995

A SUFFICIENT CONDITION FOR HAMILTONIAN CYCLES IN TOUGH GRAPHS

VU DINH HOA

Abstract. *We prove in this paper a conjecture of Bauer and Schmeichel [5] that every 1-tough graph G on $n \geq 3$ vertices with $\sigma_3(G) \geq n+\kappa-2$ is Hamiltonian, where $\sigma_3(G) = \min\{d(x)+d(y)+d(z) : \{x,y,z\}$ is an independent set of three vertices$\}$ and κ is the connectivity number of G.*

1. INTRODUCTION

In this paper we consider only finite undirected graphs without loops or multiple edges. For undefined terms we refer to [3]. Herein $V(G)$, $\kappa(G)$, $\alpha(G)$ and $\omega(G)$ denote respectively the vertex set, vertex connectivity number, independence number and the number of the components of a graph G. We let $n = |V(G)|$ throughout the paper. Following Chvátal [7] we define a graph G to be 1-tough if $\omega(G-S) \leq |S|$ for every subset S of $V(G)$ with $\omega(G-S) > 1$. If $v \in V(G)$ and $H \subseteq V(G)$ then $G[H]$ is the spanning graph of H in G and $N_H(v)$ is the set of all vertices in H adjacent to v. We denote $|N_H(v)|$ by $d_H(v)$. We let

$$\sigma_k(G) = \min\left\{\sum_1^k d(v_i) : \{v_1, v_2, \ldots, v_k\} \text{ is an independent set of vertices}\right\}$$

if $k \leq \alpha$, and $\sigma_k(G) = \infty$ if $k > \alpha$. For σ_1 we use the more common notation δ. If no ambiguity can arise we write sometime κ, α... instead of $\kappa(G)$, $\alpha(G)$, etc...

We begin with a result of Häggkvist and Nicoghossian [6].

Theorem 1. *Let G be a 2-connected graph with minimum degree $\delta \geq \dfrac{n+\kappa}{3}$. Then G is Hamiltonian.*

The following Theorem of Bauer, Broersma, Veldman and Rao [4] generalizes Theorem 1.

Theorem 2. *Let G be a 2-connected graph on $n \geq 3$ vertices with $\sigma_3 \geq n + \kappa$. The G is Hamiltonian.*

Clearly all 1-tough graphs on three or more vertices are 2-connected. Hence a natural question is whether the degree condition in Theorem 1 can be improved if G is assumed to be 1-tough. This is the content of a result of Bauer and Schmeichel [5].

Theorem 3. *Let G be a 1-tough graph with* $\delta \geq \dfrac{n+\kappa-2}{3}\cdot$ *Then G is Hamiltonian.*

In this paper we prove a conjecture of Bauer and Schmeichel [5].

Theorem 4. *Let G be a 1-tough graph with $\sigma_3 \geq n + \kappa - 2$. Then G is Hamiltonian.*

The following class of graphs, originally given in [1], shows that our result is best possible if $\kappa = 2$. For $n = 3r + 1 \geq 7$, let us construct the graph H_n from $3K_r + K_1$ by choosing one vertex from each copy of K_r, say u, v and w, and by adding the edges uv, uw and vw. The H_n is a 1-tough nonhamiltonian graph on $n = 3r + 1$ vertices. Clearly, $\kappa(H_n) = 2$ and $\sigma_3 = 3r = n + \kappa - 3$.

The proof of Theorem 4 relies on the following Theorem, conjectured by Jung and weibing.

Theorem 5 (Theorem 2.12 in [9]). *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq \max(n, 3\alpha - 5)$. Then G is Hamiltonian.*

A part of our argument is facilited by a recent result of Ainouche and Christofides [2]. For nonadjacent vertices u and v in a graph G, we denote by $\alpha_{uv}(G)$ the cardinality of a largest set of independent vertices in G containing u and v. Let $\ell_{uv}(G) = |N(u) \cap N(v)|$.

Theorem 6. *Let u and v be nonadjacent vertices of a graph G. Suppose $\alpha_{uv}(G) \leq \ell_{uv}(G)$. Then G is Hamiltonian if and only if $G + uv$ is Hamiltonian.*

2. PRELIMINARIES

Let C be a cycle or a path in G with an assigned orientation. If x and y are two vertices of C then $x \overrightarrow{C} y$ denotes the path on C from x to y, inclusively both x and y, following the assigned orientation. The same vertices in reverse order are given by $y \overleftarrow{C} x$. We will consider $x \overrightarrow{C} y$ and $y \overleftarrow{C} x$ both as a path and as a vertex set. If c is a vertex on C, then c^+ and c^- are its sucessor and predecessor on C, respectively, according to the assigned orientation. If X is a set of vertices on C then $X^+ := \{x^+ : x \in X\}$. If G is a nonhamiltonian graph, we set $\mu(C) = \max\{d(v) : v \in V(G) - V(C)\}$ and $\mu(G) := \max\{\mu(C) : C \text{ is a longest cycle in } G\}$.

