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A REPRESENTATION THEOREM FOR SKOROHOD MARTINGALES

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Abstract. *A new representation of anticipating martingales is given via a two-parameter stochastic integral. Its advantages are shown. A kind of moment inequalities for the martingales is presented.*

INTRODUCTION

An integrable process $X = X_t, \quad 0 \le t \le 1$ will be called a Skorohod martingale, or simply an S-martingale, if $E\{X_t - X_s | \mathcal{F}_{[s,t]^c}\} = 0$ for all $s < t$, where $\mathcal{F}_{[s,t]^c}$ denotes the σ-field generated by the increments of the Brownian motion on the complement of the interval $[s,t]$.

This notion arises from the following property of the Skorohod stochastic integral (see [6], Proposition 5.1): if $u = \{u_t, \quad 0 \le t \le 1\}$ is a Skorohod integrable process such that there exists the indefinite integral $\int_0^t u_r dW_r$ for every $t \in [0,1]$, then for all $s < t$

$$E\left\{\int_s^t u_r dW_r | \mathcal{F}_{[s,t]^c}\right\} = 0.$$

Conversely, in [4], we proved that an S-martingale can be represented by the Skorohod stochastic integral under a slight hypothesis. In the present paper we shall give a new representation of Skorohod martingales via a two-parameter stochastic integral and show its advantages in characterizing smooth Wiener functionals without using their Wiener chaos expansions and in giving a new sufficient condition for $f(\int_0^t u_s dW_s)$ to be an S-quasimartingale, where the function f belongs to Class $C^2(R)$. Other related results are also discussed in Section 1.

In Section 2, we present a moment inequality for S-martingales and its application to deducing a sufficient condition for an S-martingale to have a continuous version.

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1. A REPRESENTATION RESULT

Our basic probability space $(\Omega, \mathcal{F}, P)$ will be the canonical Wiener space associated with the standard Brownian motion $\{W_t,\ 0 \le t \le 1\}$ on the unit interval $[0,1]$. For $0 \le t \le 1$ let $(\mathcal{F}_t)$ denote the right-continuous completion of the σ-field $\sigma(W_s,\ 0 \le s \le t)$. In the same way we define $(\mathcal{F}^t)$ in terms of the σ-field $\sigma(W_1 - W_s,\ 1-t \le s \le 1)$.

Put

$$T = \{(s,t) : 0 \le s,t \quad \text{and } s+t \le 1\},$$

$$R_{st} = \{(u,v) \in T \quad \text{such that } u \le s \text{ and } v \le t\}.$$

Let $\Phi = \{\phi_{uv}, (u,v) \in T\}$ be a stochastic process such that:

For all $0 \le t \le 1$, the process $\{\phi_{uv}, (u,v) \in R_{t,1-t}\}$ is a predictable process w.r.t. the filtration $\{\mathcal{F}_u \vee \mathcal{F}^u, (u,v) \in R_{t,1-t}\}$ (see [2]) and (1.1)

$$E \int_{R_{t,1-t}} \phi_{uv}^2 du dv < +\infty. \tag{1.2}$$

Under these conditions, $X_{st} := \int_{R_{st}} \phi_{uv} dW_u dW^v$ is well-defined for any $(s,t) \in T$ as a two parameter stochastic integral, where W^v denotes $W_1 - W_{1-v}$. For fixed t, $\{W_u, 0 \le u \le t\}$ and $\{W^v, 0 \le v \le 1-t\}$ are two independent Brownian motions, and so $\{X_{uv}, (u,v) \in R_{t,1-t}\}$ is a square-integrable continuous two-parameter martingale.

