Tạp chí Toán học Journal of Mathematics **22**(3 & 4)(1994), 83-109

NUMERICAL SOLUTION FOR NONLINEAR PERIODIC BOUNDARY-VALUE PROBLEMS

PHAM KY ANH *and* BUI DUC TIEN

Abstract. *An approximate method for nonlinear periodic boundary-value problems (PB-VP) is discussed. Under suitable hypotheses, the stability and convergence of the scheme are established. Some numerical examples are given.*

1. INTRODUCTION

The most frequently used method for the numerical solution of nonlinear PBVP is the finite-difference method. This method is, at least conceptually, easy to perform. However, it may be difficult to solve the obtained systems of nonlinear equations and to verify the stability of the scheme.

This paper deals with a numerical method for solving the following PBVP:

$$y^{(n)} = g(t, y, \dot{y}, ..., y^{(n)}), \quad t \in (0, \omega), \tag{1.1}$$

$$y^{(i)}(0) = y^{(i)}(\omega) \quad (i = \overline{0, n-1}); \quad y, g \in \mathbb{R}^1. \tag{1.2}$$

The simple case, when $n = 1$, has been studied in our previous work [3]. For recent presentations of the Seidel - Newton method, see [1-7].

Problem (1.1), (1.2) may be written in the vector form

$$\dot{x} = Ax + f(t, x, \dot{x}); \quad x(0) = x(\omega), \tag{1.3}$$

where $x = (y, \dot{y}, ..., y^{(n-1)})^T$; $f = (0, 0, ..., g)^T$; $x, f \in \mathbb{R}^n$, and

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Discretizing (1.3) we obtain a system of nonlinear equations with a linear singular part:

$$(x_{k+1}-x_k)/h = Ax_k + f(t_k, x_k, (x_{k+1}-x_k)/h)\,, \tag{1.4}$$

$$x_0 = x_N\,, \tag{1.5}$$

$$k=\overline{0,N-1};\quad t_i = ih \quad (i=\overline{0,N});\quad Nh=\omega.$$

We shall use the notation $|.|$ for the max-norm of vectors and the corresponding norm of matrices. The transpose of a matrix or vector will bear a super script T, and the scalar product in $\mathbb{R}^n$ will be denoted by $\langle .,. \rangle$. For any linear operator A, $\mathcal{N}(A)$ and $\mathcal{R}(A)$ will denote the null space and the range of A respectively.

Problem (1.4), (1.5) is reduced to the operator equation

$$\mathcal{A}_h x_h = F_h(x_h)\,, \tag{1.6}$$

where $[\mathcal{A}_h x_h]_k := (x_{k+1}-x_k)/h - Ax_k$; $[F_h(x_h)]_k := f(t_k, x_k, (x_{k+1}-x_k)/h)$; $(k=\overline{0,N-1})$; $\mathcal{A}_h, F_h : X_h \longrightarrow Y_h$,

$$X_h = \{x_h = (x_0, ..., x_N) \in \mathbb{R}^{(N+1)n} : x_0 = x_N\}\,;$$
$$Y_h = \{y_h = (y_0, ..., y_{N-1}) \in \mathbb{R}^{Nn}\}\,;$$

$$||y_h||_h = \max_{0\le j\le N-1} |y_j|\,;\quad |||x_h|||_h = \max_{0\le j\le N} |x_j| + h^{-1} \max_{0\le j\le N-1} |x_{j+1}-x_j|\,;$$

$\dim X_h = \dim Y_h = Nn$.

We always assume that $C_k^i = 0,\ \forall i > k$. Denote by $\tilde{e} \in Y_h$ and $e \in \mathbb{R}^{(N+1)n}$ the vectors, whose elements are : $\tilde{e}_k = (C_k^{n-1}h^{n-1}, C_k^{n-2}h^{n-2}, ..., C_k^1 h, 1)^T$ and $(e)_k = e_1 \quad (k=\overline{0,N})$ respectively, where $e_1 = (1,0,...,0)^T \in \mathbb{R}^n$.

A simple computation shows that

$$(E+hA)^k = \begin{pmatrix} 1 & hC_k^1 & h^2C_k^2 & \dots & h^{n-1}C_k^{n-1} \\ 0 & 1 & hC_k^1 & \dots & h^{n-2}C_k^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \qquad (k=\overline{0,N}).$$

The remainder of the paper is organized as follows: in Section 2, an abstract Seidel - Newton method is described. Then, in Section 3, some basic properties of the linear and nonlinear difference operators are studied. In Section 4, we state some existence and uniqueness theorems for continuous problem (1.3) (see [2,6]). Section 5 deals with an iterative method for solving nonlinear system (1.4)–(1.5). Section 6 is concerned with the stability and convergence of scheme (1.4)–(1.5). Finally, some numerical results are presented in Section 7 and we conclude with a discussion of the results.

2. SEIDEL - NEWTON METHOD

Let us consider the problem

$$\mathcal{A}x = F(x)\,, \tag{2.1}$$

where $\mathcal{A} : X \longrightarrow Y$ is a bounded linear Fredholm operator (of index zero), $F : X \longrightarrow Y$ is a possibly nonlinear operator, X and Y are Banach spaces.

Together with (2.1) we consider the following "discrete" problem

$$\mathcal{A}_h x = F_h(x)\,, \tag{2.2}$$

where $\mathcal{A}_h, F_h : X_h \longrightarrow Y_h$ are linear and possibly nonlinear operators respectively, and X_h, Y_h are finite-dimensional Banach spaces. We always suppose that $\dim X_h = \dim Y_h < \infty$, hence $\mathcal{A}_h$ is a Fredholm operator (of index zero).

Consider the diagram

$$\begin{array}{ccc} X_0 \subset X & \xrightarrow{\mathcal{A},F} & Y \\ \pi_h \big\downarrow & & \big\downarrow \tau_h \\ X_h & \xrightarrow{\mathcal{A}_h,F_h} & Y_h \end{array} \tag{2.3}$$

where X_0 is a set containing a solution x^* of "continuous" problem (2.1). π_h, τ_h are bounded linear operators which satisfy

$$||\pi_h x||_h \longrightarrow ||x|| \quad (h \longrightarrow 0)\,, \quad \forall x \in X\,; \quad ||\tau_h y||_h \longrightarrow ||y|| \quad (h \longrightarrow 0)\,, \quad \forall y \in Y.$$

Suppose that the discrete and continuous problems (2.1), (2.2) are compatible in the following sense:

a) Diagram (2.3) is asympotically commutative on X_0, i.e.

$$||\mathcal{A}_h(\pi_h x) - \tau_h(\mathcal{A}x)||_h \longrightarrow 0 \quad (h \longrightarrow 0)\,, \quad \forall x \in X_0\,,$$

$$||F_h(\pi_h x) - \tau_h(F(x))||_h \longrightarrow 0 \quad (h \longrightarrow 0)\,, \quad \forall x \in X_0.$$

b) The scheme (2.2) is stable, i.e. $\exists \varepsilon_0, h_0 > 0, \forall h < h_0, \forall y_h \in Y_h : ||y_h||_h < \varepsilon_0$, the perturbed problem $\mathcal{A}_h x = F_h(x) + y_h$ has a unique solution $x_h \in \Omega_h \subset X_h$.

Moreover, suppose that $\forall h < h_0, \pi_h x^* \in \Omega_h$, and $||\bar{x}_h - x_h||_h \le C_1||y_h||_h$, where $\bar{x}_h$ is a solution of problem (2.2). Then under some additional hypotheses we can apply the Seidel - Newton method to discrete problem (2.2). Let $x_h^{(m)}$ be a m-th approximation of $\bar{x}_h$ defined by the Seidel - Newton method and let

$$||x_h^{(m)} - \bar{x}_h||_h \le C_2 q_h^m,$$

where $0 \le q_h \le q < 1$. Then $||x_h^{(m)} - \pi_h x^*||_h \le \varphi(h) + C_2 q_h^m$, where $\varphi(h) := C_1\{||\mathcal{A}_h \pi_h x^* - \tau_h \mathcal{A} x^*||_h + ||F_h(\pi_h x^*) - \tau_h F(x^*)||_h\}$.

In many cases an estimate for $\varphi(h)$ can be found. Thus, to find x^* with an accuracy $\varepsilon > 0$, we first choose the sufficiently small step $h > 0$ such that $\varphi(h) < \varepsilon/2$. With that fixed h we make $m \ge m_0(h)$ iterations to get $C_2 q_h^m < \varepsilon/2$.

Before concluding this section, we collect some facts which will be of later use. To simplify notations, we are going to state results for problem (2.1). The same results hold for problem (2.2).

Since $\mathcal{A}$ is a Fredholm operator, X, Y can be decomposed into direct sums of closed subspaces: $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$, where $Y_1 = \mathcal{R}(\mathcal{A}) \subset Y$, $X_2 = \mathcal{N}(\mathcal{A}) \subset X$, and codim $Y_1 =$ dim $X_2 < \infty$. Further, the restriction $\hat{\mathcal{A}}$ of $\mathcal{A}$ to X_1 has a bounded inverse. Let P and Q be the bounded linear projections satisfying conditions

$$\mathcal{R}(P) = \mathcal{N}(Q) = Y_1; \quad \mathcal{R}(Q) = \mathcal{N}(P) = Y_2.$$

Theorem 1.1 (see [1-3]). *Assume that the mapping $F : \Omega \subset X \longrightarrow Y$ is continuously Fréchet differentiable in an open set Ω including the closed ball S with center at x_0 and radius $r > 0$ and satisfies the conditions*

$$||PF'(x)|| \le \alpha; \quad ||QF'(x)|| \le \beta; \quad ||[QF'(x)]^{-1}_{X_2}|| \le \gamma;$$

$$||QF'(x) - QF'(y)|| \le \rho(||x - y||) \quad (\forall x, y \in S),$$

where $\rho(t)$ is a continuous, nondecreasing function and $\rho(0) = 0$, $\rho(t) \ge 0$.