The following lemmas are proved in [3].

Lemma 1 (Theorem 5 in [3]). *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. Then every longest cycle in G is a dominating cycle.*

Lemma 2 (Lemma 8 [3]). *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. Suppose C is a longest cycle in G. If $v_0 \in V(G) - V(C)$ and $A = N(v_0)$, then $(V(G) - V(C)) \cup A^+$ is an independent set of vertices.*

The next lemma is proved in [8].

Lemma 3 (Lemma 9 in [8]). *Let G be a 1-tough graph on $n \geq 3$ vertices with $\sigma_3 \geq n$. If G is nonhamiltonian then G has a longest cycle C such that C avoids a vertex $v_0 \in V(G) - V(C)$ with $d(v_0) = \mu(G)$ and $|N(v_0)^+ \cap N(v_0)^-| \geq \sigma_3 - n + 4$.*

The next lemma can easily be proved with arguments similar to those used in the proof of Case 2 of Theorem 2 in [5]. Therefore, its proof is ommited.

Lemma 4. *Let G be a 1-tough graph with $\sigma_3 \geq n + \kappa - 2$. If there exists a cut set S of κ vertices and an independent set T of α vertices such that $d(t) = \dfrac{n + \kappa - 2}{3}$ for any $t \in T$ and $T - (S \cup V(G_i)) \neq \emptyset$ for any component G_i of $G - S$. Then G is Hamiltonian.*

In what follows we assume that C is a longest cycle in a 1-tough graph nonhamiltonian graph G on $n \geq 3$ vertices with $\sigma_3 \geq n$, $v_0 \in V(G) - V(C)$ with $d(v_0) = \mu(G)$, $v_1, v_2, \ldots, v_k$ the vertives of $N(v_0)$ on consecutive order, $u_i = v_i^+$ and $w_i = v_{i+1}^-$ (indices modulo k). We denote, for convenience, $\mathcal{F} = \{i :$ there exists $j \neq i$ such that $u_i w_j \in E(G)\}$.

Lemma 5 (Lemma 4 in [8]). *If $u_i = w_i$ then $N(u_i) \subset V(C)$ and $(V(G) - V(C)) \cup N(v_0)^+ \cup N(u_i)^+$ is an independent set of vertices.*

Lemma 6 (Lemma 5 in [8]). *If $\alpha = |(V(G) - V(C)) \cup N(v_0)^+|$ then $\mathcal{F} \neq \emptyset$.*

Lemma 7 (Lemma 6 in [8]). *Suppose $u_1 = w_1$ and $N(u_1) = N(v_0)$. Moreover, suppose $\mathcal{F} \neq \emptyset$. Let $i_0 = \max \mathcal{F}$ and $j_0 \neq i_0$ such that $u_{i_0} w_{j_0} \in E(G)$. Then $d(u_{j_0}) + 2d(v_0) \leq \ell(C) + x$, where x is the number of the vertices $u_i = w_i$ such that $|N(u_i) \cap N(v_0)| \leq d(v_0) - 2$.*

Lemma 8. *Let $G = (A, B, E)$ be a bipartite graph with $A = \{a_1, a_2, \ldots, a_{n-1}\}$, $B = \{b_1, b_2, \ldots, b_n\}$ $(n \geq 3)$ and $\underline{S} = \{b_1, b_2, \ldots, b_k\}$ for some $2 \leq k < n$ such that*

1) $b_i a_j \in E(G)$ for any $i > k$ and $j \leq n - 1$;

2) $d(b_i) \geq k - 1$ for any $i \leq k$;

3) $|N(\underline{S})| \geq k$.

Then G contains a Hamiltonian path joining two different vertices of $\underline{S}$ in G.

Proof. The proof is by induction on k. The statement is trivially true for $k = 2$. Now, let $k > 2$ and G is a graph satisfying the hypotheses of the lemma. We distinguish two cases:

Case 1: There exists a subset $\underline{S}'$ of $k - 1$ elements of $\underline{S}$ such that $|N(\underline{S}')| = k - 1$.