Let us introduce the process $X = \{X_t,\ 0 \le t \le 1\}$ defined by

$$X_t = X_{t,1-t} = \int_{R_{t,1-t}} \phi_{uv} dW_u dW^v. \tag{1.3}$$

Proposition 1.1. *Under the above assumptions, the process X is an S-martingale. Such an S-martingale of the form (1.3) will be called an outward martingale.*

Proof. Indeed, for all $0 \le s < t \le 1$ we have

$$X_t - X_s = \int_{[s,t]\times[0,1-t]} \phi_{uv} dW_u dW^v - \int_{[0,s]\times[1-t,1-s]} \phi_{uv} dW_u dW^v.$$

Since

$$E\{dW_u | \mathcal{F}_s \vee \mathcal{F}^{1-t}\} = 0 \quad \text{for all} \quad u \in [s,t],$$

$$E\{dW^v | \mathcal{F}_s \vee \mathcal{F}^{1-t}\} = 0 \quad \text{for all} \quad v \in [1-t, 1-s],$$

we have

$$E\{X_t - X_s | \mathcal{F}_s \vee \mathcal{F}^{1-t}\} = 0 \quad \text{q.e.d.}$$

Suppose that $X = \{X_t, 0 \le t \le 1\}$ is an integrable process. Let us recall that X is said to be a (forward) martingale (resp. a backward martingale) iff X_t is $\mathcal{F}_t$-(resp. $\mathcal{F}^{1-t}$- measurable) for all t and

$$E\{X_t - X_s | \mathcal{F}_s\} = 0 \quad \text{for all} \quad s < t,$$

$$(\text{resp. } E\{X_t - X_s | \mathcal{F}^{1-t}\} = 0 \quad \text{for all} \quad s < t).$$

Theorem 1.2. *Suppose that $X = \{X_t, 0 \le t \le 1\}$ is a square-integrable S-martingale. Then there exists a unique decomposition*

$$X = X^{(1)} + X^{(2)} + X^{(3)}, \tag{1.4}$$

where

$X^{(1)}$ is a forward martingale with $EX_0 = EX_0^{(1)}$,

$X^{(2)}$ is a backward martingale,

$X^{(3)}$ is an outward martingale,

$X^{(1)}$ and $X^{(2)}$ are given by

$$X_t^{(1)} = E\{X_1 | \mathcal{F}_t\}, X_t^{(2)} = E\{X_0 | \mathcal{F}_{1-t}\} - EX_0; \quad 0 \le t \le 1. \tag{1.5}$$

Proof. It is shown in [4] that a square-integrable S-martingale X has the following form

$$X_t = EX_0 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} I_n(h_{n,k} \cdot 1_{A_n^k(t)}) \right); \quad (0 \le t \le 1). \tag{1.6}$$

Here we put

$$T_n = \{(t_1, \ldots, t_n) : 0 < t_1 < \ldots < t_n < 1\},$$

$$A_n^k(t) = \{(t_1, ..., t_n) \in T_n : t_k < t < t_{k+1}\},$$

and $h_{n,k}$ are deterministic functions on T_n satisfying

$$\Lambda_n(h_{n,k}^2 \cdot 1_{A_n^k(t)}) := \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} h_{n,k}^2 \cdot 1_{A_n^k(t)} dt_1 ... dt_n < +\infty \quad \text{for all} \quad 0 \leq t \leq 1. \tag{1.7}$$

$I_n(f)$ denotes the multiple Ito integral of the deterministic function f (see [3]):

$$I_n(f) := \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} f(t_1, ..., t_n) dW_{t_1} ... dW_{t_n}.$$

Now we put

$$X_t^{(1)} = EX_0 + \sum_{n=1}^{\infty} I_n(h_{n,n} \cdot 1_{A_n^n(t)}) = E\{X_1 | \mathcal{F}_t\},$$

$$X_t^{(1)} = \sum_{n=1}^{\infty} I_n(h_{n,0} \cdot 1_{A_n^0(t)}) = E\{X_0 | \mathcal{F}_{1-t}\} - EX_0.$$

Clearly $X^{(1)}$ is a forward martingale, and $X^{(2)}$ is a backward martingale. On the other hand,

$$\phi_{uv} := h_{2,1}(u, 1-v) + \sum_{n=3}^{\infty} \sum_{k=1}^{n-1} I_{n-2}(h_{n,k}(t_1, ..., t_{k-1}, u, 1-v, t_k, ..., t_{n-2})), (u, v) \in T, \tag{1.8}$$

clearly satisfies hypotheses (1.1) and (1.2), and for every $0 \leq t \leq 1$,

$$X_t - X_t^{(1)} - X_t^{(2)} = \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} I_n(h_{n,k} \cdot 1_{A_n^k(t)}) \right) = \int_{R_{t,1-t}} \phi_{uv} dW_u dW^v.$$

Therefore $X^{(3)} = X - X^{(1)} - X^{(2)}$ is an outward martingale. The uniqueness of the above decomposition, if for instance we assume that $EX_0 = EX_0^{(1)}$, follows from the representation (1.5). q.e.d.

