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ON THE SMOOTHNESS OF GLOBAL SOLUTIONS OF HAMILTON-JACOBI EQUATIONS *

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Abstract. *We investigate the smoothness of global solutions of Cauchy problem for some Hamilton-Jacobi equations.*

It is well-known that the Cauchy problems for Hamilton-Jacobi equations do not have classical solutions in general, even if the Hamiltonians and initial conditions are smooth. Most of authors have approached these problems by looking for generalized solutions, usually locally Lipschitz real functions which satisfy the Hamilton-Jacobi equations almost everywhere or continuous functions which satisfy some differential inequalities.

In [7] Subbotin has defined minimax solution for a class of Hamilton-Jacobi equations. He has also proved that a minimax solution of Hamilton-Jacobi equation is also a viscosity solution [7]. The notion of viscosity solution was introduced by Crandall and Lions [4]. Moreover, in some cases if the initial condition is convex and Lipschitz, then minimax solution is a Lipschitz solution, see [3]. Therefore these notions of solutions are equivalent under suitable hypotheses on Hamiltonians and initial conditions.

Nevertheless by the definition, the minimax solutions or the Lipschitz solutions are only continuous or Lipschitz functions so we do not know much about their differentiability and whether they are classical solutions.

This paper consists of two sections. In §1 we prove that the Lipschitz solutions are classical solutions basing on Hopf's formula with some appropriate assumptions. In §2 the smoothness of minimax solutions for some concrete Hamilton-Jacobi equations is investigated.

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1. LIPSCHITZ SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

Consider the Cauchy problem

$$u_t + H(u_x) = 0, \tag{1}$$

$$u(0,x) = f(x), \tag{2}$$

with $(t,x) \in G = (0,T) \times R$.

We denote by Lip G the set of all locally Lipschitz continuous functions u defined on G. Recall that a classical (resp. Lipschitz) solution of (1), (2) is a function $u \in C^1(G)$ (resp. $u \in \text{Lip } G$) that satisfies (2) on $\{t = 0, x \in R\}$ and (1) in G (resp. almost everywhere in G).

In [5], Hopf proved the following theorem

Theorem 1.1. *If H is continuous and f is convex and Lipschitz, then*

$$u(t,x) = \max_p \min_y \{f(y) + p(x-y) - tH(p)\} \tag{3}$$

is the Lipschitz solution of problem (1), (2). Furthermore, u is Lipschitz in x with the same Lipschitz constant as f for all $t > 0$.

Let $f^*(p) = \max\limits_{y \in R} \{py - f(y)\}$ be the conjugate function of f.

Since f is convex and Lipschitz so f^* is convex and $D = \text{Dom } f^* = \{p \in R : f^*(p) < +\infty\}$ is bounded [6, p.134]. Therefore D is an interval with finite end-points a, b ($a \leq b$) and then

$$\begin{aligned} u(t,x) &= \max_{p \in R} \min_{y \in R} \{f(y) + p(x-y) - tH(p)\} \\ &= \max_{p \in D} \{px - f^*(p) - tH(p)\}. \end{aligned} \tag{4}$$

We shall prove the following theorem

Theorem 1.2. *Suppose that the assumptions of Theorem 1.1 hold, moreover f^*, $H \in C^2(\text{int } D)$ and*

$$f^{*\prime\prime}(p) + tH''(p) > 0, \quad \forall (t,p) \in (0,T) \times \text{int } D. \tag{5}$$

Then the Lipschitz solution (3) is the classical solution of problem (1), (2).

The proof of Theorem 1.2 follows the lemma below. For a given function u of n variables $(x_1, ..., x_n)$, we denote $(u_{x_1}(x), ..., u_{x_n}(x))$ by $u'(x)$, where $x = (x_1, ..., x_n)$.