We assume, without loss of generality, that $\underline{S}' = \{b_1, b_2, \ldots, b_{k-1}\}$ and $N(\underline{S}') = \{a_1, a_2, \ldots, a_{k-1}\}$. By 2), the spanning graph of $\underline{S}' \cup N(\underline{S}')$ is a complete bipartite graph on $2(k-1)$ vertices. By 3), there exists a vertex of $A - N(\underline{S}')$, say a_k, adjacent to b_k. Now,

$b_k a_k b_{k+1} a_{k+1} \dots a_{n-1} b_n a_1 b_1 a_2 b_2 \dots a_{k-1} b_{k-1}$ is a Hamiltonian path joining two vertices of $\underline{S}$.

Case 2: For any subset $\underline{S}' \subset \underline{S}$ *with* $|\underline{S}'| = k-1$ *we have* $|N(\underline{S}')| > k-1$.

To prove that G contains a Hamiltonian path joining two vertices of $\underline{S}$ we assume the contrary and show that this assumption leads to a contradiction. A path W joining two vertices of $\underline{S}$ such that $\underline{S} \subseteq V(W)$ is called an 0-path. We claim that G contains no 0-path. Suppose, to the contrary, that G contains an 0-path W. Without loss od generality we may assume that $\ell(W) \geq \ell(W')$ for any 0-path W'. We give a direction on W and assume, without loss of generality, that $b_{i+1} \in b_i \overrightarrow{W} b_{i+2}$ for any $i < k-1$. Since G contains no Hamiltonian path joining two vertices of $\underline{S}$, $B - V(W) \neq \emptyset$ and $A - V(W) \neq \emptyset$. Let $b^* \in B - V(W)$ and $a^* \in A - V(W)$. By the maximality of $\ell(W)$, $b_i^+ = b_{i+1}^-$. Otherwise, $b_i^{++} \notin \underline{S}$ for some $b_i \in \underline{S} \cap V(W)$ and therefore $b_1 \overrightarrow{W} b_i b_i^+ b^* a^* b_i^{++} \overrightarrow{W} b_k$ is a longer 0-path than W, a contradiction. For convenience we may assume that $a_i = b_i^+$ for any $i \leq k-1$. we claim that $N(b_1) = \{a_1, a_2, \dots, a_{k-1}\}$. Otherwise, there exists some $a_{i_0} \notin \{a_1, a_2, \dots, a_{k-1}\}$ adjacent to b_1 and therefore $b_1 a_{i_0} b^* a_1 \overrightarrow{W} b_k$ is a longer 0-path than W, a contradiction. Similarly, $N(b_k) = \{a_1, a_2, \dots, a_{k-1}\}$. By 3), there exists some $b_{j_0} \in \underline{S}$ and some $a'_{i_0} \notin V(W)$ such that $b_{j_0} a'_{i_0} \in E$ and therefore $b_{j_0} a'_{i_0} b^* b_{j_0}^- \overleftarrow{W} b_1 b_{j_0}^+ \overrightarrow{W} b_k$ is a longer 0-path than W, a contradiction. Thus, G contains no 0-path.

Let $\underline{S}' = \{b_1, b_2, \dots, b_{k-1}\}$ and choose some $a_{i_0} \in N(b_k)$. We set $G' = G - \{b_k, a_{i_0}\}$. It is easy to see that the graph G' satisfies conditions 1) - 3) for $k-1$ and therefore there is in G' a Hamiltonian path joining two vertices of $\underline{S}'$. A path W in G' joining two vertices of $\underline{S}'$ such that $\underline{S}' \subseteq V(W)$ is called a 1-path. Let W be a 1-path in G' such that $\ell(W) \leq \ell(W')$ for any 1-path W'. We give a direction on W and assume, without loss of generality, that $b_{i+1} \in b_i \overrightarrow{W} b_{i+2}$ for any $i < k-1$. By the minimality of $\ell(W)$, $N(b_1) \cap (V(W) - \underline{S}'^+) = \emptyset$. Otherwise, there exists some $a^* \in N(b_1) \cap (V(W) - \underline{S}'^+)$. We set $i_0 = \max\{i : b_i \in b_1 \overrightarrow{W} a^* \text{ and } b_i \in \underline{S}'\}$ and the path $b_{i_0} \overleftarrow{W} b_1 a^* \overrightarrow{W} b_{k-1}$ would be a shorter 1-path than W, a contradiction. Thus, $N(b_1) \cap V(W) \subseteq \underline{S}'^+$ and therefore $|N(b_1) \cap V(W)| \leq k-2$, implying that there exists in G some $a^* \notin V(W)$ adjacent to b_1. Since there is no 0-path in G, $a^* \neq a_{i_0}$. Since $|B| - |A| = 1$, there exists some $b^* \in B$ such that $b^* \notin S \cup V(W)$. Now the path $b_k a_{i_0} b^* a^* b_1 \overrightarrow{W} b_{k-1}$ would be an 0-path, a contradiction.