If the coefficients α, β, γ, the radius r and the initial approximation x_0 satisfy the relations

$$q := 2\alpha\beta\gamma||\hat{\mathcal{A}}^{-1}|| + \gamma\int_0^1 \rho(\delta t)dt < 1 \quad \textit{and} \quad 2\delta(1-q)^{-1} < r,$$

where $\delta := \beta\gamma||\hat{\mathcal{A}}^{-1}||\,||\mathcal{A}x - PF(x_0)|| + \gamma||QF(x_0)||$, then the sequence $\{x_n\}$, constructed by the Seidel - Newton method

$$u_{n+1} = -\hat{\mathcal{A}}^{-1} PF(x_n);$$
$$\tilde{x}_n = u_{n+1} + v_n; \quad v_{n+1} = v_n - [QF'(\tilde{x}_n)]^{-1}_{X_2} QF(\tilde{x}_n);$$
$$x_{n+1} = u_{n+1} + v_{n+1} \quad (u_n \in X_1; \quad v_n \in X_2)$$

converges to a solution $x^* \in S(x_0, r)$ of (2.1) at the rate:

$$||x_n - x^*|| \le 2\delta(1-q)^{-1} q^n < rq^n.$$

Besides, if $\dim \mathcal{N}(\mathcal{A}) = 1$, then x^* is the unique solution of problem (2.1) in the set $\Omega_0 = \{x \in S(x_0, r) : ||v - v_0|| \le r/2\}$, where v, v_0 are projections of x, x_0 onto $X_2 = \mathcal{N}(\mathcal{A})$ respectively.

3. LINEAR AND NONLINEAR DIFFERENCE OPERATORS

3.1. Basic properties of the linear difference operator

Theorem 3.1. *The mapping $\mathcal{A}_h : X_h \longrightarrow Y_h$ is bounded linear Fredholm operator (index zero). Moreover,* $\mathcal{N}(\mathcal{A}_h) = X_2^h \equiv \text{span}\,\{e\}$; $\mathcal{R}(\mathcal{A}_h) = Y_1^h \equiv \{y_h \in Y_h : \sum_{k=0}^{N-1} y_{k,n} = 0\}$; $X_h = X_1^h \oplus X_2^h$, $X_1^h = \{x_h : \langle \sum_{k=0}^{N} (E + hA)^{N-k} x_k, e_1\rangle = 0\}$, $Y_h = Y_1^h \oplus Y_2^h$, $Y_2^h = \text{span}\,\{\tilde{e}\}$.

Proof. First, we prove the following relations:

i) $\mathcal{N}(\mathcal{A}_h) = X_2^h$.

ii) $\mathcal{R}(\mathcal{A}_h) = Y_1^h$.

iii) $X_h = X_1^h \oplus X_2^h$.

iv) $Y_h = Y_1^h \oplus Y_2^h$.

Then, from (i)-(iv) it is obvious that $\mathcal{A}_h$ is a Fredholm operator.

i) Note that $x_h \in \mathcal{N}(\mathcal{A}_h)$ iff $\forall k = \overline{0, N-1}$ $(x_{k+1} - x_k)/h - Ax_k = 0$ or $x_{k+1} = (E + hA)x_k$, $k = \overline{0, N-1}$. If $x_h \in \mathcal{N}(\mathcal{A}_h)$ then $Bx_0 = 0$, where $B := (E + hA)^N - E$, hence $x_0 = \alpha e_1$ and therefore $x_h = \alpha e$.

Conversely, if $x_h = \alpha e$ then $x_h \in \mathcal{N}(\mathcal{A}_h)$.

ii) The relation $y_h \in \mathcal{R}(\mathcal{A}_h)$ means that there exists an element $x_0 \in \mathbb{R}^n$ such that

$$x_{k+1} = (E + hA)^{k+1} x_0 + h \sum_{s=0}^{k} (E + hA)^{k-s} y_s \quad (k = \overline{0, N-1}),$$

$$Bx_0 + h \sum_{i=0}^{N-1} (E + hA)^{N-k-1} y_i = 0.$$

Multiplying both sides of the last equality by e_n and noticing that $\langle Bx_0, e_n \rangle = 0$, we get

$$h \sum_{i=0}^{N-1} \langle (E + hA)^{N-1-i} y_i, e_n \rangle = 0.$$

Therefore $\displaystyle\sum_{i=0}^{N-1} y_{i,n} = 0.$

iii) First, we show that $X_1^h \cap X_2^h = \{0\}$. Indeed, if $x_h \in X_1^h \cap X_2^h$ then $x_h = \alpha e$.

Since $x_h \in X_1^h$, we have $0 = \alpha \sum_{k=0}^{N} \langle (E + hA)^{N-k} e_1, e_1 \rangle = \alpha$, thus $x_h = 0$.

Let $x_h \in X_h, \ddot{x}_h = \alpha e \in X_2^h, \dot{x}_h := x_h - \ddot{x}_h$. Choosing $\alpha \in \mathbb{R}^1$ such that $\dot{x}_h \in X_1$, i.e.

$$0 = \langle e_1, \sum_{k=0}^{N} (E+hA)^{N-k} \dot{x}_k \rangle = \langle e_1, \sum_{k=0}^{N} (E+hA)^{N-k} x_k \rangle - \alpha \sum_{k=0}^{N} \langle (E + hA)^{N-k} e_1, e_1 \rangle$$

we obtain $\alpha = \dfrac{1}{N+1} \sum_{k=0}^{N} \langle (E + hA)^{N-k} x_k, e_1 \rangle$. Thus the relation $X_h = X_1^h \oplus X_2^h$ has been proved.

iv) Finally, let $y_h \in Y_1^h \cap Y_2^h$ then $y_h = \alpha \tilde{e}$. The inclusion $y_h \in Y_1^h$ implies that $0 = \alpha \sum_{k=0}^{N-1} \tilde{e}_{k,n} = \alpha N$, hence $\alpha = 0$, and therefore $y_h = 0$.

For every $y_h \in Y_h$, we put $\ddot{y}_h = \alpha \tilde{e} \in Y_2^h$ and choose α such that $\dot{y}_h := y_h - \ddot{y}_h \in Y_1^h$. It is obvious that $\alpha = \dfrac{1}{N} \sum_{k=0}^{N-1} y_{k,n}$, and hence the relation (iv) has been proved.

Theorem 3.2. *The restriction $\hat{\mathcal{A}}_h$ of the operator $\mathcal{A}_h$ to X_1^h has a uniformly bounded inverse defined by the formula*

$$x_h := \hat{\mathcal{A}}_h^{-1} y_h\,; \quad x_{k+1} = (E + hA)^{k+1} x_0 + h\sum_{i=0}^{k}(E + hA)^{k-i} y_i \quad (k = \overline{0, N-1})\,;$$

$x_0 = (\xi_1, ..., \xi_n)^T \in \mathbb{R}^n$:

$$\begin{pmatrix} hC_N^1 & h^2C_N^2 & \dots & h^{n-1}C_N^{n-1} \\ 0 & hC_N^1 & \dots & h^{n-2}C_N^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & hC_N^1 \end{pmatrix} \begin{pmatrix} \xi_2 \\ \xi_3 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-1} \end{pmatrix}, \tag{3.1}$$

$$\xi_1 = -\eta_1 - \frac{1}{N+1}\hat{\eta}_1, \tag{3.2}$$

where $\eta = -h\sum_{k=0}^{N-1}(E + hA)^{N-k-1} y_k\,, \quad \hat{\eta} = h\sum_{k=1}^{N}\sum_{s=0}^{k-1}(E + hA)^{N-s-1} y_s\,.$

Finally there holds the estimation

$$||\hat{\mathcal{A}}_h^{-1}|| \leq \rho_1 \tag{3.3}$$

(ρ_1 will be shown just below).

Proof. Suppose that $y_h \in Y_1^h$ and $\eta := -h\sum_{k=0}^{N-1}(E + hA)^{N-k-1} y_k$. As $y_h \in Y_1^h$ we have $\eta = \sum_{i=1}^{n-1}\eta_i e_i$. Let $\hat{\eta} := h\sum_{k=1}^{N}\sum_{i=0}^{k-1}(E + hA)^{N-i-1} y_i = \sum_{i=1}^{n}\hat{\eta}_i e_i$.

Putting $x_{k+1} = (E+hA)^{k+1}x_0 + h\sum_{i=0}^{k}(E+hA)^{k-i} y_i \quad (k = \overline{0, N-1})$, where $x_0 = (\xi_1, ..., \xi_n)^T$ and using relation (3.1) we have $\mathcal{A}_h x_h = y_h\,, x_h \in X_h$. Further, choosing ξ_1 such that $x_h \in X_1^h$ and noting that $x_0 = x_N = (E + hA)^N x_0 - \eta$, we get $0 = \langle e_1, \sum_{k=0}^{N}(E + hA)^{N-k} x_k\rangle = \langle (N+1)(E + hA)^N x_0 + h\sum_{k=0}^{N}\sum_{s=0}^{k-1}(E + hA)^{N-s-1} y_s, e_1\rangle = \langle (N+1)(x_0 + \eta)\,, e_1\rangle + \langle \hat{\eta}, e_1\rangle$, thus $\xi_1 = -\eta_1 - \frac{1}{N+1}\hat{\eta}_1$.

Setting $q := |E + hA| = 1 + h$, we have

$$|\eta| \le h \sum_{k=0}^{N-1} |E + hA|^{N-k-1} |y_k| \le h \sum_{k=0}^{N-1} q^{N-k-1} ||y_h||_h$$
$$= [(1+h)^N - 1]||y_h||_h \le (e^\omega - 1)||y_h||_h.$$

Note that $|\hat{\eta}| \le h\sum_{k=1}^{N}\sum_{i=0}^{k-1} q^{N-i-1}||y_h||_h \le \Big(|\omega - 1|\frac{(1+h)^N}{h} + \frac{1}{h}\Big)||y_h||_h$. Hence $|\hat{\eta_1}| \le \Big\{\frac{|\omega-1|}{\omega}e^\omega + \frac{1}{\omega}\Big\}||y_h||_h$.