Lemma 1.3 *Let V be an open set in R^n, F be a relatively closed subset of V. Suppose that $u \in C(V) \cap C^1(V \setminus F)$ and $v \in C^1(V)$; $u|_F = v|_F$ and $\lim_{\substack{x \to y \\ x \in V \setminus F}} u'(x) = v'(y)$, $\forall y \in F$. Then $u \in C^1(V)$.*

Proof. Without loss of generality we can assume that $v = 0$. Let y be a given point of F. If $x \in F$ then $u(x) - u(y) = 0$. If $x \in V \setminus F$ (and close enough to y), we denote by z the point of $F \cap [x, y]$ nearest to x. From the mean-value theorem and by assumption, it follows that

$$|u(x) - u(y)| = |u(x) - u(z)| \leq ||x - z|| \sup_{t \in (0,1)} ||u'(z + t(x - z))|| = o||x - y||$$

as $x \to y$.

Thus u is differentiable at y and $u'(y) = 0$, $\forall y \in F$. Hence, $u \in C^1(V)$. □

Proof of Theorem 1.2. Let $c(t, x, p) = x - f^{*\prime}(p) - tH'(p)$, $(t, x, p) \in G \times \text{int } D$. Then

$$\begin{aligned} F_1 &= \{(t, x) \in G / \ \forall p \in \text{int } D : c(t, x, p) \leq 0\}, \\ F_2 &= \{(t, x) \in G / \ \forall p \in \text{int } D : c(t, x, p) \geq 0\} \end{aligned}$$

are closed in G. Moreover $F_1 \cap F_2 = \phi$ since $\dfrac{\partial c}{\partial p}(t, x, p) = -f^{*\prime\prime}(p) - tH''(p) < 0$ by (5).

We put $F = F_1 \cup F_2$. For $(t, x) \in G \setminus F$, it is easy to see that the function $p \to px - f^*(p) - tH(p)$ attains its maximum at one unique point $\bar{p} = \bar{p}(t, x) \in \text{int } D$. Then $c(t, x, \bar{p}(t, x)) = 0$. Applying the implicit function theorem on the function $(t, x, p) \to c(t, x, p)$ we deduce that $\bar{p} = \bar{p}(t, x)$ is continuously differentiable in $G \setminus F$. Therefore the solution (4)

$$u(t, x) = x\bar{p}(t, x) - f^*(\bar{p}(t, x)) - tH(\bar{p}(t, x))$$

is continuously differentiable in $G \setminus F$.

According to the lower semicontinuity of f^* we see that if $F_1 \neq \phi$, then $a \in D$ and

$$\forall (t, x) \in F_1 : \quad u(t, x) = ax - f^*(a) - tH(a).$$

Analogously, if $F_2 \neq \phi$, then $b \in D$ and

$$\forall (t,x) \in F_2: \quad u(t,x) = bx - f^*(b) - tH(b).$$

Let $(t_0, x_0) \in F_1$. We shall prove that $\bar{p}(t,x) \to a$ as $G \setminus F \ni (t,x) \to (t_0, x_0)$. Indeed, if $\bar{p}(t,x) \not\to a$ then $\exists \epsilon > 0 \quad (\epsilon < \frac{b-a}{2}), \forall n \in N, \quad \exists (t_n, x_n) \in G \setminus F$: $|t_n - t_0| + |x_n - x_0| < \frac{1}{n}$ and $b \geq \bar{p}(t_n, x_n) \geq a + \epsilon$.

The sequence $q(t_n, x_n) = \min\{\bar{p}(t_n, x_n), \frac{a+b}{2}\}$ is bounded, so there exists a subsequence denoted also by $q(t_n, x_n)$ tending to $a' \in [a + \epsilon, \frac{a+b}{2}]$. From the inequalities

$$c(t_n, x_n, q(t_n, x_n)) \geq 0, \quad \forall n \in N,$$

passing $n \to \infty$, we obtain $c(t_0, x_0, a') \geq 0 \geq c(t_0, x_0, a + 0)$. This contradicts with $\frac{\partial c}{\partial p}(t_0, x_0, p) < 0, \quad \forall p \in \text{int } D.$

Now, let $v(t,x) = ax - f^*(a) - tH(a), \quad v \in C^1(G \setminus F_2)$. Some simple calculations show that

$$u_t(t,x) = -H(\bar{p}(t,x)) \to -H(a) = v_t(t_0, x_0),$$
$$u_x(t,x) = \bar{p}(t,x) \to a = v_x(t,x),$$

as $G \setminus F \ni (t,x) \to (t_0, x_0)$.

Applying Lemma 1.3 we immediately deduce that $u \in C^1(G \setminus F_2)$.