The proof is complete.

3. PROOF OF THEOREM 4

The proof is by contradiction. Suppose that G is a 1-tough non-hamiltonian graph on $n \geq 3$ vertices with $\sigma_3 \geq n + \kappa - 2$ and suppose that, additionally, $|E(G)|$ is maximum. Let S be a fixed cut set of κ vertices and suppose that $G_1, G_2, \ldots, G_t$ are the components of $G - S$ $(t \geq 2)$. Let T be an independent set of α vertices $t_1, t_2, \ldots, t_\alpha$ such that $d(t_1) \geq d(t_2) \geq \cdots \geq d(t_\alpha)$.

Claim 1.

1) $\alpha \geq \dfrac{n+\kappa+4}{3}$.

2) $\dfrac{n}{2} \geq \alpha \geq \kappa + 4$.

3) $\ell_{t_i t_j} \geq \kappa$ *for* $t_i \neq t_j \in T - \{t_\alpha\}$.

Proof. Since G is a 1-tough graph, $\kappa \geq 2$ and therefore $\sigma_3 \geq n$. By Theorem 5, $\sigma_3 \leq 3\alpha - 6$, implying by $\sigma_3 \geq n+\kappa-2$ that $\alpha \geq \dfrac{n+\kappa+4}{3}$ and $n + \kappa - 2 \geq \dfrac{3}{2}(n + \kappa - \alpha)$. By $\alpha \leq \dfrac{n}{2}$ since G is a 1-tough graph and by $\alpha \geq \dfrac{n+\kappa+4}{3}$ we get $\alpha \geq \kappa + 4$. If $t_i \neq t_j \in T - \{t_\alpha\}$ then $d(t_i) + d(t_j) \geq \dfrac{2}{3}\sigma_3$ and $N(t_i) \cup N(t_j) \subseteq V(G) - T$. Thus,

$$\begin{aligned}
\ell_{t_i t_j} &= |N(t_i) \cap N(t_j)| = d(t_i) + d(t_j) - |N(t_i) \cup N(t_j)| \\
&\geq \frac{2}{3}(n + \kappa - 2) - (n - \alpha) \\
&\geq (n + \kappa - \alpha) - (n - \alpha) \\
&= \kappa. \qquad (1)
\end{aligned}$$

Claim 2. *There exists some i, say $i = 1$, such that* $T - \{t_\alpha\} \subseteq S \cup V(G_1)$.

Proof. Suppose, to the contrary, that there exists $i \neq j$, say $i = 1$ and $j = 2$, such that $(T - \{t_\alpha\}) \cap V(G_1) \neq \emptyset$ and $(T - \{t_\alpha\}) \cap V(G_2) \neq \emptyset$.

Choose t_{i_j} in $(T - \{t_\alpha\}) \cap V(G_j)$ $(j = 1, 2)$. Clearly, $N(T_{i_1}) \cap N(t_{i_2}) \subseteq S$, implying by 3)of Claim 1 that $\ell_{t_i t_j} = \kappa$ and all inequalities in (1) are in fact equalities. Thus, $d(t_{i_1}) = d(t_{i_2}) = d(t_\alpha) = \frac{n+\kappa-2}{3}$ and $S \cap T = \emptyset$, and therefore $T - (S \cup V(G_i)) \neq \emptyset$ for any component G_i of $G - S$. Since t_{i_1} and t_{i_2} are arbitrary chosen, $d(t) = d(t_\alpha) = \frac{n+\kappa-2}{3}$ for any $t \in T$. By Lemma 4, G is Hamiltonian, a contradiction.

Claim 3. *$t = 2$ and G_2 is a complete graph.*

Proof. Let $A = V(G) - V(G_1) - S$, $n_1 = |V(G_1)|$ and $n_2 = |A|$ so that $n = n_1 + n_2 + \kappa$. It suffices to show that $G|A|$ is complete. Suppose, to the contrary, that there exist $v_1, v_2 \in A$ such that $v_1v_2 \notin E(G)$. By Claim 2 and by 2) of Claim 1, $|T \cap V(G_1)| \geq 3$. So we can choose $t_1 \neq t_2$ in $T \cap V(G_1)$. Then $n+\kappa-2 \leq d(t_1)+d(v_1)+d(v_2) = n_1+\kappa-(\alpha-1)+2(n_2+\kappa-2)$, implying by $n = n_1+\kappa+n_2$ that $n_2 \geq \alpha-\kappa+1$. Similarly, $n_1 \geq 2\alpha-\kappa-3$ because of $n+\kappa-2 \leq d(t_1)+d(t_2)+d(v_1) \leq 2(n_1+\kappa-\alpha+1)+(n_2+\kappa-1)$. Thus, $n_1+n_2 \geq (\alpha-\kappa+1)+(2\alpha-\kappa-3)$ and therefore $\sigma_3 \geq n+\kappa-2 \geq 3\alpha-4$, which contradicts 1) of Claim 1.

