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THE ACTION OF THE MOD P STEENROD OPERATIONS ON THE MODULAR INVARIANTS OF LINEAR GROUPS

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Abstract. *Let p be an odd prime. The purpose of this paper is to compute the action of the mod p Steenrod operations on the Dickson and Mui invariants of the general linear group $GL(n, Z/p)$ and of the subgroup T_n consisting of all upper triangular matrices with 1 on the main diagonal. Our method is extended from that of [13] to the case of odd primes.*

1. INTRODUCTION AND STATEMENT OF RESULTS

For p an odd prime, let $GL_n = GL(n, Z/p)$ and let T_n be the Sylow p-subgroup of GL_n consisting of all upper triangular matrices with 1 on the main diagonal. These two groups act on the elementary abelian p-group $E^n \cong (Z/p)^n$ and so on its cohomology $H^*(BE^n) = H^*(BE^n; Z/p)$ in the usual manner.

As is well known, $H^*(BE^n) \cong E(x_1, \dots, x_n) \otimes P(y_1, \dots, y_n)$, where $|x_i| = 1$, $y_i = \beta(x_i)$, with β the Bockstein homomorphism. Here and in what follows, $E(., \dots, .)$ and $P(., \dots, .)$ are respectively the exterior algebra and the polynomial algebra over Z/p generated by the variables indicated. Under this identification, the action of GL_n and T_n on $H^*(BE^n)$ is identified with their canonical action on $E(x_1, \dots, x_n) \otimes P(y_1, \dots, y_n)$. The invariants of this action are determined by Dickson [2] and Mui [3] as follows.

Set $[e_1, \dots, e_n] = \det\left(y_i^{p^{e_j}}\right)$ for every sequence of non-negative integers $(e_1, \dots, e_n)$. In particular, we define

$$L_{n,s} = [0, \dots, \hat{s}, \dots, n], \quad 0 \le s \le n, \quad L_n = L_{n,n} = [0, \dots, n-1].$$

The Dickson invariant $Q_{n,s}$ and the Mui invariant V_n are defined by

$$Q_{n,s} = L_{n,s}/L_n \quad (0 \le s < n),$$
$$V_n = L_n/L_{n-1} = \prod_{\lambda_i \in Z/p} (\lambda_1 y_1 + \cdots + \lambda_{n-1} y_{n-1} + y_n).$$

They are related each other by

$$Q_{n,s} = Q_{n-1,s-1}^p + V_n^{p-1} Q_{n-1,s}, \quad V_n = \sum_{s=0}^{n-1} (-1)^{n-1+s} Q_{n-1,s} y_n^{p^s},$$

For $0 \le k \le n$, we set

$$[k; e_{k+1}, \ldots, e_n] = (1/k!) \begin{vmatrix} x_1 & \cdots & x_n \\ \cdot & \cdots & \cdot \\ x_1 & \cdots & x_n \\ y_1^{p^{e_{k+1}}} & \cdots & y_n^{p^{e_{k+1}}} \\ \cdot & \cdots & \cdot \\ y_1^{p^{e_n}} & \cdots & y_n^{p^{e_n}} \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} x \\ \cdot \\ x \end{matrix}}\right\} k \text{ rows} \\ \\ \\ \\ \end{matrix}.$$

Here the right-hand side is first computed in

$$E_Z(x_1, \ldots, x_n) \otimes P_Z(y_1, \ldots, y_n)$$

and then is projected to

$$E(x_1, \ldots, x_n) \otimes P(y_1, \ldots, y_n).$$

In particular we define the Mui invariant

$$M_{n,s_1,\ldots,s_k} = [k; 0, \ldots, \hat{s}_1, \ldots, \hat{s}_k, \ldots, n-1]$$

for $0 \le s_1 < \cdots < s_k \le n-1$.

Then one has

$$(H^*BE^n)^{GL^n} = P(Q_{n,0}, \ldots, Q_{n,n-1}) \oplus$$
$$\sum_{k=1}^{n} \oplus \sum_{0 \le s_1 < \cdots < s_k < n} M_{n,s_1,\ldots,s_k} L_n^{p-2} P(Q_{n,0}, \ldots, Q_{n,n-1}),$$

$$
\begin{aligned}
(H^*BE^n)^{T_n} &= P(V_1,\dots,V_n)\oplus \\
&\sum_{k=1}^{n}\oplus\sum_{s=k}^{n}\oplus\sum_{0\le s_1<\dots<s_k=s-1} M_{s,s_1,\dots,s_k}P(V_1,\dots,V_n).
\end{aligned}
$$

(see [2] and [3] for details).

We recall further that the mod p Steenrod algebra $A = A(p)$ acts on $H^*(BE^n)$ by means of the Cartan formula together with the relations

$$
\begin{aligned}
&\beta x_i = y_i, \quad P^0 x_i = x_i, \quad P^j x_i = 0,\ 1\le j,\\
&\beta y_i = 0, \quad P^0 y_i = y_i, \quad P^1 y_i = y_i^p, \quad P^j y_i = 0,\ 1< j.
\end{aligned}
$$

Since this action commutes with the actions of GL_n and T_n, it induces a natural action of A on $H^*(BE^n)^G$ for $G = GL_n$ or T_n.

In this paper, we determine explicitly the action of the Steerod operations on Dickson and Mui invariants which are described above. Below we formulate the main results of this paper in the form of those theorems (theorems A, B, C). Their proofs are given in Section 3. In Section 2 we recall some well-known results which arised in proving the main theorems.