Now we shall estimate the norm of x_0. We have

$$|x_0| = \max_{1\le i\le n} |\xi_i| \le \max\{|\xi_1|, |\mathcal{C}_N^{-1}|\,|\eta|\},$$

where

$$\mathcal{C}_N := \begin{pmatrix} hC_N^1 & h^2C_N^2 & \dots & h^{n-1}C_N^{n-1} \\ 0 & hC_N^1 & \dots & h^{n-2}C_N^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & hC_N^1 \end{pmatrix}.$$

Since $|\xi_1| \le |\eta_1| + \frac{|\hat{\eta}_1|}{N+1} \le \Big\{e^\omega - 1 + \frac{|\omega-1|}{\omega}e^\omega + \frac{1}{\omega}\Big\}||y_h||_h$ and $|\eta| \le (e^\omega - 1)||y_h||_h$ it follows that $|x_0| \le C_0||y_h||_h$, where

$$C_0 := \max\Big\{e^\omega - 1 + \frac{|\omega-1|}{\omega}e^\omega + \frac{1}{\omega}\,; |\mathcal{C}_N^{-1}|(e^\omega - 1)\Big\}. \tag{3.4}$$

From

$$|x_{k+1}| \le |E+hA|^{k+1}|x_0| + h\sum_{i=0}^{k}|E+hA|^{k-i}||y_h||_h$$
$$\le \Big(q^{k+1}C_0 + h\frac{q^{k+1}-1}{q-1}\Big)||y_h||_h$$
$$= [(C_0+1)q^{k+1} - 1]||y_h||_h \qquad (k = \overline{0, N-1}),$$

it follows that $\max_{0\le k\le N} |x_k| \le [(C_0+1)q^N - 1]||y_h||_h$. Thus

$$|||x_h|||_h \le \max_k |x_k| + \frac{1}{h}\max_j |x_{j+1} - x_j| \le [2(C_0+1)e^\omega - 1]||y_h||_h.$$

Summing up, we obtain that $||\hat{\mathcal{A}}_h^{-1}|| \le \rho_1 := 2(C_0+1)e^\omega - 1$, where C_0 is defined by (3.4).

3.2. Nonlinear difference operator

3.2.1. Bounded linear projections in Y_h

Let $Q_h y_h := \dfrac{1}{N}\sum_{k=0}^{N-1} y_{k,n}\tilde{e} \in Y_2^h$. We first show that Q_h is a bounded linear projection with $\mathcal{R}(Q_h) = Y_2^h$ and $\mathcal{N}(Q_h) = Y_1^h$.

Indeed, $Q_h^2 y_h = Q_h\left[\dfrac{1}{N}\sum_{k=0}^{N-1} y_{k,n}\tilde{e}\right] = \dfrac{1}{N}\sum_{k=0}^{N-1} y_{k,n}\dfrac{1}{N}\sum_{k=0}^{N-1}\tilde{e}_{k,n}\tilde{e} = Q_h y_h$, therefore $Q_h^2 = Q_h$.

Note that

$$\begin{aligned}||\tilde{e}||_h &= \max_{0\le k\le N-1}|\tilde{e}_k| \le \max_{0\le k\le n-1} C_N^k h^k = \max_{0\le k\le n-1}\frac{N(N-1)\cdots(N-k+1)}{k!}\frac{\omega^k}{N^k}\\ &= \max_{0\le k\le n-1}\frac{\omega^k}{k!}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{k-1}{N}\right) \le \max_{0\le k\le n-1}\frac{\omega^k}{k!}.\end{aligned}$$

Since $C_1 \equiv \max \omega^k/k! = \begin{cases}\omega^{n-1}/(n-1)!; & \omega \ge n\\ \omega^{n_0-1}/(n-1)!; & 0 \le n_0 - 1 \le \omega < n_0 \le n,\end{cases}$

and $||Q_h y_h||_h \le \frac{1}{N}N||y_h||_h\,||\tilde{e}||_h$ we have $||Q_h|| \le C_1$.

Clearly, $P_h := I_h - Q_h$ is also a bounded linear projection and $\mathcal{R}(P_h) = Y_1^h$, $\mathcal{N}(P_h) = Y_2^h$. Moreover $||P_h|| \le 1 + C_1$.

3.2.2. Some hypotheses on g and its derivatives

In the sequel we assume that the following hypotheses are satisfied:

h1) $g(t,\xi_1,\ldots,\xi_{n+1}) : [0,\omega]\times\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^1;\quad g, \dfrac{\partial g}{\partial\xi_i} \in C(\triangle);$

$$\left|\frac{\partial g}{\partial\xi_i}(t,\xi)\right| \le a \qquad (i=\overline{1,n+1}),\quad \text{for all} \quad (t,\xi)\in\triangle,$$

where $\triangle = \{(t,\xi)\in[0,\omega]\times\mathbb{R}^{n+1} : |\xi_i| \le R,\quad (i=\overline{1,n+1})\}$.

h2) $\left|\dfrac{\partial g}{\partial\xi_i}(t,\xi) - \dfrac{\partial g}{\partial\xi_i}(t,\tilde{\xi})\right| \le L\sum_{j=1}^{n+1}|\xi_j - \tilde{\xi}_j| \qquad (t,\xi),(t,\tilde{\xi})\in\triangle \quad (i=\overline{1,n+1}).$

h3) $\exists a \in C[0,\omega] : \displaystyle\int_0^\omega a(s)\,ds > 0$ and $\forall(t,\xi)\in\triangle,\quad \dfrac{\partial g}{\partial\xi_1} \ge a(t).$

Set $\Omega_h = \{x_h \in X_h : |||x_h|||_h \le R\}$.

Theorem 3.3. *Suppose that the function $g : \Delta \longrightarrow \mathbb{R}^1$ is continuous in the first variable and continuously differentiable in remaining variables. Moreover, assume that the conditions (h1)-(h3) hold. Then the operator $F_h(x_h)$ is continuously differentiable (in the Fréchet sense) and its derivative satisfies the inequalities:*

$$||F_h'(x_h) - F_h'(\tilde{x}_h)|| \leq Ln^2|||x_h - \tilde{x}_h|||_h, \quad \forall x_h, \tilde{x}_h \in \Omega_h,$$

$$||P_hF_h'(x_h)|| \leq \alpha := (1 + C_1)na\,; \quad ||Q_hF_h'(x_h)|| \leq \beta := C_1na.$$

Furthermore, the restriction of $[Q_hF_h'(x_h)]$ to X_2^h is given by the formula

$$\forall v_h = \lambda e, \quad [Q_hF_h'(x_h)]_{X_2^h}\, v_h = \frac{\lambda\tilde{e}}{N}\sum_{k=0}^{N-1}\frac{\partial g}{\partial \xi_1}\left(t_k, x_k, \frac{x_{k+1,n} - x_{k,n}}{h}\right).$$

If we denote by $\gamma = \left(\int_0^\omega a(s)ds\right)^{-1}$ and let N be sufficiently large such that $h\sum_{k=0}^{N-1} a(kh) \geq \gamma^{-1} - \varepsilon > 0$ and $|C_N^{-1}| < |C^{-1}| + \varepsilon$, where

$$C = \begin{pmatrix} \omega & \omega^2/2! & \dots & \omega^{n-1}/(n-1)! \\ 0 & \omega & \dots & \omega^{n-2}/(n-2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega \end{pmatrix}.$$

Then $\forall x_h \in \Omega_h, \quad |||[Q_hF_h'(x_h)]^{-1}_{X_2^h}|| \leq \dfrac{\gamma}{1 - \gamma\varepsilon}\cdot\dfrac{\omega}{C_1}.$

Proof. Let $x_h = (x_0, x_1, ..., x_N) \in \Omega_h$, $x_k = (x_{k,1}, ..., x_{k,n})^T \in \mathbb{R}^n$, $v_h = (v_0, v_1, ..., v_N) \in X_h$, $v_k = (v_{k,1}, ..., v_{k,n})^T \in \mathbb{R}^n$, $(k = \overline{0,N})$. From $[F_h'(x_h)v_h]_j = (0, ..., 0, \eta_{j,n})^T$ $(j = \overline{0, N-1})$, where

$$\eta_{j,n} = \sum_{i=1}^n \frac{\partial g}{\partial \xi_i}\left(t_j, x_j, \frac{x_{j+1,n} - x_{j,n}}{h}\right) v_{j,i} + \frac{1}{h}\frac{\partial g}{\partial \xi_{n+1}}\left(t_j, x_j, \frac{x_{j+1,n} - x_{j,n}}{h}\right)(v_{j+1,n} - v_{j,n})$$

and using hypothesis (h1) we get

$$|[F_h'(x_h)v_h]_j| = |\eta_{j,n}| \leq a\Big(\sum_{i=1}^n |v_{j,i}| + \frac{1}{h}|v_{j+1} - v_j|\Big) \leq na|||v_h|||_h.$$

Therefore $||F_h'(x_h)|| \le na$.