From the relation (1.8) we have immediately

$$E \int_T \phi_{uv}^2 du dv = \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \Lambda_n(h_{n,k}^2). \tag{1.9}$$

Suppose that $u = \{u_t, 0 \leq t \leq 1\}$ is a square-integrable measurable process and that for every $0 \leq t \leq 1$,

$$u_t = Eu_t + \sum_{n=1}^{\infty} I_n(f_n(t|\cdot)) .$$

The space $L^{1,2}$ (resp., $L^{2,2}$) consists of all such processes u verifying

$$||u||_{1,2}^2 := \int_0^1 (Eu_t)^2 dt + \sum_{n=1}^{\infty} n \cdot \int_0^1 \Lambda_n(f_n(t|\cdot)^2) dt < +\infty, \tag{1.10}$$

(resp.,

$$||u||_{2,2}^2 := \int_0^1 (Eu_t)^2 dt + \sum_{n=1}^{\infty} n^2 \cdot \int_0^1 \Lambda_n(f_n(t|\cdot)^2) dt < +\infty), \tag{1.11}$$

see [6].

A square-integrable Wiener functional

$$\xi = E\xi + \sum_{n=1}^{\infty} I_n(f_n)$$

is said to be smooth if and only if (see [6])

$$||D\xi||^2 := \sum_{n=1}^{\infty} (n-1)\Lambda_n(f_n^2) < +\infty.$$

Proposition 1.3. *Suppose that $X_t = \int_0^t udW = \delta(u \cdot 1_{[0,t]}), 0 \leq t \leq 1$, where $u \in L^{2,2}$, and let $X^{(3)}$ and ϕ be defined as in Theorem 1.2. Then*

$$E\int_T \phi_{uv}^2 dudv < +\infty. \tag{1.12}$$

Proof. We use the presentation (1.6).

In [4] the following relation has been obtained for the function Λ_n, appearing in (1.7)

$$\int_0^1 \Lambda_n(f_n(t|\cdot)^2)dt = \Lambda_n(h_{n,0}^2 + (h_{n,1} - h_{n,0})^2 + \ldots + (h_{n,n} - h_{n,n-1})^2). \quad (1.13)$$

On the other hand

$$\Lambda_n(h_{n,k}^2) \leq (k+1)\Lambda_n(h_{n,0}^2 + (h_{n,1} - h_{n,0})^2 + \ldots + (h_{n,k} - h_{n,k-1})^2)$$

for all $k = 1, \ldots, n-1$.

Hence

$$\sum_{k=1}^{n-1} \Lambda_n(h_{n,k}^2) \leq \frac{n(n+1)}{2} \int_0^1 \Lambda_n(f_n(t|\cdot)^2)dt.$$

Therefore, (1.9) implies

$$E\int_T \phi_{u,v}^2 dudv \leq \sum_{n=2}^{\infty} \frac{n(n+1)}{2} \int_0^1 \Lambda_n(f_n(t|\cdot)^2)dt.$$

Since $u \in L^{2,2}$,

$$\sum_{n=2}^{\infty} n^2 \int_0^1 \Lambda_n(f_n(t|\cdot)^2)dt < +\infty,$$

and so we get the desired conclusion

$$E\int_T \phi_{u,v}^2 dudv < +\infty. \qquad \text{q.e.d.}$$

An integrable process $M = \{M_t, 0 \leq t \leq 1\}$ is said to be an S-quasimartingale if and only if

$$\sup_\tau \sum_{i=0}^{m} E|E\{M_{\tau_{i+1}} - M_{\tau_i} | \mathcal{F}_{\tau_i} \vee \mathcal{F}^{1-\tau_{i+1}}\}| < +\infty,$$

where the supremum is taken over all finite partitions $0 = \tau_0 < \tau_1 < \ldots < \tau_{m+1} = 1$ of $[0,1]$. The following theorem specifies some properties of an S-martingale X implied by condition (1.12).