Suppose t is a non-negative interger. Let $\alpha_i = \alpha_i(t)$ be the i-th coefficient in the p-adic expansion of t. That means

$$t = \alpha_0 + \alpha_1 p^1 + \dots,$$

with $0 \le \alpha_i < p$. By convention, $\alpha_i = 0$ for $i < 0$. We have

Theorem A. $\beta(V_n) = 0$,

$$
P^t(V_n) = \begin{cases}
V_n^p & t = p^{n-1},\\[4pt]
\dfrac{(-1)^{\alpha_{n-2}}\alpha_{n-2}!}{\prod\limits_{0\le i\le n-2}(\alpha_i-\alpha_{i-1})!}V_n\prod\limits_{i=0}^{n-2}Q_{n-1,i}^{\alpha_i-\alpha_{i-1}} & t < p^{n-1},\ \alpha_i \ge \alpha_{i-1}\ \text{for } i \le n-2,\\[4pt]
0 & \text{otherwise.}
\end{cases}
$$

Theorem B. $\beta(Q_{n,s}) = 0$,

$$P^t(Q_{n,s}) = \begin{cases} Q^p_{n,s} & t = p^n - p^s, \\ \dfrac{(-1)^{\alpha_{n-1}}\alpha_{n-1}!(\alpha_s+1)}{(\alpha_s+1-\alpha_{s-1})!\prod\limits_{s\neq i<n}(\alpha_i-\alpha_{i-1})!} Q_{n,s}\prod\limits_{i=0}^{n-1} Q_{n,i}^{\alpha_i-\alpha_{i-1}} & \begin{array}{l} t < p^n - p^s, \\ \alpha_i \geq \alpha_{i-1} \\ \text{for } s \neq i < n, \\ \alpha_s + 1 \geq \alpha_{s-1}, \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem C.

(i) *For* $0 \leq s_1 < \cdots < s_k \leq n-1$,

$$\beta(M_{n,s_1,\ldots,s_k}) = \begin{cases} (-1)^{k-1}M_{n,s_2,\ldots,s_k} & s_1 = 0 \\ 0 & s_1 > 0, \end{cases}$$

$$P^t(M_{n,s_1,\ldots,s_k}) = \begin{cases} M_{n,t_1,\ldots,t_k} & \begin{array}{l} t = \sum\limits_{i=1}^{k}\dfrac{p^{s_i}-p^{t_i}}{p-1}, \\ \text{with } s_{i-1} < t_i \leq s_i, \end{array} \\ (-1)^{k+1}\sum\limits_{i=1}^{k+1}(-1)^i M_{n,t_1,\ldots,\hat{t}_i,\ldots,t_{k+1}} Q_{n,t_i} & \begin{array}{l} t = \dfrac{p^n - p^{t_{k+1}}}{p-1} \\ \quad + \sum\limits_{i=1}^{k}\dfrac{p^{s_i}-p^{t_i}}{p-1}, \\ \text{with } s_{i-1} < t_i \leq s_i, \\ s_k < t_{k+1} < n, \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Here, by convention, $s_0 = -1$.

(ii) $\beta(L_n) = 0$,

$$P^t(L_n) = \begin{cases} L_nQ_{n,s} & t = \dfrac{p^n - p^s}{p-1}, \\ 0 & \textit{otherwise.} \end{cases}$$

Part (ii) was first proved in Mui [3, II.5.5]. It is worth noticing that the above action has been studied by many authors. Their results can be divided into two kinds: either they are only valid for t, a power of p, or they are given only by inductive procedures. (See Singer [14], Campbell [1] for results concerning Theorem A, and Madsen [6], May [10, I.3], Madsen-Milgram [7, Chap. 3]. Mann [8], Mann-Milgram [9], Smith-Switzer [16], Wikerson [18], Singer [15] for results related to Theorem B.)

The results analogous to Theorem A, B, C for $p = 2$ were given by Hai-Hung [12] and Hung [13].

It should be noted that the method which we use in this paper is very elementary. It is extended from that of the first author's work [13]. Roughly speaking, we mainly read off the action of Steenrod operations from the expansion of Mui invariants in terms of Dickson and Mui invariants of fewer variables.

2. PRELIMINARIES

Recall that the Steenrod algebra $A = A(p)$ has an additive basis consisting of the admissible monomials in P^i's and β. Let A_* be the dual of A and τ_s, $\xi_k \in A_*$ the duals respectively of $\beta P^{p^{s-1}} \dots P^pP^1$, $P^{p^{k-1}} \dots P^pP^1$ with respect to that basis. Milnor showed in [11] that, as an algebra,

$$A_* \cong E(\tau_0, \tau_1, \dots) \otimes P(\xi_1, \xi_2, \dots).$$

So A_* has a basis, in the Milnor sense, consisting of all the monomials of the form

$$\tau_S\xi^R = \tau_{s_1} \dots \tau_{s_k}\xi_1^{r_1} \dots \xi_n^{r_n},$$

with $S = (s_1, \dots, s_k)$, $s_1 < \dots < s_k$ and $R = (r_1, \dots, r_n)$.

We denote by $St^{S,R}$ the dual of $\tau_S \xi^R$ with respect to the Milnor basis. In particular, $St^{(0),\emptyset} = \beta$ and $St^{\emptyset,(t)} = P^t$.

In [5], Mui described $St^{S,R}$ in terms of Dickson and Mui invariants by means of the homomorphism $d_m^* P_m$, a generalization of $d^* P$, which was first studied by Steenrod [17].