Further, $\{[F_h'(x_h) - F_h'(\tilde{x}_h)]v_h\}_j = (0, ..., 0, \eta_{j,n} - \tilde{\eta}_{j,n})^T \quad (j = \overline{0, N-1})$, and from hypothesis (h2) we have

$$
\begin{aligned}
&|\{[F_h'(x_h) - F_h'(\tilde{x}_h)]v_h\}_j| \\
&= |\eta_{j,n} - \tilde{\eta}_{j,n}| \le \sum_{i=1}^{n} \Big| \frac{\partial g}{\partial \xi_i}\left(t_j, x_j, \frac{x_{j+1,n} - x_j}{h}\right) \\
&- \frac{\partial g}{\partial \xi_i}\left(t_j, \tilde{x}_j, \frac{\tilde{x}_{j+1,n} - \tilde{x}_{j,n}}{h}\right)\Big| \, |v_{j,i}| + \frac{1}{h}\Big| \frac{\partial g}{\partial \xi_{n+1}}\left(t_j, x_j, \frac{x_{j+1,n} - x_{j,n}}{h}\right) \\
&- \frac{\partial g}{\partial \xi_{n+1}}\left(t_j, \tilde{x}_j, \frac{\tilde{x}_{j+1,n} - \tilde{x}_{j,n}}{h}\right)\Big| \, |v_{j+1,n} - v_{j,n}| \\
&\le \sum_{i=1}^{n} \left\{ L\left(\sum_{s=1}^{n} |x_{j,s} - \tilde{x}_{j,s}| + \left| \frac{x_{j+1,n} - \tilde{x}_{j+1,n}}{h} + \frac{x_{j,n} - \tilde{x}_{j,n}}{h} \right| \right) |v_{j,i}| \right\} \\
&+ L\left(\sum_{s=1}^{n} |x_{j,s} - \tilde{x}_{j,s}| + \left| \frac{x_{j+1,n} - \tilde{x}_{j+1,n}}{h} + \frac{x_{j,n} - \tilde{x}_{j,n}}{h} \right| \right)\left(\left| \frac{v_{j+1,n} - v_{j,n}}{h} \right| \right) \\
&= L\left(\sum_{s=1}^{n} |x_{j,s} - \tilde{x}_{j,s}| + \left| \frac{x_{j+1,n} - \tilde{x}_{j+1,n}}{h} + \frac{x_{j,n} - \tilde{x}_{j,n}}{h} \right| \right) \\
&\quad \times \left(\sum_{i=1}^{n} |v_{j,i}| + \left| \frac{v_{j+1,n} - v_{j,n}}{h} \right| \right) \\
&\le L \sum_{s=1}^{n} \left(|x_j - \tilde{x}_j| + \left| \frac{(x_{j+1} - \tilde{x}_{j+1}) - (x_j - \tilde{x}_j)}{h} \right| \right) \sum_{i=1}^{n} \left(|v_j| + \left| \frac{v_{j+1} - v_j}{h} \right| \right) \\
&\le Ln^2 |||x_h - \tilde{x}_h|||_h \; |||v_h|||_h.
\end{aligned}
$$

These inequalities imply that $||F_h'(x_h) - F_h'(\tilde{x}_h)||_h \le Ln^2 |||x_h - \tilde{x}_h|||_h$.

Finally, for

$$
\begin{aligned}
v_h = \lambda e, \quad [F_h'(x_h)v_h]_{k,n} &= \eta_{k,n} \\
&= \lambda \sum_{i=1}^{n} \frac{\partial g}{\partial \xi_i}\left(t_k, x_k, \frac{x_{k+1,n} - x_{k,n}}{h}\right) e_{1,i} \\
&\quad + \frac{\lambda}{h} \frac{\partial g}{\partial \xi_{n+1}}\left(t_k, x_k, \frac{x_{k+1,n} - x_{k,n}}{h}\right)(e_{1,n} - e_{1,n}) \\
&= \lambda \frac{\partial g}{\partial \xi_1}\left(t_k, x_k, \frac{x_{k+1,n} - x_{k,n}}{h}\right).
\end{aligned}
$$

Thus

$$[Q_h F_h'(x_h)]_{X_2^h} v_h = \frac{1}{N}\sum_{k=0}^{N-1} [F_h'(x_h)v_h]_{k,n}\tilde{e}$$

$$= \frac{\lambda}{N}\sum_{k=0}^{N-1} \frac{\partial g}{\partial \xi_1}\left(t_k, x_k, \frac{x_{k+1,n}-x_{k,n}}{h}\right)\tilde{e}.$$

Using the last hypothesis (h3) we get

$$|||[Q_h F_h'(x_h)]_{X_2^h} v_h||_h = \frac{|||v_h|||_h}{\omega}||\tilde{e}||_h h \left|\sum_{k=0}^{N-1}\frac{\partial g}{\partial \xi_1}\left(t_k, x_k, \frac{x_{k+1,n}-x_{k,n}}{h}\right)\right|$$

$$\geq \frac{C_1}{\omega} h \sum_{k=0}^{N-1} a(th)|||v_h|||_h \geq \frac{C_1}{\omega}\left(\frac{1-\gamma\varepsilon}{\gamma}\right)|||v_h|||_h.$$

For sufficiently large N we have $|||[Q_h F_h'(x_h)]_{X_2^h}^{-1}|| \leq \dfrac{\omega}{C_1}\dfrac{\gamma}{1-\gamma\varepsilon}$.

Noting that $C_1 = \max\limits_{0\leq k\leq n-1} \dfrac{\omega^k}{k!} \geq \omega$ we can rewrite the last inequality as

$$|||[Q_h F_h'(x_h)]_{X_2^h}^{-1}|| \leq \gamma/(1-\gamma\varepsilon).$$

4. ANALYSIS OF THE CONTINUOUS PROBLEM

In order to show the stability of difference scheme (1.4) and to prove the convergence of an iterative process for solving (1.4), we should consider problem (1.3), which can be written in the operator form

$$\mathcal{A}x = F(x)\,, \tag{4.1}$$

where $\mathcal{A}, F : X \longrightarrow Y$; $\mathcal{A}x = \dot{x} - Ax$; $F(x) = f(t, x, x)$; $X = \{x \in C^1([0,\omega], \mathbb{R}^n) : x(0) = x(\omega)\}$; $Y = C([0,\omega], \mathbb{R}^n)$; $|||x||| = ||x|| + ||\dot{x}||$; $||y|| = \max\limits_t |y(t)|$.

In this section we only state some results. The reader is refered to [1-3] for details.

4.1. Linear Fredholm operator

Theorem 4.1. *The mapping $\mathcal{A} : X \longrightarrow Y$ is a bounded linear Fredholm operator with $\mathcal{N}(\mathcal{A}) = X_2 =$ span $\{e_1\}$; $\mathbb{R}(\mathcal{A}) = Y_1 = \left\{y \in Y : \int_0^\omega y_n(s)ds = 0\right\}$. Further X, Y are decomposed into direct sums of closed subspaces :*

$$X = X_1 \oplus X_2, \quad Y = Y_1 \oplus Y_2,$$

where

$$X_1 = \left\{x \in X : \sum_{k=1}^{n} \int_0^\omega \frac{(\omega - s)^{k-1}}{(k-1)!} x_k(s)ds = 0\right\},$$

$$Y_2 = \text{ span } \{\tilde{e}(t)\}; \quad \tilde{e}(t) := \left(\frac{t^{n-1}}{(n-1)!}, \frac{t^{n-2}}{(n-2)!}, ..., t, 1\right)^T.$$

Moreover, the restriction $\hat{\mathcal{A}}$ of the operator $\mathcal{A}$ to X_1 has a uniformly bounded inverse defined by the formula

$$\hat{\mathcal{A}}^{-1} y = e^{tA}\xi + \int_0^t e^{(t-s)A} y(s)ds \quad (\forall y \in Y_1),$$

where $\xi = (\xi_1, ..., \xi_n)^T$, whose components $\xi_2, ..., \xi_n$ are determined from a linear triangular system:

$$\begin{pmatrix} \omega & \omega^2/2! & \dots & \omega^{n-1}/(n-1)! \\ 0 & \omega & \dots & \omega^{n-2}/(n-2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega \end{pmatrix} \begin{pmatrix} \xi_2 \\ \xi_3 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-1} \end{pmatrix}$$

and $\xi_1 = -\eta_1 - \frac{1}{\omega}\sum_{i=1}^{n} \int_0^\omega \frac{(\omega - s)^i}{(i-1)!} y_i(s)ds$, where $\eta_i = -\sum_{j=0}^{n-i} \int_0^\omega \frac{(\omega - s)^j}{j!} y_{i+j}(s)ds$ $(i = \overline{1, n-1})$.

Finally, the estimation $||\hat{\mathcal{A}}^{-1}|| \le \rho_0 = 2C_2(1 + C_3) + 1$ holds, where $C_2 = \sum_{k=0}^{n} \frac{\omega^k}{k!}$ and $C_3 = C_2 \max(2, |C^{-1}|)$.

4.2. Projections in Y

Consider an operator $Q : Y \longrightarrow Y_2//Y_1$, defined by $Qy := \frac{1}{\omega}\{\tilde{e}(s)\}\int_0^\omega y_n(s)ds$. Clearly, $\mathcal{N}(Q) = Y_1, \mathcal{R}(Q) = Y_2$, and $Q^2y = Qy$, hence Q is a bounded linear projection in Y. From $||\tilde{e}|| = \max_{\substack{0\le t\le\omega \\ 0\le k\le n-1}} \frac{t^k}{k!} \le \max_{0\le k\le n-1} \frac{\omega^k}{k!} = C_1$ it follows $|Qy| \le \frac{1}{\omega}|\tilde{e}(t)|\omega\ ||y|| \le ||\tilde{e}||\ ||y||$, therefore $||Q|| \le C_1$.

Set $P := I - Q$. Obviously, P is also a bounded linear projection in Y and $\mathcal{R}(P) = Y_1, \mathcal{N}(P) = Y_2, ||P|| \le 1 + C_1$.

4.3. Nonlinear differential operator

Theorem 4.2. *Assume that the function g satisfies hypotheses (h1-h3) in Section 2, then the operator $F(x) = f(t,x,\dot{x}) = (0,...,0,g(t,x,\dot{x}_n))^T$ is continuously differentiable in Fréchet sense. Moreover, $||F'(x)|| \le na$ and $||F'(x) - F'(y)|| \le Ln^2|||x-y|||$, $\forall x,y \in S = \{x \in X : |||x||| \le R\}$. Further, the restriction $[QF'(x)]_{X_2}$ of the operator $[QF'(x)]$ to X_2 is of the form*

$$[QF'(x)]_{X_2}v = \frac{\lambda\tilde{e}(t)}{\omega}\int_0^\omega \frac{\partial g}{\partial\xi_1}(t,x,\dot{x}_n)ds\,,$$

where $v = \lambda e_1 \in X_2$. Finally, $|||[QF'(x)]_{X_2}^{-1}|| \le \frac{\omega}{||\tilde{e}||}\gamma \le \gamma$.