Theorem 1.4. *Suppose that X is a square-integrable S-martingale with decomposition (1.4), such that (1.12) holds. Then*

1) X has a Skorohod integral representation, i.e., there exists a unique process $u \in L^2([0,1] \times \Omega)$ such that $u.1_{[0,t]}$ is Skorohod integrable for all t and

$$X_t = \delta(u \cdot 1_{[0,t]}), \quad (0 \le t \le 1). \tag{1.14}$$

Moreover,

$$\lim_{|\tau| \to 0} E\left(\sum_{j=0}^{m} (X_{\tau_{j+1}} - X_{\tau_j})^2 \right) = E \int_0^1 u_s^2 ds. \tag{1.15}$$

2) $X^2 := \{X_t^2, \quad 0 \le t \le 1\}$ is an L^1- continuous S-quasimartingale.

Proof.

1) For $s < t$ we have

$$\begin{aligned} E(X_t - X_s)^2 &\le 3E\{(X_t^{(1)} - X_s^{(1)})^2 + (X_t^{(2)} - X_s^{(2)})^2 + (X_t^{(3)} - X_s^{(3)})^2\} \\ &= 3E\{(X_t^{(1)})^2 - (X_s^{(1)})^2 + (X_s^{(2)})^2 - (X_t^{(2)})^2 + ((X_t^{(3)}) - (X_s^{(3)}))^2\}. \end{aligned}$$

On the other hand,

$$E((X_t^{(3)}) - (X_s^{(3)}))^2 \le 2\left\{ E \int_{[s,t] \times [0,1-t]} \phi_{uv}^2 du dv + E \int_{[0,s] \times [1-t,1-s]} \phi_{uv}^2 du dv \right\}.$$

Therefore, for any partition $0 = \tau_0 < \tau_1 < ... < \tau_{m+1} = 1$ of $[0,1]$ we have

$$E\left\{ \sum_{j=0}^{m} (X_{\tau_{j+1}} - X_{\tau_j})^2 \right\} \le 3E\{(X_1^{(1)})^2 + (X_0^{(2)})^2\} + 12E \int_T \phi_{uv}^2 du dv. \tag{1.16}$$

From Proposition 2.3 of [4], it follows that X has a Skorohod integral representation, i.e., there exists a unique process $u \in L^2([0,1] \times \Omega)$ such that (1.14) holds. In this case, it is easy to see that

$$\lim_{|\tau| \to 0} E\left(\sum_{j=0}^{m} (X_{\tau_{j+1}} - X_{\tau_j})^2 \right) = E \int_0^1 u_s^2 ds.$$

2) For any $s < t$, we have

$$E\{X_t^2 - X_s^2 | \mathcal{F}_s \vee \mathcal{F}^{1-t}\} =$$

$$E\{(X_t - X_s)^2 | \mathcal{F}_s \vee \mathcal{F}^{1-t}\} + 2E\{(X_t - X_s)(X_s - E\{X_s | \mathcal{F}_s \vee \mathcal{F}^{1-t}\}) | \mathcal{F}_s \vee \mathcal{F}^{1-t}\}.$$

Therefore

$$E|E\{X_t^2 - X_s^2 | \mathcal{F}_s \vee \mathcal{F}^{1-t}\}|$$

$$\le E(X_t - X_s)^2 + 2E|(X_t - X_s)(X_s - E\{X_s|\mathcal{F}_s \vee \mathcal{F}^{1-t}\})|$$

$$\le 2E(X_t - X_s)^2 + E(X_s - E\{X_s|\mathcal{F}_s \vee \mathcal{F}^{1-t}\})^2. \tag{1.17}$$