Throughout this paper, the cohomology is always taken with coefficients in Z/p. Suppose X is a topological space. Let

$$P_m : H^i(X) \to H^{p^m \cdot i}(E\Sigma_{p^m} \times_{\Sigma_{p^m}} X^{p^m})$$

be the Steenrod power map which sends u to $1 \otimes u^{p^m}$ at cochain level. Let $E^m \subset \Sigma_{p^m}$ be the regular permutation representation of $E^m \cong (Z/p)^m$. This inclusion together with the diagonal map of X and the Künneth formula induces the homomorphism

$$d_m^* : H^*\big(E\Sigma_{p^m} \times_{\Sigma_{p^m}} X^{p^m}\big) \to H^*(BE^m) \otimes H^*(X).$$

One can easily get $\operatorname{Im} d_m^* \subset H^*(BE^m)^{\widetilde{SL}_m} \otimes H^*(X)$, where

$$\widetilde{SL}_m = \big\{w \in GL_m;\ (\det w)^h = 1,\ h = (p-1)/2\big\}.$$

In [5], Mui determined explicitly $H^*(BE^m)^{\widetilde{SL}_m}$. In particular, this invariant algebra contains $\tilde{L}_m = L_m^h$, $\tilde{M}_{m,s} = M_{m,s} L_m^{h-1}$. Note that $Q_{m,0} = \tilde{L}_m^2$. The following beautiful result has been proved in [5].

Theorem 2.1 ([5]). *Set* $\mu(q) = (h!)^q (-1)^{hq(q-1)/2}$ *and* $h = (p-1)/2$. *Then*

$$d_m^* P_m(x) =$$
$$\mu(q)^m \sum (-1)^{r(S,R)} \tilde{M}_{m,s_1} \dots \tilde{M}_{m,s_k} . \tilde{L}_m^{r_0} Q_{m,1}^{r_1} \dots Q_{m,m-1}^{r_{m-1}} \otimes St^{S,R}(X)$$

for $x \in H^q(X)$. *Here the sum is taken over all* (S,R) *with* $R = (r_1, \dots, r_m)$, $r_i \geq 0$, $S = (s_1, \dots, s_k)$, $0 \leq s_1 < \dots < s_k < m$, *and* $r_0 = q - k - 2(r_1 + \dots + r_m)$, $r(S,R) = k + s_1 + \dots + s_k + r_1 + 2r_2 + \dots + mr_m$.

For $m = 1$, $d_1^* P_1$ is nothing but $d^* P$ of Steenrod [17]. One has

$$d_1^* P_1(x) = \mu(q) \sum_{\substack{e=0,1 \\ 0 \leq 2i \leq q-e}} (-1)^{e+i} x_1^e y_1^{(q-2i)h-e} \otimes \beta^e P^i(x).$$

Moreover, the following result holds.

Proposition 2.2 ([3], [5]).

(i) $d_m^*P_m$ *is a natural monomorphism preserving cup product up to a sign, or more precisely,* $d_m^*P_m(uv) = (-1)^{mhqr} d_m^*P_m(u) d_m^*P_m(v)$, *where* $q = \dim u$, $r = \dim v$.

(ii) $d_m^*P_m = d_{m-s}^*P_{m-s}d_s^*P_s$, $0 \leq s \leq m$.

(iii) $d_m^*P_m(y) = V_{m+1}(y_1, \ldots, y_m, y)$, *where* y *is the generator of* $H^2(BZ/p)$.

Theorem 2.1 and Proposition 2.2 will be used to prove Lemmas 3.1 and 3.2, in the next Section which supply the main techniques in our computation of $P^t(V_n)$ and $P^t(Q_{n,s})$. The following results, given respectively in [2] and [3] are also crucial for these lemmas:

$$P(y_1, \ldots, y_n)^{GL_n} = P(Q_{n,0}, \ldots, Q_{n,n-1}),$$

$$P(y_1, \ldots, y_n)^{T_n} = P(V_1, \ldots, V_n).$$

3. PROOF OF THE RESULTS

We start with a corollary of Theorem 2.1.

Lemma 3.0.

$$\begin{aligned} &d_m^*P_m(V_n) = \\ &(-1)^m \sum_{R=(r_1,\ldots,r_m)} (-1)^{r(\emptyset,R)} \tilde{L}_m^{r_0} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_{m-1}} \otimes St^{\emptyset,R}(V_n), \\ &d_m^*P_m(Q_{n,s}) = \\ &\qquad \sum_{R=(r_1,\ldots,r_m)} (-1)^{r(\emptyset,R)} \tilde{L}_m^{r'_0} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_m-1} \otimes St^{\emptyset,R}(Q_{n,s}), \end{aligned}$$

where

$$\begin{aligned} r_0 &= 2p^{n-1} - 2(r_1 + \cdots + r_m), \\ r'_0 &= 2(p^n - p^s) - 2(r_1 + \cdots + r_m), \end{aligned}$$

$$St^{\emptyset,R}(V_n) \in P(V_1,\dots,V_n),$$
$$St^{\emptyset,R}(Q_{n,s}) \in P(Q_{n,0},\dots,Q_{n,n-1}).$$

Proof. Note that $\mu(2p^{n-1}) = -1$, $\mu(2p^n - 2p^s) = 1$. So, applying Theorem 2.1, we need only to show why the invariants $M_{m,s}$'s do not appear in the right-hand sides.