Claim 4.

1) $\frac{n_1+\kappa}{2} - 1 \leq d(t_i) \leq \frac{n_1+\kappa+1}{2}$ for any $t_i \in T \cap V(G_1)$. The equality $d(t_i) = \frac{n_1+\kappa}{2} - 1$ or $d(t_i) = \frac{n_1+\kappa+1}{2}$ holds only for at most one of the vertices in $T \cap V(G_1)$.

2) If $T \cap V(G_2) = \emptyset$ then $d(t) = \frac{n_1+\kappa-1}{2} \geq \alpha - 1$ for any $t \in T \cap V(G_1)$ and $N(w) = S \cup V(G_2) - \{w\}$ for any $w \in V(G_2)$.

3) If $T \cap V(G_2) \neq \emptyset$ and $T \cap S \neq \emptyset$ then $T \cap S = \{t^*\}$ for some $t^* \in T$ and $d(t) = \frac{n_1+\kappa}{2} \geq \alpha - 1$ for any $t \in T \cap V(G_1)$ and $N(t_\alpha) = S \cup V(G_2) - \{t_\alpha, t^*\}$.

4) If $T \cap V(G_2) \neq \emptyset$ and $T \cap S = \emptyset$ then $n_1 \geq 2\alpha - \kappa - 1$ and $d(v) \geq \alpha - 1$ for any $v \in V(G_1)$.

Proof. First note that for any $t_i \neq t_j \in T \cap V(G_1)$ we have:

$$\alpha = \alpha_{t_it_j} > \ell_{t_it_j} = d(t_i) + d(t_j) - |N(t_i) \cup N(t_j)|. \qquad (*)$$

1) Setting $|N(t_i) \cup N(t_j)| \leq |S \cup V(G_1) - T| \leq n_1+\kappa-\alpha+1$ in (*) we get $(\alpha-1) \geq d(t_i)+d(t_j)-(n_1+\kappa-\alpha+1)$ and, consequently, $n_1+\kappa \geq d(t_i)+d(t_j)$. Moreover, $d(t_i)+d(t_j) \geq n_1+\kappa-1$ since $n+\kappa-2 \leq$

$d(t_i)+d(t_j)+d(w)$ by $d(w) \leq n_2+\kappa-1$ where w is an arbitrary vertex of $V(G_2)$. Thus, $n_1+\kappa \geq d(t_i)+d(t_j) \geq n_1+\kappa-1$ for any $t_i \neq t_j$ in $T \cap V(G_1)$. We easily get that $\frac{n_1+\kappa+1}{2} \geq d(t_i) \geq \frac{n_1+\kappa}{2}-1$ for any $t_i \in T \cap V(G_1)$ and that the equality $d(t_i) = \frac{n_1+\kappa}{2}-1$ or $d(t_i) = \frac{n_1+\kappa+1}{2}$ holds only for at most one of the vertices in $T \cap V(G_1)$.

2) Setting $|N(t_i) \cup N(t_j)| \leq n_1+\kappa-\alpha$ by $T \cap V(G_2) = \emptyset$, $d(t_i)+d(t_j) \geq n+\kappa-2-d(w)$ and $d(w) \leq n_2+\kappa-1$ in (*) where w is an arbitrary vertex of G_2, we get $\alpha > \ell_{t_it_j} \geq \alpha-1$. Thus, $\ell_{t_it_j} = \alpha-1$ and, consequently, $d(t_i) \geq \alpha-1$, $d(w) = n_2+\kappa-1$ and $d(t_i)+d(t_j) = (n+\kappa-2)-(n_2+\kappa-1) = n_1+\kappa-1$. Using $|T \cap V(G_1)| \geq 3$ we choose $t_k \in T \cap V(G_1)-\{t_i,t_j\}$. Similarly, $d(t_i)+d(t_k) = d(t_i)+d(t_j) = d(t_j)+d(t_k) = n_1+\kappa-1$ and therefore $d(t_i) = d(t_j) = d(t_k) = \frac{n_1+\kappa-1}{2}$. Thus, $d(t_i) = \frac{n_1+\kappa-1}{2} \geq \alpha-1$ and $N(w) = S \cup V(G_2)-\{w\}$ for any $t_i \in T \cap V(G_1)$ and $w \in V(G_2)$.