4.4. The Seidel - Newton method for nonlinear PBVP

Suppose that the m-th approximate solution is found $(m \ge 0)$. Let

$$g_m(t) := g(t, y_m, \dot{y}_m, ..., y_m^{(n)})\,;$$

$$z_k^{(m)}(t) := -\frac{t^{n-k}}{\omega(n-k)!}\int_0^\omega g_m(s)ds \quad (k = \overline{1, n-1})\,;$$

$$z_n^{(m)}(t) := g_m(t) - \frac{1}{\omega}\int_0^\omega g_m(s)ds\,;$$

$$x^{(m)} := (y_m, \dot{y}_m, ..., y_m^{(n-1)})^T \in \mathbb{R}^n\,;$$

$$F^{(m)} = f(t, x^{(m)}, \dot{x}^{(m)}) = (0,...,0,g_m(t))^T\,;$$

$$z^{(m)} := PF^{(m)} = F^{(m)} - QF^{(m)} = F^{(m)} - \frac{\tilde{e}(t)}{\omega}\int_0^\omega g_m(s)ds\,;$$

$$\eta^{(m)} := -\int_0^\omega e^{(\omega-s)A}z^{(m)}(s)ds.$$

Then

$$\eta_n^{(m)} = 0,$$

$$\eta_i^{(m)} := -\sum_{j=0}^{n-i} \int_0^\omega \frac{(\omega - s)^j}{j!} z_{i+j}^{(m)}(s)ds$$

$$= -\sum_{k=i}^{n} \int_0^\omega \frac{(\omega - s)^{k-i}}{(k-i)!} z_k^{(m)}(s)ds \quad (i = \overline{1, n-1}).$$

The vector $\xi^{(m)} = (\xi_1^{(m)}, \dots, \xi_n^{(m)})^T$ is determined as follows

$$\xi_1^{(m)} = -\eta_1^{(m)} - \frac{1}{\omega} \sum_{i=1}^{n} \int_0^\omega \frac{(\omega - s)^i}{(i-1)!} z_i^{(m)}(s)ds,$$

and $\xi_2^{(m)}, \dots, \xi_n^{(m)}$ are defined from the following linear triangular system

$$\begin{pmatrix} \omega & \omega^2/2! & \dots & \omega^{n-1}/(n-1)! \\ 0 & \omega & \dots & \omega^{n-2}/(n-2)! \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega \end{pmatrix} \begin{pmatrix} \xi_2^{(m)} \\ \xi_3^{(m)} \\ \vdots \\ \xi_n^{(m)} \end{pmatrix} = \begin{pmatrix} \eta_1^{(m)} \\ \eta_2^{(m)} \\ \vdots \\ \eta_{n-1}^{(m)} \end{pmatrix}.$$

Set $u_k^{(m+1)} = (e^{tA}\xi^{(m)})_k + \int_0^t \{e^{(t-s)A} z^{(m)}\}_k ds \quad (k = \overline{0, N})$, and put

$$\lambda^{(m+1)} = \lambda^{(m)} - \frac{\displaystyle\int_0^\omega g_m(s)ds}{\displaystyle\int_0^\omega \frac{\partial g_m}{\partial \xi_1}(s)ds}$$

$$= \lambda^{(m)} - \frac{\displaystyle\int_0^\omega g_m(s, u_1^{(m+1)} + \lambda^{(m)}, u_2^{(m+1)}, \dots, u_n^{(m+1)}, \dot{u}_n^{(m+1)})ds}{\displaystyle\int_0^\omega \frac{\partial g_m}{\partial \xi_1}(s, u_1^{(m+1)} + \lambda^{(m)}, u_2^{(m+1)}, \dots, u_n^{(m+1)}, \dot{u}_n^{(m+1)})ds};$$

Finally, let $x^{(m+1)} = u^{(m+1)} + \lambda^{(m+1)} e_1$. Note that $y_{m+1} = u_1^{(m+1)} + \lambda_{m+1}$, then $\frac{d^k}{dt^k} y_{m+1}(t) = \frac{d^k}{dt^k} u_1^{(m+1)} = u_{k+1}^{(m+1)}(t) \quad (k = \overline{1, N-1})$.

Let $\alpha := (1 + C_1)na$; $\beta := C_1 na$; $l = C_1 L n^2$;

Suppose that $q := 2\alpha\beta\gamma\rho_0 + \ell\delta\gamma/2 < 1$ and $2\delta(1-q)^{-1} < r$, where $r := R - |||x^{(0)}|||$, $x^{(0)} = (y^{(0)}, \dot{y}^{(0)}, \dots, \frac{d^{n-1}}{dt^{n-1}} y^{(0)})^T$, Put $\delta = \beta\gamma\rho_0||\mathcal{A}x^{(0)} - PF(x^{(0)})|| + \gamma||QF(x^{(0)})||$. Now we shall estimate the residual δ. For this purpose, let

$$g_0(t) = g\left(t, y^{(0)}, \dot{y}^{(0)}, ..., \frac{d^n}{dt^n}y^{(0)}\right). \quad F^{(0)} = (0, ..., 0, g_0(t)).$$

Then $QF^{(0)} = \dfrac{e(t)}{\omega}\displaystyle\int_0^\omega g_0(s)ds,$ and therefore $||QF^{(0)}|| \le \dfrac{C_1}{\omega}\left|\displaystyle\int_0^\omega g_0(s)ds\right|.$

Since

$$\mathcal{A}x^{(0)} - PF(x^{(0)}) = \dot{x}^{(0)} - Ax^{(0)} - F(x^{(0)}) + QF(x^{(0)}),$$

$$\dot{x}^{(0)} - Ax^{(0)} = \left(0, ..., 0, \frac{d^n}{dt^n}y^{(0)}\right)^T,$$

$$-PF(x^{(0)}) = QF(x^{(0)}) - F(x^{(0)})$$

$$= \left(\frac{t^{n-1}}{\omega(n-1)!}\int_0^\omega g_0 ds, ..., \frac{t}{\omega}\int_0^\omega g_0 ds, \frac{1}{\omega}\int_0^\omega g_0 ds - g_0\right),$$

it follows

$$\mathcal{A}x^{(0)} - PF(x^{(0)}) = \left(\frac{t^{n-1}}{\omega(n-1)!}\int_0^\omega g_0 ds, ..., \frac{t}{\omega}\int_0^\omega g_0 ds, \frac{1}{\omega}\int_0^\omega g_0 ds - g_0 + \frac{d^n}{dt^n}y^{(0)}\right).$$

We put

$$G_0 := \frac{1}{\omega}\left|\int_0^\omega g\left(t, y^{(0)}, \dot{y}^{(0)}, ..., \frac{d^n}{dt^n}y^{(0)}\right)dt\right|;\ \hat{G}_0 := \max\left|y_0^{(n)} - g_0 + \frac{1}{\omega}\int_0^\omega g_0 ds\right|.$$

Thus $||\mathcal{A}x^{(0)} - PF(x^{(0)})||$

$$= \max_t \max\left\{\max_{0\le k\le n-1}\frac{t^k}{k!\omega}\left|\int_0^\omega g_0 ds\right|, \left|\frac{1}{\omega}\int_0^\omega g_0 ds - g_0 + \frac{d^n}{dt^n}y^{(0)}\right|\right\}.$$

Consequently $||\mathcal{A}x^{(0)} - PF(x^{(0)})|| \le \max\{C_1G_0, \hat{G}_0\} = C_4$ and $\delta \le \beta\gamma\rho_0 C_4 + \gamma C_1 G_0$.

Theorem 4.3. *Assume that all conditions of Theorem 4.2 are fulfilled. Moreover, suppose that the coefficients α, β, γ, radius r and the residual δ satisfy the relations*

$$q < 1; \quad 2\delta(1-q)^{-1} < r.$$

Then there exists a unique solution $x^* \in S(x^{(0)}, r) \subset X$ *and the sequence* $\{x^{(m)}\}$, *constructed by the Seidel - Newton method, converges to* x^* *at the rate:*

$$|||x^{(m)} - x^*||| \leq rq^m .$$

4.5. Comparison with the discrete problem

Note that $\rho_0 = 2C_2(1 + C_3) + 1 < 2e^\omega(1 + e^\omega \max\,(2, |\mathcal{C}^{-1}|) + 1$ and $\rho_1 = 2(1+C_0)e^\omega - 1 < 2e^\omega(1+C_0)+1$, where $C_0 = \max\{e^\omega - 1 + |\frac{\omega-1}{\omega}|\,e^\omega + \frac{1}{\omega};\ |\mathcal{C}^{-1}|(e^\omega - 1)\}$.

Now for definiteness, we suppose that $\omega \geq 1$.

Since $|\mathcal{C}_N^{-1}|(e^\omega - 1) \underset{N\to\infty}{\longrightarrow} |\mathcal{C}^{-1}|(e^\omega - 1) < |\mathcal{C}^{-1}|e^\omega$, for sufficiently large N, we have $|\mathcal{C}_N^{-1}|(e^\omega - 1) < |\mathcal{C}^{-1}|e^\omega$. It is obvious that $\rho_1, \rho_0 < \hat{\rho} := 2e^\omega(1 + e^\omega \max\,(2, |\mathcal{C}^{-1}|)) + 1$.

A comparison table

The discrete case	*The continuous case*
$\|\|Q_h\|\| \leq C_1\,; \|\|P_h\|\| \leq 1 + C_1$	$\|\|Q\|\| \leq C_1\,; \|\|P\|\| \leq 1 + C_1$
$\|\|P_hF_h'(x_h)\|\| \leq (1 + C_1)na = \alpha$	$\|\|PF'(x)\|\| \leq (1 + C_1)na = \alpha$
$\|\|Q_hF_h'(x_h)\|\| \leq C_1na = \beta$	$\|\|QF'(x)\|\| \leq C_1na = \beta$
$\|\|[Q_hF_h(x_h)]^{-1}_{X_2^h}\|\| \leq \gamma_h := (h\sum_{k=0}^{N-1} a(kh))^{-1}$	$\|\|[QF'(x)]^{-1}_{X_2}\|\| \leq \gamma$
$l_h = C_1Ln^2$	$l = C_1Ln^2$.

5. ITERATIVE METHOD FOR DISCRETE PROBLEMS

We shall use the Seidel - Newton method for solving discrete problem (2.2). The reader is refered to Theorem 1.1 and [1-3] for details.

Let the initial approximation be chosen as $x_h^{(0)} = (x_{h,0}^{(0)}, ..., x_{h,N}^{(0)})$, where $x_{h,k}^{(0)} = (y_0(t_k), \dot{y}_0(t_k), ..., y_0^{(n-1)}(t_k))^T$ for some $y_0 \in C^n[0,\omega]$.