Remark that $P(y_1,\dots,y_n)$ and so $P(V_1,\dots,V_n)$ as well as $P(Q_{n,0},\dots,Q_{n,n-1})$ is closed under the action of any P^i, and is annihilated by β. Thus, by twice dualizing, we get

$$St^{\emptyset,R}(V_n) \in P(V_1,\dots,V_n),$$
$$St^{\emptyset,R}(Q_{n,s}) \in P(Q_{n,0},\dots,Q_{n,n-1}),$$
$$St^{S,R}(V_n) = 0, \quad St^{S,R}(Q_{n,s}) = 0,$$

for any $S \neq \emptyset$. The lemma follows.

Lemma 3.1. *There exists uniquely an expansion*

$$V_{m+n}(z_1,\dots,z_m,y_1,\dots,y_n) = \sum_{R=(r_1,\dots,r_m)} Q_{m,0}^{p^{n-1}-(r_1+\cdots+r_m)} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_{m-1}} \varphi^R(V_1,\dots,V_n)$$

with $\varphi^R \in P(V_1,\dots,V_n)$, $Q_{m,s} = Q_{m,s}(z_1,\dots,z_m)$ *for* $0 \le s < m$, $V_r = V_r(y_1,\dots,y_r)$ *for* $1 \le r \le n$. *Furthermore,*

$$St^{\emptyset,R}V_n = (-1)^{m+r(\emptyset,R)}\varphi^R(V_1,\dots,V_n), \quad St^{S,R}V_n = 0,$$

for any $S \neq \emptyset$.

Proof. From Proposition 2.2, we have

$$d_m^* P_m V_n(y_1,\dots,y_n) = V_{m+n}(z_1,\dots,z_m,y_1,\dots,y_n).$$

Recall that $\tilde{L}_m^2 = Q_{m,0}$. So, according to Lemma 3.0, there exists such an expansion. The uniqueness of the expansion follows from the algebraic independence of $V_1,\dots,V_n$ over $P(z_1,\dots,z_m)$. The last equations also follow from Lemma 3.0.

Proof of Theorem A.

By Lemma 3.1, $\beta(V_n) = St^{(0),\emptyset}(V_n) = 0$. Setting $V_n' = V_n(y_1, \dots, y_{n-1}, y)$ and applying Lemma 3.1 with $m = 1$, one gets

$$\begin{aligned}
V_{n+1}(y, y_1, \dots, y_n) &= \prod_\lambda V_n(y_1, \dots, y_{n-1}, \lambda y + y_n)\\
&= \prod_\lambda \Big[\sum_{s=0}^{n-1} (-1)^{n-1+s} Q_{n-1,s}(\lambda y + y_n)^{p^s}\Big]\\
&= \prod_\lambda (V_n + \lambda V_n')\\
&= V_2(V_n', V_n)\\
&= V_n^p - V_n \cdot V_n'^{p-1}\\
&= V_n^p - V_n \cdot \sum_{\substack{I=(i_{s_1},\dots,i_{s_u})\\ k=i_{s_1}+\dots+i_{s_u}\le p-1\\ 0\le s_1<\dots<s_u<n-1}} (-1)^{\nu(I)} c(I) Q_{n-1,s_1}^{i_{s_1}} \dots\\
&\quad \dots Q_{n-1,s_u}^{i_{s_u}} y^{i_{s_1}p^{s_1}+\dots+i_{s_u}p^{s_u}+(p-1-k)p^n},
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\nu(I) &= s_1 i_{s_1} + \dots + s_u i_{s_u} + (p-1-k)(n-1),\\
c(I) &= \frac{(p-1)!}{i_{s_1}! \dots i_{s_u}!(p-1-k)!} = (-1)^k \frac{k!}{i_{s_1}! \dots i_{s_u}!}.
\end{aligned}$$

On the other hand, applying Lemma 3.1 with $m = 1$, $R = (t)$, $Q_{1,0} = y^{p-1}$, we get

$$V_{n+1}(y, y_1, \dots, y_n) = -\sum_t (-1)^t y^{(p^{n-1}-t)(p-1)} P^t(V_n). \tag{3.2}$$

Comparing (3.1) with (3.2) and using the uniqueness of the expansion of V_{n+1} showed in Lemma 3.1, we obtain

$$P^t(V_n) = \begin{cases}
V_n & t = 0,\\
V_n^p & t = p^{n-1},\\
(-1)^{t+\nu(I)} c(I) Q_{n-1,s_1}^{i_{s_1}} \dots Q_{n-1,s_u}^{i_{s_u}} V_n & (p^{n-1} - t)(p-1) =\\
& i_{s_1}p^{s_1} + \dots i_{s_u}p^{s_u} +\\
& (p-1-k)p^n,\\
0 & \text{otherwise.}
\end{cases}$$

Since $(p^{n-1}-t)(p-1) = i_{s_1}p^{s_1} + \cdots + i_{s_u}p^{s_u} + (p-1-k)p^n$, we easily get

$$t = \frac{kp^{n-1} - i_{s_1}p^{s_1} - \cdots - i_{s_u}p^{s_u}}{p-1},$$

with $0 \leq k = i_{s_1} + \cdots + i_{s_u} \leq p-1$, In particular, $t = \nu(I) \pmod 2$.

It is easy to express $k, i_{s_1}, \ldots, i_{s_u}$ in terms of $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$, the coefficients in the p-adic expansion of t, completing the proof of the theorem.