On the other hand,

$$X_s - E\{X_s|\mathcal{F}_s \vee \mathcal{F}^{1-t}\} = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} I_n(h_{n,k} \cdot 1_{\{t_k<s<t_{k+1}<t\}}).$$

Hence

$$E(X_s - E\{X_s|\mathcal{F}_s \vee \mathcal{F}^{1-t}\})^2 = \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} \Lambda_n(h_{n,k}^2 \cdot 1_{\{t_k<s<t_{k+1}<t\}}).$$

Let $0 = \tau_0 < \ldots < \tau_{m+1} = 1$ be a partition of $[0,1]$, and put

$$B(n,k) = \{(t_1, \ldots, t_n) \in T_n$$

such that there exists $i : t_k < \tau_i < t_{k+1} < \tau_{i+1}\}$. We have

$$\sum_{j=0}^{m} E(X_{\tau_j} - E\{X_{\tau_j}|\mathcal{F}_{\tau_j} \vee \mathcal{F}^{1-\tau_{j+1}}\})^2 \le \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} \Lambda_n(h_{n,k}^2 \cdot 1_{B(n,k)})$$

$$\le \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} \Lambda_n(h_{n,k}^2) = E\int_T \phi_{uv}^2 \, du dv. \tag{1.18}$$

Therefore, from (1.16)-(1.18),

$$\sum_{j=0}^{m} E|E\{X_{\tau_{j+1}}^2 - X_{\tau_j}^2 | \mathcal{F}_{\tau_j} \vee \mathcal{F}^{1-\tau_{j+1}}\}|$$

$$\le 2\sum_{j=0}^{m} E(X_{\tau_{j+1}} - X_{\tau_j})^2 + E\int_T \phi_{uv}^2 \, du dv$$

$$\le 6E\{(X_1^{(1)})^2 + (X_0^{(2)})^2\} + 25E\int_T \phi_{uv}^2 \, du dv.$$

The right side of the above inequality does not depend on the choice of the partition, and so X^2 is an S- quasimartingale.

To show L^1-continuity of X^2, we first note

$$E|X_t^2 - X_s^2| \le (E(X_t - X_s)^2)^{1/2} \cdot (E(X_t + X_s)^2)^{1/2}. \tag{1.19}$$

Moreover,

$$E(X_t + X_s)^2 \le 2E(X_t^2 + X_s^2) \le 4\left((EX_0)^2 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \Lambda_n(h_{n,k}^2)\right). \tag{1.20}$$

If we define $X_{p,t} = \sum_{n=p}^{\infty}\left(\sum_{k=0}^{\infty} I_n(h_{n,k}\cdot 1_{A_n^k(t)})\right)$, then

$$E(X_{p,t} - X_{p,s})^2 \le 4\sum_{n=p}^{\infty}\sum_{k=0}^{n} \Lambda_n(h_{n,k}^2). \tag{1.21}$$

Thus, the left side of this inequality tends to zero as $p \to \infty$, uniformly in s and t. On the other hand

$$E(X_t - X_s)^2 = E(X_{p,t} - X_{p,s})^2 + A,$$

where

$$\begin{aligned} A &= E(X_t - X_{p,t} - X_s + X_{p,s})^2 \\ &\le \sum_{n=1}^{p-1}(n+1)\cdot\left(\sum_{k=0}^{n} \Lambda_n(h_{n,k}^2\cdot 1_{A_n^k(s)\triangle A_n^k(t)})\right) \end{aligned} \tag{1.22}$$

which tends to zero as p fixed and $(t-s) \to 0$, since the Lebesgue measure of the symmetric difference $A_n^k(s)\triangle A_n^k(t)$ tends to zero as $(t-s)\to 0$. The L^1-continuity of X^2 now follows from (1.19) -(1.22). q.e.d.