3) Setting $|N(t_i) \cup N(t_j)| \leq n_1+\kappa-\alpha+1$ by $T \cap V(G_2) \neq \emptyset$, $d(t_i)+d(t_j) \geq n+\kappa-2-d(t_\alpha)$ and $d(t_\alpha) \geq n_2+\kappa-|S \cap T|-1$ in (*), we get $\alpha > \ell_{t_it_j} \geq |S \cap T|+\alpha-2$. By $S \cap T \neq \emptyset$, in fact, $|S \cap T| = 1$ and $\ell_{t_it_j} = \alpha-1$ and consequently, $d(t_i) \geq \alpha-1$, $d(t_\alpha) = n_2+\kappa-2$ and $d(t_i)+d(t_j) = (n+\kappa-2)-(n_2+\kappa-2) = n_1+\kappa$. Similarly as in 2), we get $d(t_i) = \frac{n_1+\kappa}{2} \geq \alpha-1$ and $N(t_\alpha) = S \cup V(G_2)-\{t_\alpha, t^*\}$.

4) By $T \cap S = \emptyset, \alpha = w(G-(V(G_1) \cup S-T)) \leq |V(G_1) \cup S-T| = n_1+\kappa-\alpha+1$ since G is a 1-tough graph, and therefore $n_1 \geq 2\alpha-\kappa-1$.

Now, suppose, to the contrary, that $d(v) \leq \alpha-2$ for some $v \in V(G_1)$. Then $d(v) \leq \frac{n_1+\kappa-3}{2}$ by $n_1 \geq 2\alpha-\kappa-1$ and therefore $v \in V(G_1)-T$ since 1). Moreover, $d(v)+d(t_i)+d(t_\alpha) < n+\kappa-2$ since $d(t_i) \leq \frac{n_1+\kappa+1}{2}$ by 1) and $d(t_\alpha) \leq n_2+\kappa-1$, and therefore $vt_i \in E(G)$ for any $\alpha-1 \geq i \geq 2$, implying by $d(v) \leq \alpha-2$ that, in fact, $d(v) = \alpha-2$ and $N(v) = \{t_2, t_3, \ldots, t_{\alpha-1}\}$. Hence, $vt_1 \notin E(G)$ and therefore $n+\kappa-2 \leq d(v)+d(t_\alpha)+d(t_1) \leq \frac{n_1+\kappa-3}{2}+(n_2+\kappa-1)+\frac{n_1+\kappa+1}{2} = n+\kappa-2$ and consequently, $d(t_1) = \frac{n_1+\kappa+1}{2}$

and $d(v) = \alpha - 2 = \frac{n_1 + \kappa - 3}{2}$. But $|V(G_1) \cup S - T| = \frac{n_1 + \kappa + 1}{2}$ by $\alpha - 2 = \frac{n_1 + \kappa - 3}{2}$ and therefore $N(t_1) = V(G_1) \cup S - T$, which contradicts $vt_1 \notin E(G)$.

Claim 5. $\sigma_2(G_1) \geq 2(\alpha - 1)$.

Proof. For any two different nonadjacent vertices u and v in G_1 choose w in G_2, specialy $w = t_0$ if $V(G_2) \cap T \neq \emptyset$. We easily get $d(u) + d(v) \geq 2(\alpha - 1)$ by 2) - 4) of Claim 4 and by $d(u) + d(v) \geq \sigma_3 - d(w)$. Thus Claim 5 is true.