Lemma 3.2. *There exists uniquely an expansion*

$$\begin{aligned} &V_{m+n+1}(z_1, \ldots, z_m, y_1, \ldots, y_n, z) = \\ &\quad = \sum_{s=0}^{n} P_s(z_1, \ldots, z_m, y_1, \ldots, y_n) V_{m+1}^{p^s}(z_1, \ldots, z_m, z), \end{aligned}$$

with $P_s \in P(z_1, \ldots, z_m)^{GL_m} \otimes P(y_1, \ldots, y_n)^{GL_n}$. *Furthermore, if*

$$\begin{aligned} P_s = \sum_{R=(r_1,\ldots,r_m)} & Q_{m,0}^{p^n-p^s-(r_1+\cdots+r_m)} Q_{m,1}^{r_1} \cdots \\ & \cdots Q_{m,m-1}^{r_{m-1}} \psi_s^R(Q_{n,0}, \ldots, Q_{n,n-1}), \end{aligned}$$

with $Q_{m,r} = Q_{m,r}(z_1, \ldots, z_m)$, $Q_{n,t} = Q_{n,t}(y_1, \ldots, y_n)$, *then*

$$St^{\emptyset,R} Q_{n,s} = (-1)^{n+s+r(\emptyset,R)} \psi_s^R(Q_{n,0}, \ldots, Q_{n,n-1}),$$

$$St^{S,R} Q_{n,s} = 0,$$

for any $S \neq \emptyset$.

Proof. Recall that $V_{n+1}(y_1, \ldots, y_n, z) = \sum (-1)^{n+s} Q_{n,s} z^{p^s}$. By using Theorem 2.1 and Proposition 2.2, we obtain

$$\begin{aligned} &V_{m+n+1}(z_1, \ldots, z_m, y_1, \ldots, y_n, z) = \\ &\quad = \sum_{s=0}^{n} (-1)^{n+s} (d_m^* P_m Q_{n,s}) V_{m+1}^{p^s}(z_1, \ldots, z_m, z). \end{aligned}$$

This is the desired expansion with

$$\begin{aligned} &P_s(z_1, \ldots, z_m, y_1, \ldots, y_n) = \\ &= (-1)^{n+s} d_m^* P_m Q_{n,s} \in P(\tilde{L}_m, Q_{m,1}, \ldots, Q_{m,m-1}) \otimes P(y_1, \ldots, y_n)^{GL_n}. \end{aligned}$$

Moreover, combine Lemma 3.0 with the fact that $\tilde{L}_m^2 = Q_{m,0}$, one gets

$$d_m^* P_m Q_{n,s} = \sum_R (-1)^{r(\emptyset,R)} Q_{m,0}^{r_0} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_{m-1}} \otimes St^{\emptyset,R} Q_{n,s}.$$

So

$$\begin{aligned} &St^{\emptyset,R} Q_{n,s} = (-1)^{n+s+r(\emptyset,R)} \psi_s^R (Q_{n,0}, \ldots ; Q_{n,n-1}), \\ &St^{S,R} Q_{n,s} = 0 \qquad \text{for } S \neq \emptyset. \end{aligned}$$

To prove the uniqueness of such an expansion, assume that

$$\sum_{s=0}^{n} F_s(z_1, \ldots, z_m, y_1, \ldots, y_n) V_{m+1}^{p^s}(z_1, \ldots, z_m, z) = 0$$

with the polynomials F_s of the indicated variables. Observing the coefficient of the leading term $z^{p^{m+n}}$, we have $F_n = 0$. By descending induction on n, we get $F_n = \cdots = F_0 = 0$. The lemma follows.

The following lemma can easily be proved by induction on k, so we omit its proof.

Lemma 3.3. *Set* $V = V_2(y, z) = z(z^{p-1} - y^{p-1})$, *then*

$$z^{p^k} = z y^{p^k - 1} + \sum_{s=0}^{k-1} V^{p^s} y^{p^k - p^{s+1}}$$

for $k > 0$.

Proof of Theorem B.

From Lemma 3.2, $\beta(Q_{n,s}) = St^{(0),\emptyset}(Q_{n,s}) = 0$. We now apply Lemma 3.2 with $m = 1$. Denote

$$V_{n+1} = V_{n+1}(y_1, \ldots, y_n, z), \quad V'_{n+1} = V_{n+1}(y_1, \ldots, y_n, y).$$

We have, as seen in the proof of Theorem A,

$$\begin{aligned} V_{n+2} = V_{n+2}(y, y_1, \ldots, y_n, z) &= V_2(V'_{n+1}, V_{n+1}) = V_{n+1}^p - V_{n+1} V'^{\,p-1}_{n+1} \\ &= (-1)^n \Big\{ \sum_{i=0}^{n} (-1)^i Q_{n,i}^p z^{p^{i+1}} \\ &\quad - \Big[\sum_{i=0}^{n} (-1)^i Q_{n,i} z^{p^i} \Big] \Big[\sum_{i=0}^{n} (-1)^i Q_{n,i} y^{p^i} \Big]^{p-1} \Big\}. \end{aligned}$$

By Lemma 3.3, we have

$$V_{n+2} = (-1)^n\Big\{\sum_{i=0}^{n}(-1)^i Q_{n,i}^p \sum_{s=0}^{i} V^{p^s} y^{p^{i+1}-p^{s+1}}$$

$$-\Big[\sum_{i=1}^{n}(-1)^i Q_{n,i}\sum_{s=0}^{i-1} V^{p^s} y^{p^i-p^{s+1}}\Big]\Big[\sum_{i=0}^{n}(-1)^i Q_{n,i} y^{p^i}\Big]^{p-1}\Big\} + Fz,$$

with

$$F = (-1)^n\Big\{\sum_{i=0}^{n}(-1)^i Q_{n,i}^p y^{p^{i+1}-1}$$

$$-\Big[\sum_{i=0}^{n}(-1)^i Q_{n,i} y^{p^i-1}\Big]\Big[\sum_{i=0}^{n}(-1)^i Q_{n,i} y^{p^i}\Big]^{p-1}\Big\}.$$