Let us now consider the particular case of a constant process $X_t \equiv \xi$, $(0 \le t \le 1)$, where $\xi \in L^2(\Omega)$. According to Theorem 1.2, we have

$$\begin{aligned} X_t^{(1)} &= E\{\xi|\mathcal{F}_t\}, \\ X_t^{(2)} &= E\{\xi|\mathcal{F}^{1-t}\} - E\xi, \\ X_t^{(3)} &= \sum_{n=2}^{\infty} I_n(h_n\cdot 1_{\cup_{k=1}^{n-1} A_n^k(t)}) = \sum_{n=2}^{\infty} I_n(h_n\cdot 1_{\{t_1<t<t_n\}}), \end{aligned} \tag{1.23}$$

where

$$\xi = E\xi + \sum_{n=1}^{\infty} I_n(h_n)$$

is the Wiener chaos expansion of ξ.

Conversely, given a square-integrable S-martingale

$$X_t = EX_0 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} I_n(h_{n,k}\cdot 1_{A_n^k(t)}), \quad 0 \le t \le 1,$$

the existence of a random variable $\xi \in L^2$ such that $X^{(3)}$ is given by (1.23) is clearly equivalent to

$$h_{n,1} = h_{n,2} = \dots = h_{n,n-1} (:= h_n)$$

for all $n = 2, 3, \dots$ and

$$\sum_{n=2}^{\infty} \Lambda_n(h_n^2) < +\infty.$$

This implies the following characterization of smooth Wiener functionals:

Corollary 1.5. *Let*

$$\int_{R_{t,1-t}} \phi_{uv} dW_u dW^v \quad (0 \le t \le 1)$$

be the process $X^{(3)}$ associated to $\xi \in L^2$ via (1.23). Then $\xi \in D^{1,2}$ if and only if $E\int_T \phi_{uv}^2 dudv < +\infty$, and we have

$$||D\xi||^2 = E\int_T \phi_{uv}^2 dudv.$$

Let us now show that the process $X^{(3)}$ associated to

$$\xi = E\xi + \sum_{n=1}^{\infty} I_n(h_n) \in L^2$$

can be represented as a Skorohod integral (1.14):

$$X_t^{(3)} = \delta(uI_{[0,t]}) \quad (0 \le t \le 1).$$

In fact, define

$$f_n(t|t_1,\dots,t_n) = \begin{cases} h_{n+1}(t,t_1,\dots,t_n) & \text{if } t < t_1 \\ -h_{n+1}(t_1,\dots,t_n,t) & \text{if } t_n < t \\ 0 & \text{otherwise} \end{cases}$$

and put

$$u_t = \sum_{n=1}^{\infty} I_n(f_n(t|.)), \quad 0 \le t \le 1.$$

Then $u \cdot I_{[0,t]}$ is Skorohod integrable, and (1.14) holds. Moreover, we have

$$\int_0^1 \Lambda_n(f_n(t|\cdot))^2 dt = 2\Lambda_{n+1}(h_{n+1}^2).$$

Therefore, $u \in L^{1,2}$ if and only if $\xi \in D^{1,2}$, and this is equivalent to

$$E \int_T \phi_{uv}^2 dudv < +\infty.$$

Thus, we get a class of examples where $u \in L^{1,2} \backslash L^{2,2}$, but nevertheless we have (1.12).

In [5] we proved that if $u \in L^{2,2}, X_t = \int_0^t u_s dW_s,\ 0 \le t \le 1$, and f is a function of class C^2 with an uniformly continuous and bounded second derivative then $f(X)$ is an S- quasimartingale. Moreover, in that case, $f(X)$ has a Doob-Meyer decomposition

$$f(X_t) = M_t + A_t, \quad 0 \le t \le 1,$$

where the variation part, $(A_t)_{0 \le t \le 1}$, is given in an explicit form.

In the following, based on Theorem 1.2, we shall show that $f(X)$ is still an S-quasimartingale even if function f is only supposed to belong to the class C^2 with bounded second derivative. However, in this case, we could not have an explicit representation for the variation part $(A_t)_{0 \le t \le 1}$.

Theorem 1.6. *Suppose that X is a square-integrable S-martingale with decomposition (1.4), such that (1.12) holds and f is a function of class C^2 with a bounded second derivative. Then $(f(X_t),\ 0 \le t \le 1)$ is an S-quasimartingale.*

Proof. Let m be a positive real number such that

$$|f''(x)| \le m$$

for all $x \in R$. For $s < t$, we denote

$$\bar{X}_{st} = X_s^{(1)} + X_t^{(2)} + X_{st}^{(3)}.$$

Clearly, $\bar{X}_{st}$ is $\mathcal{F}_s \vee \mathcal{F}^{1-t}$-measurable.