Now, a longest cycle C and a vertex $v_0 \in V(G) - V(C)$ are chosen such that $d(v_0) = \mu(G)$ and $|N(v_0)^+ \cap N)v_0)^-| \geq \sigma_3 - n + 4$. Let $T^* := \{v_0\} \cup (N(v_0)^+ \cap N(v_0)^-)$ and $u_{i_1}, u_{i_2}, \ldots, u_{i_s}$ the vertices of $T^* - \{v_0\}$ such that $d(u_{i_1}) \geq \cdots \geq d(u_{i_s})$. Using $|T^*| \geq \kappa + 3$ because of $\sigma_3 - n + 4 \geq \kappa + 2$ and $|T^* \cap V(G_2)| \leq 1$ since G_2 is a complete graph by Claim 3, we get $|T^* \cap V(G_1)| \geq 2$, say u^*, $u^{**} \in T^* \cap V(G_1)$. By Claim 5 and by the maximality of $d(v_0)$, $d(v_0) \geq (d(u^*) + d(u^{**}))/2 \geq \alpha - 1$. Since $\{v_0\} \cup N(v_0)^+$ is an independent set of vertices, in fact, $d(v_0) = d(u^*) = d(u^{**}) = \alpha - 1$. Note that if $u_i = w_i$ then $N(u_i) \subseteq N(v_0)$ by Lemma 5. It follows that $N(u^*) = N(u^{**}) = N(v_0)$ by $d(v_0) = d(u^*) = d(u^{**})$ proved above. Let $T := \{v_0\} \cup N(v_0)^+$. Using $\mathcal{F} \neq \emptyset$ by Lemma 6 we determine $i_0 = \max \mathcal{F}$ and $j_0 \neq i_0$ such that $u_{i_0} w_{j_0} \in E(G)$. By Lemma 7, $d(u_{j_0}) + 2d(v_0) \leq \ell(C) + x$, where x is the number of the vertices $u_i = w_i$ such that $d(u_i) \leq d(v_0) - 2$. By $\sigma_3 \geq n$, $x \geq 1$ and therefore $d(u_{i_s}) + d(v_0) + d(u_{j_0}) \leq \ell(C) + x - 2$. Thus, $x \geq 3$ and therefore $\alpha - 3 \geq d(u_{i_{s-2}}) \geq d(u_{i_{s-1}}) \geq d(u_{i_s})$. Hence $T \cap V(G_2) = \emptyset$ since, otherwise, $\{u_{i_{s-1}}, u_{i_{s-2}}\} \cap V(G_1) \neq \emptyset$ and therefore $d(u_{i_{s-2}}) \geq \alpha - 1$ by 3) and 4) of Claim 4, a contradiction. Now, by 2) of Claim 4 for $t = u^*$, and by $d(u^*) = \alpha - 1$, $d(u^*) = \frac{n_1 + \kappa - 1}{2} = \alpha - 1$ and therefore $N(u^*) = V(G_1) \cup S - T$ since $|V(G_1) \cup S - T| = \alpha - 1$ by $\frac{n_1 + \kappa - 1}{2} = \alpha - 1$. It follows by 2) of Claim 4 that $N(t) = V(G_1) \cup S - T$ for any $t \in T \cap V(G_1)$. Let $v_1, v_2, \ldots, v_{n-1}$ the vertices of $V(G_1) \cup S - T$, $t_{i_1}, t_{i_2}, \ldots, t_{i_r}$ the vertices of $S \cap T$, $t_{i_{r+1}}, t_{i_{r+2}}, \ldots, t_{i_a}$ the vertices of $T \cap V(G_1)$ and $w_1, \ldots, w_{n_2}$ the vertices of G_2. Clearly, $r \geq 1$ by $T \cap V(G_2) = \emptyset$. Moreover, $r \geq 2$ by $\omega(G - (V(G_1) \cup S - T)) \leq |V(G_1) \cup S - T| = \alpha - 1$ since G is a 1-tough graph. Let $A = V(G_1) \cup S - T$, $B = T$, $\underline{S} = T \cap S$ and $G^* = (A, B; E)$ the bipartite graph obtained from $G[A \cup B]$ by deleting all edges of $G[A]$.