Since

$$Fy = (-1)^n\Big\{\sum_{i=0}^{n}(-1)^i Q_{n,i}^p y^{p^{i+1}}$$

$$-\Big[\sum_{i=0}^{n}(-1)^i Q_{n,i} y^{p^i}\Big]\Big[\sum_{i=0}^{n}(-1)^i Q_{n,i} y^{p^i}\Big]^{p-1}\Big\} = 0.$$

so $F = 0$. This yields

$$V_{n+2} = (-1)^n\Big\{\sum_{s=0}^{n} V^{p^s}\Big[\sum_{i=s}^{n}(-1)^i Q_{n,i}^p y^{p^{i+1}-p^{s+1}}$$

$$-\Big(\sum_{i=s+1}^{n}(-1)^i Q_{n,i} y^{p^i-p^{s+1}}\Big)\Big(\sum_{i=0}^{n}(-1)^i Q_{n,i} y^{p^i}\Big)^{p-1}\Big]\Big\}.$$

Therefore by Lemma 3.2, we get

$$d_1^* P_1 Q_{n,s} = (-1)^s\Big\{\sum_{i=s}^{n}(-1)^i Q_{n,i}^p y^{p^{i+1}-p^{s+1}}$$

$$-\Big[\sum_{i=s+1}^{n}(-1)^i Q_{n,i} y^{p^i-p^{s+1}}\Big]\Big[\sum_{i=0}^{n}(-1)^i Q_{n,i} y^{p^i}\Big]^{p-1}\Big\}.$$

Hence

$$\begin{aligned}
d &= d_1^* P_1 Q_{n,s} \cdot y^{p^{s+1}} \\
&= \sum_t (-1)^t y^{(p^n - p^s - t)(p-1) + p^{s+1}} P^t(Q_{n,s}) \\
&= (-1)^s \Big\{ \sum_{i=s}^{n} (-1)^i Q_{n,i}^p y^{p^{i+1}} \\
&\qquad - \Big[\sum_{i=s+1}^{n} (-1)^i Q_{n,i} y^{p^i} \Big] \Big[\sum_{i=0}^{n} (-1)^i Q_{n,i} y^{p^i} \Big]^{p-1} \Big\} \\
&= (-1)^s \Big\{ \sum_{i=s}^{n} (-1)^i Q_{n,i}^p y^{p^{i+1}} - \Big[\sum_{i=s+1}^{n} (-1)^i Q_{n,i} y^{p^i} \Big] \cdot \\
&\qquad \Big[\sum_{\substack{I=(i_{s_1},\dots,i_{s_u}) \\ 0 \le k = i_{s_1} + \dots + i_{s_u} < p \\ 0 \le s_1 < \dots < s_u < n}} (-1)^{\kappa(I)} c(I) Q_{n,s_1}^{i_{s_1}} \dots \\
&\qquad \dots Q_{n,s_u}^{i_{s_u}} y^{i_{s_1} p^{s_1} + \dots + i_{s_u} p^{s_u} + (p-1-k)p^n} \Big] \Big\},
\end{aligned}$$

where $\kappa(I) = s_1 i_{s_1} + \dots + s_u i_{s_u} + (p-1-k)n$, and $c(I)$ is defined as in the proof of Theorem A.

$$\begin{aligned}
d = (-1)^s \Big\{ & \sum_{i=s}^{n} (-1)^i Q_{n,i}^p y^{p^{i+1}} \\
& - \sum_{\substack{I=(i_{s_1},\dots,i_{s_u}) \\ 0 \le k = i_{s_1} + \dots + i_{s_u} < p \\ 0 \le s_1 < \dots < s_u < n}} (-1)^{\kappa(I)+n} c(I) Q_{n,s_1}^{i_{s_1}} \dots \\
& \dots Q_{n,s_u}^{i_{s_u}} y^{i_{s_1} p^{s_1} + \dots + i_{s_u} p^{s_u} + (p-k)p^n} \\
& - \sum_{\substack{I=(i_{s_1},\dots,i_{s_u}) \\ 1 \le k = i_{s_1} + \dots + i_{s_u} \le p \\ 0 \le s_1 < \dots < s_m \le s < \\ < s_{m+1} < \dots < s_u < n}} (-1)^{\kappa(I)+n} \sum_{j=m+1}^{u} \frac{(p-1)!}{i_{s_1}! \dots (i_{s_j} - 1)! \dots i_{s_u}! (p-k)!} \\
& \cdot Q_{n,s_1}^{i_{s_1}} \dots Q_{n,s_u}^{i_{s_u}} y^{i_{s_1} p^{s_1} + \dots + i_{s_u} p^{s_u} + (p-k)p^n} \Big\} =
\end{aligned}$$

$$= (-1)^s \Big\{ (-1)^s Q_{n,s}^p y^{p^{s+1}}$$

$$- \sum_{\substack{I=(i_{s_1},\dots,i_{s_u}) \\ 1\le k=i_{s_1}+\dots+i_{s_u}\le p \\ 0\le s_1<\dots<s_m\le s< \\ <s_{m+1}<\dots<s_u<n}} (-1)^{\kappa(I)+n+k} \frac{k! - (k-1)!(i_{s_{m+1}} + \dots + i_{s_u})}{i_{s_1}!\dots i_{s_u}!}$$

$$\cdot Q_{n,s_1}^{i_{s_1}} \dots Q_{n,s_u}^{i_{s_u}} y^{i_{s_1}p^{s_1}+\dots+i_{s_u}p^{s_u}+(p-k)p^n} \Big\}.$$