By the mean-value theorem, we have

$$\frac{x_{h,k+1}^{(0)} - x_{h,k}^{(0)}}{h} = (\dot{y}_0(\xi_{k1}), \ddot{y}_0(\xi_{k2}), ..., y_0^{(n)}(\xi_{kn}))^T,$$

where $t_k < \xi_{kn} < t_{k+1}$ $(k = \overline{0, N-1})$, and hence $[\mathcal{A}_hx_h] = \frac{x_{h,k+1}^{(0)} - x_{h,k}^{(0)}}{h} - [Ax_h^{(0)}]_k = (\dot{y}_0(\xi_{k1}) - \dot{y}_0(t_k), ..., y_0^{(n-1)}(\xi_{k,n-1}) - y_0^{(n-1)}(t_k), y_0^{(n)}(\xi_{kn}))$. Further,

$$g_k := g(t_k, y_0(t_k), ..., y_0^{(n-1)}(t_k), y_0^{(n)}(\xi_{kn}))^T; \quad [F_h(x_h^{(0)})]_k = (0, ..., 0, g_k)^T.$$

Setting $g_{tb} := \frac{1}{N}\sum_{k=0}^{N-1} g_k$, we have $Q_h F_h(x_h^{(0)}) = g_{tb}\tilde{e}$ and $P_h F_h(x_h^{(0)}) = F_h(x_h^{(0)}) - g_{tb}\tilde{e}$, and therefore $[P_h F_h(x_h^{(0)})]_k = (-C_k^{n-1}h^{n-1}g_{tb}, ..., -C_k^1 h g_{tb}, g_k - g_{tb})^T$ $(0 \le k \le N-1)$. These relations show that $[\mathcal{A}_h x_h^{(0)} - P_h F_h(x_h^{(0)})]_k = (\dot{y}_0(\xi_{kn}) - \dot{y}_0(t_k), ..., y_0^{(n-1)}(\xi_{k,n-1}) - y_0^{(n-1)}(t_k), y_0^{(n)}(\xi_{kn}) - g_k)^T + g_{tb}\tilde{e}$ $(0 \le k \le N-1)$.

From the last relation we find

$$||\mathcal{A}_h x_h^{(0)} - P_h F_h(x_h^{(0)})||_h \le \max\left\{\max_k |y_0^{(n)}(\xi_{kn}) - g_k + g_{tb}|, \max_i \frac{\omega^i}{i!} g_{tb}\right\}.$$

Since $g_{tb} \longrightarrow G_0$ and $\max_k |y_0^{(n)}(\xi_{kn}) - g_k + g_{tb}| \longrightarrow \max |y_0^{(n)}(t) - g(t, y_0(t), ..., y_0^{(n)}(t)) + \frac{1}{\omega}\int_0^\omega g(s, y_0(s), ...)ds| := \hat{G}_0$, as $N \longrightarrow \infty$, for N sufficiently large, we can assume that

$$|y_0^{(i)}(\xi_{ki}) - y_0^{(i)}(t_k)| \le \max_k |y_0^{(k)}(\xi_{kn}) - g_k| \quad (i = \overline{0, n-1}; \quad k = \overline{0, N-1}).$$

Consequently, $||\mathcal{A}_h x_h^{(0)} - P_h F_h(x_h^{(0)})||_h \le C_4 := \max\{C_1 G_0, \hat{G}_0\}$.

Suppose that $q_h := 2\alpha\beta\gamma_h\hat{\rho} + \ell\gamma_h\delta_h/2 < 1$ and $2\delta_h(1-q_h)^{-1} < r \le r_h$, where $\delta_h := \beta\gamma_h\rho C_4 + \gamma_h C_1 G_0$, $r_h := R - |||x_h^{(0)}|||_h \ge R - |||x^{(0)}||| = r$, and the m-th approximate solution is found $(m > 0)$

$$x^{(m)} = (x_0^{(m)}, ..., x_N^{(m)}), \quad x_k^{(m)} = (x_{k,1}^{(m)}, ..., x_{k,n}^{(m)})^T \in \mathbb{R}^n \quad (k = \overline{0, N}).$$

Letting $g_k^{(m)} := g\left(t_k, x_k^{(m)}, \frac{x_{k+1,n}^{(m)} - x_{k,n}^{(m)}}{h}\right)$,

$$f^{(m)} = (f_0^{(m)}, ..., f_{N-1}^{(m)}), \quad f_k = (0, ..., 0, g_k^{(m)})^T \quad (k = \overline{0, N-1}),$$

and setting $g_{mtb} := \sum_{k=0}^{N-1} g_k/N$, we have $y^{(m)} = f^{(m)} - g_{mtb}\tilde{e}$. Further, put

$$\eta^{(m)} = -h\sum_{k=0}^{N-1}(E + hA)^{N-k-1}y_k^{(m)}, \; \hat{\eta}^{(m)} = h\sum_{k=1}^{N}\sum_{s=0}^{k-1}(E + hA)^{N-s-1}y_s^{(m)},$$

$$\eta^{(m)} = \sum_{i=1}^{n-1}\eta_i^{(m)} e_i, \quad \hat{\eta}^{(m)} = (\hat{\eta}_1^{(m)}, ..., \hat{\eta}_n^{(m)})^T.$$

The first component of $u_0^{(m+1)} = (\xi_1^{(m)}, ..., \xi_n^{(m)})^T$ is defined by the formula $\xi_1^{(m)} = -\eta_1^{(m)} - (\hat{\eta}_1^{(m)}/(N+1))$ and the remaining components are computed from the triangular system of linear equations

$$\begin{pmatrix} hC_N^1 & h^2C_N^2 & \dots & h^{n-1}C_N^{n-1} \\ 0 & hC_N^1 & \dots & h^{n-2}C_N^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & hC_N^1 \end{pmatrix} \begin{pmatrix} \xi_2^{(m)} \\ \xi_3^{(m)} \\ \vdots \\ \xi_n^{(m)} \end{pmatrix} = \begin{pmatrix} \eta_1^{(m)} \\ \eta_2^{(m)} \\ \vdots \\ \eta_{n-1}^{(m)} \end{pmatrix}.$$

Then $u_{k+1}^{(m+1)} = (E + hA)^{(k+1)} u_0^{(m+1)} + h \sum_{s=0}^{k} (E + hA)^{k-s} y_s^{(m)} \quad (k = \overline{0, N-1})$.

Let $\tilde{x}_k = u_k^{(m+1)} + \lambda^{(m)} e_1 \quad (k = \overline{0,N}) \quad (m \geq 0)$. Denote by $\tilde{g}_k^{(m)}$, $\dfrac{\partial \tilde{g}_k^{(m)}}{\partial y}$ the values of g and $\dfrac{\partial g}{\partial y}$, calculated at $\tilde{x}^{(m)}$. Then,

$$\lambda^{(m+1)} = \lambda^{(m)} - \left\{ \sum_{k=0}^{N-1} \frac{\partial \tilde{g}_k^{(m)}}{\partial y} \right\}^{-1} \left\{ \sum_{k=0}^{N-1} \tilde{g}_k^{(m)} \right\}.$$

Finally,

$$x_k^{(m+1)} = u_k^{(m+1)} + \lambda^{(m+1)} e_1 \quad (k = \overline{0,N}). \tag{5.1}$$

Theorem 5.1. *Suppose that the function $g : \triangle \longrightarrow \mathbb{R}^1$ satisfies hypotheses h1-h3 (in Section 3.2.2). Let the initial approximation $y^{(0)}$ of the problem (1.1)-(1.2) be chosen such that*

h4) $\max\limits_{0 \leq t \leq \omega} \max\limits_{0 \leq i \leq n} \left| \dfrac{d^i}{dt^i} y^{(0)} \right| < R$,

h5) $2\delta_0 (1 - q_0)^{-1} < r$, *where* $\delta_0 := \beta \gamma \hat{\rho} C_4 + \gamma C_1 G_0$,

h6) $q_0 = 2\alpha\beta\gamma\hat{\rho} + \ell\gamma\delta_0/2 < 1$.

Then whenever $h \leq h_0$, the iterative process for solving the nonlinear difference equations (1.4) converges to an isolated solution $\bar{x}_h$ and the following estimate holds

$$|||x^{(m)} - \bar{x}_h|||_h \leq r_h q_h^m,$$

where $0 \leq q_h \leq \tilde{q} < 1$ *and* $r_h < R$.

Proof. Putting $\delta_h = \beta\gamma_h\hat{\rho}||\mathcal{A}_h x_h^{(0)} - P_h F_h(x_h^{(0)})||_h + \gamma_h||Q_h F_h(x_h^{(0)})||_h$, $q_h := 2\alpha\beta\gamma_h\hat{\rho} + \ell\gamma_h\delta_h/2 < 1$. Then as $h \longrightarrow 0$, $\gamma_h = \left(h\sum_{k=0}^{N-1} a(kh)\right)^{-1} \longrightarrow \left(\int_0^\omega a(s)ds\right)^{-1} = \gamma$, $\delta_h \longrightarrow \delta_0, q_h \longrightarrow q_0 < 1$. Hence $h \le h_0$ implies $q_h \le \tilde{q} < 1$. Denoting $r_h := R - |||x_h^{(0)}|||_h$, we obtain $r_h \longrightarrow r > 0$, $r_h \ge r$. Since $2\delta_0(1-q_0)^{-1} < r$, we get $2\delta_h(1-q_h)^{-1} < r_h$.

Thus in the closed ball $S_h(x^{(0)}, r_h)$ there is a locally unique solution $\bar{x}_h$ and the iterative process is convergent.

6. STABILITY OF THE DIFFERENCE SCHEME AND THE CONVERGENCE OF THE ITERATIVE METHOD

Denote by X_0 a class of all differentiable vector functions on $[0,\omega]$, possessing Lipschitz continuous derivatives. Define two bounded linear projections

$$\pi_h : X \longrightarrow X_h\,, \quad (\pi_h x)_k = x(t_k) \quad (k = \overline{0,N})\,,$$

$$\tau_h : Y \longrightarrow Y_h\,, \quad (\tau_h y)_j = y(t_j) \quad (j = \overline{0, N-1}).$$

It is clear that $||\pi_h|| = 1$ and $||\tau_h|| = 1$.