By $d(t_i) = \dfrac{n_1 + \kappa - 1}{2}$ for any $t_i \in V(G_1) \cap T$ and by $\sigma_3 \geq n + \kappa - 2$, $d(t) \geq \kappa - 1 + n_2$ and therefore $d_A(t) \geq \kappa - 1$ for any $t \in \underline{S}$. If $|\underline{S}| < |S|$ or $|\underline{S}| = |S| \leq |N_A(\underline{S})|$ then G^* contains a Hamiltonian path joining two different vertices of $\underline{S}$ by Lemma 8. Let H be a Hamiltonian path in G^* joining two vertices t_{i_1} and t_{i_2} in $\underline{S}$. Then $w_1 w_2 \ldots w_{n_2} H w_1$ would be a Hamiltonian cycle in G, a contradiction (note $N(w_i) = S \cup V(G_2) - \{w_i\}$ by 2) of Claim 4). Thus, $|\underline{S}| = |S| > |N_A(\underline{S})|$, implying by $d_A(t) \geq \kappa - 1$ for any $t \in \underline{S}$ that, in fact, $|\underline{S}| = |S| = |N_A(\underline{S})| + 1$ and $d_A(t) = \kappa - 1$ for any $t \in \underline{S}$. Since G is a 1-tough graph, $\omega(G - (N_A(\underline{S}) \cup V(G_2)) \leq |N_A(\underline{S}) \cup V(G_2)| = \kappa - 1 + n_2$ and therefore $2 \leq n_2$. Let $v_1, \ldots, v_{k-1}$ the vertices of $N_A(\underline{S})$ then every vertex of S is adjacent to any vertex of $\{v_1, \ldots, v_{k-1}\}$. Now, $\{v_k, \ldots, v_{\alpha-1}\}$ is an independent set of vertices since, otherwise, say $v_k v_{k+1} \in E(G)$, and $C : t_{i_1} v_2 t_{i_2} v_3 \ldots t_{i_{\kappa-2}} v_{\kappa-1} t_{i_{\kappa+1}} v_\kappa v_{\kappa+1} t_{i_{\kappa+2}} v_{\kappa+2} \ldots t_{i_{\alpha-1}} v_{\alpha-1} t_{i_\alpha} v_1 t_{i_{\kappa-1}} w_1 t_{i_\kappa} w_2 \ldots w_{n_2} t_{i_1}$ would be a Hamiltonian cycle, a contradiction. Moreover, there is no edge joining a vertex of $\{v_1, \ldots, v_{\kappa-1}\}$ with a vertex of $\{v_\kappa, \ldots, v_{\alpha-1}\}$ since, othewise, say $v_\kappa v_{\kappa-1} \in E(G)$, and $C : t_{i_1} v_2 t_{i_2} v_3 \ldots t_{i_{\kappa-2}} v_{\kappa-1} v_\kappa t_{i_{\kappa+1}} v_{\kappa+1} t_{i_{\kappa+2}} v_{\kappa+2} \ldots t_{i_{\alpha-1}} v_{\alpha-1} t_{i_\alpha} v_1 t_{i_{\kappa-1}} w_1 t_{i_\kappa} w_2 \ldots w_{n_2} t_{i_1}$ would be a Hamiltonian cycle, a contradiction. Thus, $d(v_\kappa)$, $d(v_{\kappa+1}) \leq \alpha - \kappa$ and therefore $d(v_\kappa) + d(v_{\kappa+1}) + d(w_1) \leq 2(\alpha - \kappa) + n_2 + \kappa - 1) \leq n + \kappa - 4$, a contradiction (note that $n_1 = 2\alpha - \kappa - 1$). Thus last contradiction completes our proof.

4. FINAL REMARKS

The following conjecture will strengthen the conjecture of Bauer and Schmeichel.

Conjecture: *Let G be a 1-tough graph such that* $\sigma_3 \geq \max(n + \kappa - 4, n)$, *then G is Hamiltonian.*

Acknowledgement. The author wishes to thank the referees of this paper for their comments and many helpful suggestions.

REFERENCES

1. A. Ainouche and N. Christofides, *Conditions for the existence of Hamiltonian circuits in graphs based on vertex degrees*, J. London Math. Soc. (2) **32** (1985), 385-391.

2. A. Ainouche and N. Christofides, *Strong sufficient conditions for the existence of Hamiltonian ciecuits in undirected graphs*, J. Combin. Theory, **B 31** (1981), 339-343.

3. D. Bauer, A. Morgana, E. Schmeichel, and H. J. Veldman, *Long cycle in graphs with large degree sums*, Discrete Mathematics **79** (1989/90). 59-70.

4. D. Bauer, H, J. Broersma, H. J. Veldman and L. Rao, *A generalization of a results of Häggkvist and Nicoghossian*, J. Combin. Theory, **B 47** (1989), 237-243.

5. D. Bauer, and E. Schmeichel, *A sufficient condition for Hamiltonian cycles in 1-tough graphs*, Stevens research reports in Mathematics, 1988, Stevens Institute of Technology, Hoboken, NJ 07030.

6. R. Häggvist, and G. G. Nicoghossian, *A remark on Hamiltonian cycles*, J. of Combin. Theory, **B 30** (1981), 118-120.

7. V. Chvátal, *Tough graphs and Hamiltonian circuit*, Discrete Math. **5** (1973), 215-228.

8. Vu Dinh Hoa, *Note on a theorem of Bauer, Morgana, Veldman and Schmeichel*, J. of Graph Theory, to appear.

9. Vu Dinh Hoa, *On the length of longest dominating cycles in graphs*, Discrete Mathematics **121** (1993), 211-222.

Wundt str. 7/4L1
01217 Dresden
Germany

Received April 12, 1993
Revised September 5, 1994