By the same argument as used in the proof of Theorem A, we obtain

$$P^t(Q_{n,s}) =$$

$$\begin{cases} Q_{n,s} & t = 0, \\ Q_{n,s}^p & t = p^n - p^s, \\ (-1)^{k-1+\kappa(I)+n+s+t} \frac{(k-1)!(i_{s_1}+\dots+i_{s_u})}{i_{s_1}!\dots i_{s_u}!} Q_{n,s_1}^{i_{s_1}} \dots Q_{n,s_u}^{i_{s_u}} & (p^n - p^s - t)\times \\ & (p-1) + p^{s+1} \\ & = i_{s_1}p^{s_1} + \dots \\ & \quad +i_{s_u}p^{s_u} \\ & \quad +(p-k)p^n, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(p^n - p^s - t)(p-1) + p^{s+1} = i_{s_1}p^{s_1} + \dots + i_{s_u}p^{s_u} + (p-k)p^n$, we easily obtain

$$t = -\frac{p^n - p^s}{p-1} + \frac{kp^n - i_{s_1}p^{s_1} - \dots - i_{s_u}p^{s_u}}{p-1},$$

with $1 \le k = i_{s_1} + \dots + i_{s_u} \le p$. In particular, $t = \kappa(I) + n - s \pmod 2$.

By a routine computation we can express $k, i_{s_1}, \dots, i_{s_u}$ in terms of $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$, the coefficients in the p-adic expansion of t. This completes the proof of the theorem.

Proof of Theorem C.

We proced now to prove Theorem C. Notice first that the second part of Theorem C was proved by Mui [3, Lemma II.5.5], using the Cartan formula, as follows. Let $(e_1, \dots, e_n)$ be a sequence of non-negative integers with $e_1 < \dots < e_n$. We have

$$P^t[e_1,\dots,e_n] = \sum_{t_1+\dots+t_n=t} \begin{vmatrix} P^{t_1}y_1^{p^{e_1}} & \dots & P^{t_1}y_n^{p^{e_1}} \\ \vdots & \ddots & \vdots \\ P^{t_n}y_1^{p^{e_n}} & \dots & P^{t_n}y_n^{p^{e_n}} \end{vmatrix}$$

$$= \begin{cases} [e_1+\varepsilon_1,\dots,e_n+\varepsilon_n] & t=\varepsilon_1 p^{e_1}+\dots+\varepsilon_n p^{e_n},\ \varepsilon_i = 0,1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

It remains to prove the first part of Theorem C. Also using the Cartan formula, we have

$$\beta \begin{vmatrix} x_1 & \dots & x_n \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ x_1 & \dots & x_n \\ y_1 & \dots & y_n \\ \cdot & \dots & \cdot \\ \hat{y}_1^{p^{s_1}} & \dots & \hat{y}_n^{p^{s_1}} \\ \cdot & \dots & \cdot \\ \hat{y}_1^{p^{s_k}} & \dots & \hat{y}_n^{p^{s_k}} \\ \cdot & \dots & \cdot \\ y_1^{p^{n-1}} & \dots & y_n^{p^{n-1}} \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} x \\ \cdot \\ \cdot \\ x \end{matrix}}\right\} k \text{ rows} \\ \vphantom{\begin{matrix} y \\ \cdot \\ \hat{y} \\ \cdot \\ \hat{y} \\ \cdot \\ y \end{matrix}} \end{matrix} = (-1)^{k-1} \begin{vmatrix} x_1 & \dots & x_n \\ \cdot & \dots & \cdot \\ x_1 & \dots & x_n \\ y_1 & \dots & y_n \\ y_1 & \dots & y_n \\ \cdot & \dots & \cdot \\ \hat{y}_1^{p^{s_1}} & \dots & \hat{y}_n^{p^{s_1}} \\ \cdot & \dots & \cdot \\ \hat{y}_1^{p^{s_k}} & \dots & \hat{y}_n^{p^{s_k}} \\ \cdot & \dots & \cdot \\ y_1^{p^{n-1}} & \dots & y_n^{p^{n-1}} \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} x \\ \cdot \\ x \end{matrix}}\right\} k-1 \text{ rows} \\ \vphantom{\begin{matrix} y \\ y \\ \cdot \\ \hat{y} \\ \cdot \\ \hat{y} \\ \cdot \\ y \end{matrix}} \end{matrix} .$$

Hence, from the definition of $M_{n,s_1,\dots,s_k}$, we obtain

$$\beta M_{n,s_1,\dots,s_k} = \begin{cases} (-1)^{k-1} M_{n,s_2,\dots,s_k} & s_1 = 0, \\ 0 & s_1 > 0. \end{cases}$$