Lemma 6.1. *Difference scheme (1.4) approximates BVP (1.3) at every $x \in X_0$.*

Proof. For $x \in X_0$, by the mean value theorem we have

$$||\mathcal{A}_h\pi_h(x) - \tau_h\mathcal{A}x||_h = \max_{0\le k\le N-1}\ \max_{0\le i\le n} |\dot{x}_i(\xi_{ki}) - \dot{x}_i(t_k)|\,,$$

where $t_k < \xi_{ki} < t_{k+1}$ $(i = \overline{1,n})$.

Let the derivative $\dot{x}$ of $x \in X_0$ be Lipschitz continuous with a Lipschitz coefficient $K(x)$, then

$$||\mathcal{A}_h\pi_h(x) - \tau_h\mathcal{A}x||_h = \max_{i,k} |\dot{x}(\xi_{ki}) - \dot{x}(t_k)| \le K(x)h.$$

Similarly, using hypothesis (h1) we get

$$\begin{aligned} ||F_h(\pi_h x) - \tau_h F(x)||_h &= \max_{0\le k\le N-1} \left| g\left(t_k, x(t_k), \frac{x_n(t_{k+1}) - x_n(t_k)}{h}\right) - g(t_k, x(t_k), \dot{x}_n(t_k)) \right| \\ &\le \max_{0\le k\le N-1} a|\dot{x}_n(\xi_{kn}) - \dot{x}_n(t_k)| \le aK(x)h. \end{aligned}$$

Thus $\quad ||\mathcal{A}_h\pi_h(x) - \tau_h Ax||_h + ||F_h(\pi_h x) - \tau_h F(x)||_h \leq (1+a)K(x)h.$

Lemma 6.2. *Difference scheme (1.4) is stable, i.e., there exist $h_0 > 0$ and $\varepsilon > 0$ such that for any $h < h_0$ and $z \in Y_h$ $\;(||z||_h < \varepsilon)$ the following statements hold:*

i) The operator equation

$$\mathcal{A}_h x = F_h(x) + z \quad (||z||_h < \varepsilon) \tag{6.1}$$

has a unique solution in the set $\Omega_h = \{x \in S_h(x_0, r_h) : |||R_h(x - x_h^{(0)})|||_h \leq r_h/2\}$, where $R_h x := \dfrac{1}{N+1}\sum_{k=0}^{N}\langle (E + hA)^{N-k}x_k, e_1\rangle e.$

ii) $|||x_h - \bar{x}_h|||_h \leq C_5\; ||z||_h$,

where x_h and $\bar{x}_h$ are solutions of (6.1) and (1.6) respectively.

Proof. Let $\tilde{F}_h(x) = F_h(x) + z$, then (6.1) is of the form $\mathcal{A}_h x = \tilde{F}_h(x)$. Further, $\tilde{F}_h'(x) = F_h'(x)$, so $\tilde{F}_h$ possesses the same properties as F_h in Ω_h (cf. Theorem 3.3).

We find the residual $\quad \delta(z) = \beta\gamma_h\hat{\rho}||\mathcal{A}_h x_h - P_h\tilde{F}_h(x_h^{(0)})||_h + \gamma_h||Q_h\tilde{F}_h(x_h^{(0)})||_h$; and the expression $\quad q(z) = 2\alpha\beta\gamma_h\hat{\rho} + \ell\gamma_h\delta_h(z)/2.$

As $\quad \delta(z) \longrightarrow \delta_h \quad (z \longrightarrow 0), \quad q(z) \longrightarrow q_h < 1 \quad (z \longrightarrow 0)$, then there exists $\varepsilon > 0$, such that for every $||z||_h \leq \varepsilon$, $\; z \in Y_h$, we have $q(z) < 1$, $2\delta(z)(1 - q(z))^{-1} < r_h$. By Theorem 4.1, there exists a unique solution $x_h \in S_h$.

Observing that $u_h - \bar{u}_h = \hat{\mathcal{A}}_h^{-1}\{P_h[F_h(x_h) - F_h(\bar{x}_h)] + P_h z\}$ and using the facts that $Q_h F_h(x_h) = Q_h z$, $\;Q_h F_h(\bar{x}_h) = 0$ we come to the estimate

$$|||u_h - \bar{u}_h|||_h \leq \hat{\rho}\{\alpha|||x_h - \bar{x}_h|||_h + (1 + C_1)||z||_h\}. \tag{6.2}$$

Letting

$$\hat{x}_h = u_h + \bar{v}_h \in \Omega_h,$$
$$Q_h z = Q_h F_h(x_h) = Q_h F_h(x_h) - Q_h F_h(\hat{x}_h) + Q_h F_h(\hat{x}_h) - Q_h F_h(\bar{x}_h),$$

we have $||Q_h F_h(x_h) - Q_h F_h(\hat{x}_h)||_h \leq ||Q_h F_h(\hat{x}_h) - Q_h F_h(\bar{x}_h)||_h + ||Q_h z||_h \leq \beta|||u_h - \bar{u}_h|||_h + C_1||z||_h$. On the other hand, since dim $X_2^h =$ dim $Y_2^h = 1$, by the finite increments formula, we find

$$\begin{aligned}||Q_h F_h(x_h) - Q_h F_h(\hat{x}_h)||_h &= |||[Q_h F_h'(\hat{x}_h + \xi(v_h - \bar{v}_h))]_{X_2^h}(v_h - \bar{v}_h)||_h \\ &\geq \gamma_h^{-1}|||v_h - \bar{v}_h|||_h.\end{aligned}$$

Consequently

$$|||v_h - \bar{v}_h|||_h \leq \gamma_h||Q_hF_h(x_h) - Q_hF_h(\hat{x}_h)||_h \\ \leq \gamma_h\beta|||u_h - \bar{u}_h|||_h + \gamma_hC_1||z||_h.$$

From (6.2) and the last inequality it follows

$$\begin{aligned} |||x_h - \bar{x}_h|||_h &\leq |||u_h - \bar{u}_h|||_h + |||v_h - \bar{v}_h|||_h \\ &\leq (1+\gamma_h\beta)|||u_h - \bar{u}_h|||_h + \gamma_hC_1||z||_h \\ &\leq \hat{\rho}\{\alpha|||x_h - \bar{x}_h|||_h + (1+C_1)||z||_h\}(1+\gamma_h\beta) + \gamma_hC_1||z||_h \\ &= \alpha(1+\gamma_h\beta)\hat{\rho}|||x_h - \bar{x}_h|||_h + [\gamma_hC_1 + (1+\gamma_h\beta)(1+C_1)\hat{\rho}]||z||_h. \end{aligned}$$

Noting that $1 \leq \beta\gamma_h$ and hence $\alpha(1+\gamma_h\beta)\hat{\rho} \leq 2\alpha\beta\gamma_h\hat{\rho} \leq q_h < \tilde{q} < 1$, we get the following estimate

$$|||x_h - \bar{x}_h|||_h \leq \frac{C_1\gamma_h + (1+\gamma_h\beta)(1+C_1)\hat{\rho}}{1-\alpha(1+\gamma_h\beta)\hat{\rho}}||z||_h \leq C_5||z||_h,$$

where

$$C_5 = \frac{(\gamma+\varepsilon)C_1 + [1+(\gamma+\varepsilon)\beta](1+C_1)\hat{\rho}}{1-\tilde{q}}.$$

Set $S = \{x \in X : |||x - x_0||| \leq r, \quad |||v - v_0||| \leq r/2\}$. We are now ready to prove the main theorem.

Theorem 6.1. *Suppose that the hypotheses (h1)-(h5) are satisfied. Then*

i) Problems (1.1), (1.2) have a unique solution $x^* \in S$.

ii) Difference scheme (1.4) is stable and has a unique solution $\bar{x}_h \in S_h$ *for* $h \leq h_0$.

iii) Iterative process (4.1) converges to $\bar{x}_h$.

iv) Suppose that $x^* \in X_0$, *then there holds the estimation*

$$|||x_h^{(m)} - \pi_hx^*|||_h \leq R\tilde{q}^m + C_5h.$$

Proof. Statements (i)-(iii) have been proved in Lemmas 6.1, 6.2. To prove iv) we set $z_h := \mathcal{A}_h(\pi_hx^*) - F_h(\pi_hx^*)$, then

$$||z_h||_h \leq ||\mathcal{A}_h(\pi_hx^*) - \tau_h(Ax^*)||_h + ||F_h(\pi_hx^*) - \tau_hF(x^*)||_h \leq (1+a)K^*h.$$

Therefore $||z_h|| < \varepsilon$ for all $h \leq h_1$. By Lemma 6.2, the operator equation $\mathcal{A}_h x = F_h(x) + z_h$ has a unique solution $x_h \in S_h$.