By the Taylor decomposition theorem we have

$$|f(X_t) - f(\bar{X}_{st}) - f'(\bar{X}_{st})(X_t - \bar{X}_{st})| \le \frac{m}{2}(X_t - \bar{X}_{st})^2 \tag{1.24}$$

and

$$|f(X_s) - f(\bar{X}_{st}) - f'(\bar{X}_{st})(X_s - \bar{X}_{st})| \le \frac{m}{2}(X_s - \bar{X}_{st})^2.$$

On the other hand,

$$E\{X_t - \bar{X}_{st}|\mathcal{F}_s \vee \mathcal{F}^{1-t}\} = E\{X_s - \bar{X}_{st}|\mathcal{F}_s \vee \mathcal{F}^{1-t}\} = 0.$$

Therefore, from (1.24) we get the following estimation

$$E|E\{f(X_t) - f(X_s)|\mathcal{F}_s \vee \mathcal{F}^{1-t}\}| \le \frac{m}{2}E\{(X_s - \bar{X}_{st})^2 + (X_t - \bar{X}_{st})^2\}. \tag{1.25}$$

Meanwhile,

$$X_t - \bar{X}_{st} = X_t^{(1)} - X_s^{(1)} + \int_{[s,t]\times[0,1-t]} \phi_{uv} dW_u dW^v$$

and

$$X_s - \bar{X}_{st} = X_s^{(2)} - X_t^{(2)} + \int_{[0,s]\times[1-t,1-s]} \phi_{uv} dW_u dW^v$$

Thus, from a property of one-parameter martingales we have

$$E\{(X_t - \bar{X}_{st})^2 + (X_s - \bar{X}_{st})^2\} \le 2E\{(X_t^{(1)})^2 - (X_s^{(1)})^2 + (X_s^{(2)})^2 - (X_t^{(2)})^2\} + E\int_{\{[s,t]\times[0,1-t]\cup[0,s]\times[1-t,1-s]\}} \phi_{uv}^2 du dv. \tag{1.26}$$

Now suppose that $\tau = \{0 = \tau_0 < \tau_1 < \dots < \tau_{n+1} = 1\}$ is a partition of [0,1]. From (1.25) and (1.26) it yields that

$$\sum_{i=0}^{n} E|E\{f(X_{\tau_{i+1}} - f(X_{\tau_i})|\mathcal{F}_{\tau_i} \vee \mathcal{F}^{1-\tau_{i+1}}\}| \le mE\{(X_1^{(1)})^2 + (X_0^{(2)})^2 + 2\int_T \phi_{uv}^2 du dv\}.$$

In particular, $f(X_t)$, $0 \le t \le 1$, is an S-quasimartingale. q.e.d.

Corollary 1.7. *Suppose that $u \in L^{2,2}$ and f a function of class C^2 with a bounded second derivative. Then $f(\int_0^t u dW)$, $0 \le t \le 1$, is an S-quasimartingale.*

Proof. It follows immediately from Theorem 1.6 and Proposition 1.3. q.e.d.

2. A MOMENT INEQUALITY FOR S-MARTINGALES

Consider a square-integrable S-martingale with decomposition as in Theorem 1.2,

$$X = X^{(1)} + X^{(2)} + X^{(3)}.$$

Since $X^{(1)}$ and $X^{(2)}$ are one-parameter Brownian martingales, it is well-known that both of them have continuous version.

To study the existence of a continuous version of a square integrable S-martingale X, it is therefore enough to consider the case of an outward martingale,

$$X_t = \int_{R_{t,1-t}} \phi_{uv} dW_u dW^v, \quad 0 \leq t \leq 1.$$

Let $(A_i)_{i=1}^{\infty}$ be a partition of T into rectangles $A_i = [a_i, b_i]$ with $b_i = (t_i, 1 - t_i)$ and $A_i^0 \cap A_j^0 = \emptyset$ for all $i \neq j$.