Similarly, we have

$$P^t M_{n,s_1,\dots,s_k} =$$

$$= \begin{cases} [k;0,\dots,\hat{t}_1,\dots,\hat{t}_k,\dots,n-1] & t = \sum\limits_{1\le i\le k} \frac{p^{s_i}-p^{t_i}}{p-1}, \\ & s_{i-1} \le t_i \le s_i, \\ [k;0,\dots,\hat{t}_1,\dots,\hat{t}_{k+1},\dots,n-1] & t = \frac{p^n-p^{t_{k+1}}}{p-1} + \sum\limits_{1\le i\le k} \frac{p^{s_i}-p^{t_i}}{p-i}, \\ & s_{i-1} < t_i \le s_i,\ s_k < t_{k+1} < n. \\ 0 & \text{otherwise.} \end{cases}$$

The theorem follows by combining [3, Prop. I.4.7], which asserts that

$$[k; e_{k+1}, \ldots, e_n] = (-1)^{\frac{k(k-1)}{2}} \cdot \sum_{0 \le s_1 < \cdots < s_k \le n-1} (-1)^{s_1 + \cdots + s_k} M_{n,s_1,\ldots,s_k} [s_1, \ldots, s_k, e_{k+1}, \ldots, e_n]/L_n$$

for $e_{k+1} < \cdots < e_n \le n$, and the following obvious fact

$$[t_1, \ldots, \hat{t}_i, \ldots, t_{k+1}, 0, \hat{t}_1, \ldots, \hat{t}_{k+1}, \ldots, n] = \\ = (-1)^{\frac{k(k+1)}{2} + t_1 + \cdots + \hat{t}_i + \cdots + t_{k+1} + 1 - i} L_{n,t_i}.$$

Remark. We note that the proof of Theorem C is only based on the Cartan formula and the definition of L_n and $M_{n,s_1,\ldots,s_k}$. Using a similar argument, an alternative proof of Theorems A and B can be given. Indeed, since $V_n L_n = L_{n+1}$ and $Q_{n,s} L_n = L_{n,s}$, by the Cartan formula, we have

$$P^t(V_n) = \frac{P^t(L_{n+1})}{L_n} - \sum_{0<i\le t} P^{t-i}(V_n) \frac{P^i(L_n)}{L_n},$$

$$P^t(Q_{n,s}) = \frac{P^t(L_{n,s})}{L_n} - \sum_{0<i\le t} P^{t-i}(Q_{n,s}) \frac{P^s(L_n)}{L_n}.$$

Using (3.3) we can prove Theorems A and B by induction on k, which is determined by t as follows. Suppose $0 < t < p^m$, then t can uniquely be decomposed in the form:

$$t = \frac{kp^m - i_{s_1} p^{s_1} - \cdots - i_{s_u} p^{s_u}}{p-1} + q(t),$$

where $0 < i_{s_j} < p$, $0 \le s_1 < \cdots < s_u < m-1$, $0 < k < p$, and if

$$q(t) = q_0 + q_1 p + \cdots + q_{m-2} p^{m-2}$$

is the p-adic expansion of $q(t)$, then $q_{s_j - 1} = 0$, for $1 \le j \le u$.

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REFERENCES

1. H. E. A. Campbell, *Upper triangular invariants*, Canad. Math. Bull **28** (1985), 243-248.
2. L. E. Dickson, *A fundamental system of invariants of the general modular linear group with a solution of the form problem*, Trans. Amer. Math. Soc. **12** (1911), 75-98.
3. Huỳnh Mùi, *Modular invariants theory and cohomology algebras of symmetric groups*, J. Fac. Sci., Univ. Tokyo, Sec. IA **22** (1975), 319-369.
4. Huỳnh Mùi, *Homology operations derived from modular coinvariants*, Lecture Notes in Math., Vol. 1172, Springer-Verlag, Berlin and New York, 1984, pp. 85-115.
5. Huỳnh Mùi, *Cohomology operations derived from modular invariants*, Math. Z. **193** (1986), 151-163.
6. I. Madsen, *On the action of the Dyer-Lashof algebra in $H_*(G)$*, Pacific J. Math. **60** (1975), 235-275.
7. I. Madsen and R. I. Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Ann. of Math. Studies, No. 92, Princeton Univ. Press, 1979.
8. B. M. Mann, *The cohomology of the symmetric groups*, Trans. Amer. Math. Soc. **242** (1978), 157-184.
9. B. M. Mann and R. I. Milgram, *On the Chern classes of the regular representations of some finite groups*, Proc. Edinburgh Math. Soc. **25** (1982), 259-268.
10. J. P. May, F. R. Cohen and T. J. Lada, *The cohomology of iterated loop spaces*, Lecture Notes in Math., Vol. 533, Springer-Verlag, Berlin and New York, 1976.
11. J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. **67** (1958), 150-171.
12. Nguyên N. Hai and Nguyên H. V. Hung, *Steenrod operations on mod 2 homology of the iterated loop space*, Acta Math. Vietnam. **13** (1988), 113-126.
13. Nguyên H. V. Hung, *The action of the Steenrod square on the modular invariants of linear groups*, Proc. Amer. Math. Soc. **113** (1991), 1097-1104.
14. W. M. Singer, *On the construction of certain algebras over the Steenrod algebra*, J. Pure and Appl. Algebra **11** (1977), 53-59.
15. W. M. Singer, *The transfer in homological algebrs*, Math. Z. **202** (1989), 493-523.
16. L. Smith and R. Switser, *Realizability and non-realizability of Dickson algebras as cohomology rings*, Proc. Amer. Math. Soc. **89** (1983), 303-313.
17. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Studies, No. 50, Princeton Univ. Press, 1962.
18. C. Wilkerson, *A primer on the Dickson invariants*, Contemporary Math., Amer.

Math. Soc., Providence, R. I., [illegible] pp. 421-434.

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