To prove $\pi_h x^* = x_h$, it suffices to show that $\pi_h x^* \in S_h$. Note first that

$$|||\pi_h x^* - x_h^{(0)}|||_h = |||\pi_h(x^* - x^{(0)})|||_h$$
$$\leq ||\pi_h||\ |||x^* - x^{(0)}||| \leq |||x^* - x^{(0)}||| \leq r \leq r_h\,.$$

Further, we should verify the inequality $|||R_h(\pi_h x^* - x_h)|||_h \leq r_h/2$. First, we show that $\forall x \in X, \quad |||R_h(\pi_h x)|||_h \longrightarrow |||Rx|||$, where

$$Rx = \frac{1}{\omega}\Big(\sum_{i=1}^{n}\int_0^{\omega}\frac{(\omega - s)^{i-1}}{(i-1)!}x_i(s)ds\Big)e_1\,.$$

It is clear that

$$|||Rx||| = \frac{1}{\omega}\int_0^{\omega}\sum_{i=0}^{n-1}\left|\frac{(\omega - s)^i}{i!}x_{i+1}(s)ds\right|$$
$$= \frac{1}{\omega}\lim_{N\to\infty}\left|\sum_{i=0}^{n-1}\sum_{k=0}^{N}\frac{\omega}{N+1}\frac{(\omega - kh)^i}{i!}x_{i+1}(t_k)\right|$$
$$= \lim_{N\to\infty}\frac{1}{N+1}\left|\sum_{i=0}^{n-1}\sum_{k=0}^{N}\frac{k^i h^i}{i!}x_{i+1}(t_{N-k})\right|$$
$$= \lim_{N\to\infty}\frac{1}{N+1}\left|\sum_{i=0}^{n-1}\sum_{k=n}^{N}\frac{k^i h^i}{i!}x_{i+1}(t_{N-k})\right|\,.$$

We represent

$$|||R_h\pi_h x|||_h = \frac{1}{N+1}\left|\sum_{k=0}^{N}\langle (E + hA)^k x(t_{N-k}), e_1\rangle\right|\ |||e|||_h$$
$$= \frac{1}{N+1}\left|\sum_{k=0}^{N}\langle (E + hA)^k x(t_{N-k}), e_1\rangle\right|$$

as $|\triangle_h^{(1)} + \triangle_h^{(2)} + \triangle_h^{(3)}|$, where

$$\triangle_h^{(1)} = \frac{1}{N+1} \sum_{k=0}^{n-1} \sum_{i=0}^{k} C_k^i h^i x_{i+1}(t_{N-k}),$$

$$\triangle_h^{(2)} = \frac{1}{N+1} \sum_{k=n}^{N} \sum_{i=0}^{n-1} \frac{k^i h^i}{i!} x_{i+1}(t_{N-k}),$$

$$\triangle_h^{(3)} = \frac{1}{N+1} \sum_{k=n}^{N} \sum_{i=0}^{n-1} \frac{h^i}{i!} \left[\frac{k!}{(k-i)!} - k^i \right] x_{i+1}(t_{N-k}).$$

Clearly, as already mentioned, $\triangle_h^{(1)} \longrightarrow 0$ and $|\triangle_h^{(2)}| \longrightarrow |||Rx|||$ as $N \longrightarrow \infty$.

Estimating the third term, we have

$$\begin{aligned}
|\triangle_h^{(3)}| &\le \frac{1}{N+1} \sum_{k=n}^{N} \sum_{i=1}^{n-1} \frac{1}{i!} \left(\frac{\omega^k}{N} \right)^i \left\{ 1 - \left(1 - \frac{1}{k}\right)\left(1 - \frac{2}{k}\right) \cdots \left(1 - \frac{i-1}{k}\right) \right\} |||x||| \\
&\le \frac{1}{N+1} \sum_{k=n}^{N} \sum_{i=1}^{n-1} \frac{\omega^i}{i!} \left(\frac{k}{N} \right)^i \left\{ 1 - \left(1 - \frac{i-1}{k} \right)^{i-1} \right\} |||x||| \\
&\le \frac{C_1}{N+1} \sum_{k=n}^{N} \sum_{i=0}^{n-1} \frac{k^{i-1}(n-2)^2}{N^i} |||x||| \le \frac{C_1 (n-2)^2}{N(N+1)} \sum_{k=n}^{N} \sum_{i=1}^{n-1} \left(\frac{k}{N} \right)^{i-1} |||x||| \\
&\le \frac{C_1 (n-2)^2}{N(N+1)} (N-n)(n-1) |||x||| \le \frac{C_1 (n-2)^2 (n-1)}{N+1} |||x||| \longrightarrow 0
\end{aligned}$$

as $N \longrightarrow \infty$. Thus we have proved the relation $|||R_h(\pi_h x)|||_h \longrightarrow |||Rx|||$ $(N \longrightarrow \infty)$ for any fixed $x \in X$.

Since $|||v_h^* - v_h^{(0)}|||_h = |||R_h \pi_h (x^* - x^{(0)})|||_h \longrightarrow |||R(x^* - x^{(0)})||| < r/2$, it follows that for sufficiently small h, $|||v_h^* - v_h^{(0)}|||_h < r/2 \le r_h/2$.

Finally,

$$\begin{aligned}
|||x_h^{(m)} - \pi_h x^*|||_h &= |||x_h^{(m)} - x_h|||_h \\
&\le |||x_h^{(m)} - \bar{x}_h|||_h + |||\bar{x}_h - x_h|||_h \le r_h q_h^m + C_5 h < R\tilde{q}^m + C_5 h .
\end{aligned}$$

The proof of Theorem 6.1 is complete.

7. NUMERICAL EXAMPLES

We shall illustrate the described above method by considering a nonlinear PBVP for the Duffing's equation

$$\begin{cases} y''(t) = \kappa(y(t) + y^3(t) + y'(t) - u(t)), \quad t \in (0, 2\pi) & (7.1) \\ y(0) - y(2\pi) = y'(0) - y'(2\pi) = 0, & (7.2) \end{cases}$$

where $\kappa > 0$ is a small parameter and $u \in C[0, 2\pi]$ is given.

By putting $x(t) = (y(t), y'(t))^T$, problem (7.1), (7.2) is written in the vector form

$$\begin{cases} \dot{x} = Ax + f(t, x, \dot{x})\kappa & (7.3) \\ x(0) = x(\omega), & (7.4) \end{cases}$$

where $f = (0, x_1 + x_1^3 - x_2 - u(t))^T$, and $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

If κ is small enough (e.g. less than 10^{-2}) then all of conditions of the Theorem 6.1 will be satisfied. However, in many cases for much greater κ, the sequence of approximate solutions constructed by the Seidel - Newton method is still convergent.

Let us consider an uniform mesh of $[0, 2\pi] : \{0 = t_0 < ... < t_N = 2\pi\}$, where $t_i = ih \quad (i = \overline{0, N}), h = 2\pi/N$.

Consider problem (7.3)-(7.4) in three cases with $x^0 \equiv 0$. The estimations are given in the following tables.

1) *Case 1:* $u(t) \equiv 2$. Problem (7.1)-(7.2) has an exact solution $y^*(t) \equiv 1$, i.e. $\pi_h x^* = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}$.

κ	N = 100	200	500
0.01	$\|\|\|x_h^{(6)} - x_h^*\|\|\|_h \leq 10^{-3}$	$\|\|\|x_h^{(6)} - x_h^*\|\|\|_h \leq 10^{-3}$	$\|\|\|x_h^{(6)} - x_h^*\|\|\|_h \leq 10^{-3}$
0.001	$\|\|\|x_h^{(5)} - x_h^*\|\|\|_h \leq 10^{-3}$	$\|\|\|x_h^{(5)} - x_h^*\|\|\|_h \leq 10^{-3}$	$\|\|\|x_h^{(5)} - x_h^*\|\|\|_h \leq 10^{-3}$

2) *Case 2:* $u(t) = (1 + 1/\kappa)\sin t + \sin^3 t + \cos t$. In this case, problem (7.1)-(7.2) has an exact solution $y^*(t) = \sin\, t$, i.e. $\pi_h x^* = \begin{pmatrix} \sin t_0 & \dots & \sin t_N \\ \cos t_0 & \dots & \cos t_N \end{pmatrix}$

κ	N		
	100	200	500
0.1	$\|\|\|x_h^{(3)} - x_h^*\|\|\|_h < 0.1$	$\|\|\|x_h^{(3)} - x_h^*\|\|\|_h < 0.05$	$\|\|\|x_h^{(6)} - x_h^*\|\|\|_h < 0.02$
0.01	$\|\|\|x_h^{(1)} - x_h^*\|\|\|_h < 0.1$	$\|\|\|x_h^{(1)} - x_h^*\|\|\|_h \le 0.05$	$\|\|\|x_h^{(4)} - x_h^*\|\|\|_h \le 0.02$
0.001	$\|\|\|x_h^{(1)} - x_h^*\|\|\|_h \le 0.1$	$\|\|\|x_h^{(1)} - x_h^*\|\|\|_h \le 0.05$	$\|\|\|x_h^{(1)} - x_h^*\|\|\|_h \le 0.02$

3) *Case 3:* $u(t) = \sin t$. The exact solution is not available. When $N = 100$, $\kappa = 0.1$ the curve of the approximate solution $x^{(4)}$ is shown in figure 1 and $|||x_h^{(4)} - x_h^{(3)}|||_h < 10^{-6}$.

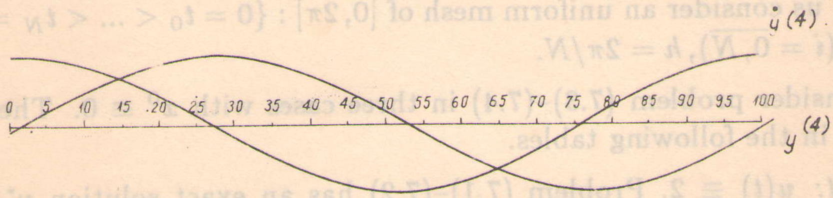

Fig. 1

REFERENCES

1. Pham Ky Anh, *On the Seidel - Newton method for solving quasilinear operator equations*, Acta Math. Viet., **7** (1982), no. 2, 111–126.

2. Pham Ky Anh and Vu Huy Tich, *An iterative method for general periodic boundary-value problems*, Ukrain. Math. J., **35** (1983), no. 3, 348–352.

3. Pham Ky Anh and Tran Dang Hong, *An approximate method for a nonlinear periodic boundary-value problem*, Acta Math. Viet., **11** (1986), no. 2, 156–171.

4. Pham Ky Anh, *An approximate method for multipoint BVPs at resonance*, Ukrain. Math. J., **39** (1987), no. 5, 619–624.

5. ———, *On two approximate methods for solving nonlinear Neumann problems*, Ukrain. Math. J., **40** (1988), no. 5, 619–625.

6. ———, *An iterative method for nonlinear BVPs at resonance*, Ukrain. Math. J., **43** (1991), no. 5, 663–674.

7. Pham Ky Anh and Bui Duc Tien, *An inexact Seidel - Newton method for nonlinear boundary-value problems*, Acta Math. Viet., **17** (1992), no. 2, 63–82.

Department of Mathematics, Mechanics and Informatics
Hanoi University
Nguyen Trai St. 90
Hanoi, Vietnam

Received June 18, 1993
Revised November 3, 1993