Put

$$K(X) = \inf_{(A_i)} \sum_{i=1}^{\infty} E\Big\{ \Big(\int_{A_i} \phi_{uv}^2 dudv \Big)^{1/2} \Big\}, \tag{2.1}$$

where the infremum is taken over all such partitions.

Note that from Jensen's inequality it follows that

$$K(X) \leq \inf_{(A_i)} \sum_{i=1}^{\infty} \Big(\int_{A_i} E\phi_{uv}^2 dudv \Big)^{1/2}.$$

As we have defined the two-parameter process

$$X_{st} = \int_{R_{st}} \phi_{uv} dW_u dW^v, \quad (s,t) \in T,$$

and now we put

$$X^* = \sup_{z \in T} |X_z|.$$

Theorem 2.1. *There exists an universal constant C such that for any outward martingales $X = \{X_t, 0 \leq t \leq 1\}$ whose corresponding process $X = \{X_{st}, (s,t) \in T\}$ is sample continuous the following inequality holds*

$$EX^* \leq C \cdot K(X).$$

Proof. Let $(A_i)_{i=1}^{\infty}$ be a partition of T into rectangles such that

$$\sum_{i=1}^{\infty} E\Big\{\Big(\int_{A_i} \phi_{uv}^2 dudv\Big)^{1/2}\Big\} < +\infty. \tag{2.2}$$

From the Burkholder-Davis-Gundy inequality for two-parameter continuous martingales ([1]) it follows that there exists an universal constant C so that for any $i = 1, 2, \ldots$

$$E\{\sup_{z\in A_i} |\triangle X[a_i, z]|\} \le CE\Big(\int_{A_i} \phi_{uv}^2 dudv\Big)^{1/2},$$

where $A_i = [a_i, b_i]$ and $\triangle X[a_i, z]$ denotes the increment of X on the rectangle $[a_i, z]$. Therefore, from (2.2)

$$E\Big\{\sup_{x\in A_i} |\triangle X[a_i, z]|\Big\} \le C \cdot \sum_{i=1}^{\infty} E\Big\{\Big(\int_{A_i} \phi_{uv}^2 dudv\Big)^{1/2}\Big\} < +\infty.$$

The desired inequality now follows from the following fact

$$X^* \le \sum_{i=1}^{\infty} \sup_{z\in A_i} |\triangle X[a_i, z]| \quad \text{a.s.} \qquad \text{q.e.d.}$$

Corollary 2.2. *Suppose that X is a square integrable S-martingale with decomposition (1.4) such that $K(X^{(3)}) < +\infty$. Then X has a continuous version.*

Proof. Without loss of generality we can assume that $X = X^{(3)}$ that means X is an outward martingale.

From $K(X) < +\infty$, we can suppose that $(A_i)_{i=1}^{\infty}$ is a partition of T into rectangles such that (2.2) holds.

For any $n = 1, 2, \ldots$ we define a new process $\{X_z^{(n)}\}, z \in T$ as follows

$$X_z^{(n)} = \int_{R_z \cap \{\cup_{i=1}^{n} A_i\}} \phi_{uv} dW_u dW^v.$$

Clearly, $X^{(n)}, n \ge 1$ are continuous-paths processes. From the above Theorem 2.1, we have for all $n < m$

$$\begin{aligned} E(\sup_{z\in T} |X_z^{(m)} - X_z^{(n)}|) &\le E\Big\{\sum_{i=n+1}^{m} \sup_{z\in A_i} |\triangle X[a_i, z]|\Big\} \\ &\le C \cdot \sum_{i=n+1}^{m} E\Big(\int_{A_i} \phi_{uv}^2 dudv\Big)^{1/2}. \end{aligned} \tag{2.3}$$

The above estimation ensures that P-almost surely $\{X^{(n)}, n = 1, 2, ...\}$ is Cauchy sequence in the space $C(T)$ and its limit clearly is a continuous version of $\{X_z, z \in T\}$ q.e.d.

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