By an analogous argument, we also see that $u \in C^1(G \setminus F_1)$. Hence $u \in C^1(G)$.□

Remark 1.4.

1. M. Bardi and L.C. Evans proved that formula (3) is the classical solution of problem (1), (2) in $(0,T) \times R^n$ under a strong assumption that f is Lipschitz, $|D^2 f| \in L^\infty(R)$, and, moreover, both H and f are convex.(cf. Section 4, [2]).
2. We consider the following example. The problem

$$u_t - (u_x)^2 = 0,$$
$$u(0,x) = |x|,$$

in G possesses a Lipschitz solution, by formula (3),

$$u(t,x) = |x| + t.$$

We see that all the conditions of Theorem 1.2 hold except for (5) and it is clear that this solution is not differentiable on $(0,T) \times \{0\}$. So the condition (5) is significant.

2. SMOOTHNESS OF MINIMAX SOLUTIONS OF CAUCHY PROBLEMS

Let α, β be positive numbers such that $\alpha^{-1} + \beta^{-1} = 1$, $T > 1$. We now consider the problem

$$u_t + (1 + |u_x|^\alpha)^{\frac{1}{\alpha}} = 0, \tag{6}$$

$$u(T, x) = \frac{|x|^\beta}{\beta}, \tag{7}$$

with $(t, x) \in G = (0, T) \times R$.

The function

$$u(t, x) = \max_{l \in R} \left(lx - \frac{|l|^\alpha}{\alpha} + (T - t)(1 + |l|^\alpha)^{\frac{1}{\alpha}} \right) \tag{8}$$

is the minimax solution of the problem (6), (7), (see [7, p.14 and p.112]). Therefore $u \in C([0, T] \times R)$.

Now we state the main result of this section.

Theorem 2.1. *For $\alpha > 1$ the minimax solution (8) is continuously differentiable on $G \setminus ((0, T - 1] \times \{0\})$ and is not differentiable at any point of $(0, T - 1) \times \{0\}$.*

Proof. For $(t, x) \in [0, T] \times R$, $l \in R$, we put

$$g(t, x, l) = lx - \frac{|l|^\alpha}{\alpha} + (T - t)(1 + |l|^\alpha)^{\frac{1}{\alpha}}.$$

We see that $\lim_{l \to \pm\infty} g(t, x, l) = -\infty, \quad \forall (t, x) \in [0, T] \times R$. Thus the function $g(t, x, \cdot)$ attains a global maximum at some $\bar{l} = \bar{l}(t, x) \in R$. Moreover, since $g(t, x, .) \in C^1(R)$ it follows that

$$\frac{\partial g}{\partial l}(t, x, \bar{l}(t, x)) = 0. \tag{9}$$

We have

$$\frac{\partial g}{\partial l}(t, x, l) = x + [(T - t)(1 + |l|^\alpha)^{\frac{1}{\alpha} - 1} - 1]|l|^{\alpha - 1} \mathrm{sgn}\, l, \tag{10}$$

and

$$\frac{\partial^2 g}{\partial l^2}(t, x, l) = |l|^{\alpha - 2} \Big(\frac{T - t}{(1 + |l|^\alpha)^{2 - \frac{1}{\alpha}}} - 1 \Big)(\alpha - 1), l \neq 0. \tag{11}$$

The proof is divided into the following cases

Case 1. $t \in (T-1, T), x \in R.$

In this case we have $\dfrac{\partial^2 g}{\partial l^2}(t,x,l) < 0, \forall l \neq 0$. Therefore $\bar{l}(t,x)$ satisfying (9) is uniquely determined. Furthermore, if $x \neq 0$ then $\bar{l}(t,x) \neq 0$. Hence by virtue of the implicit function theorem and from (9), we see that $\bar{l} \in C^1((T-1,T) \times (R \setminus \{0\}))$. Consequently the function

$$u(t,x) = \bar{l}(t,x)x - \frac{|\bar{l}(t,x)|^\alpha}{\alpha} + (T-t)(1+|\bar{l}(t,x)|^\alpha)^{\frac{1}{\alpha}} \tag{12}$$

is continuously differentiable on $(T-1,T) \times (R \setminus \{0\})$.

On the other hand

$$\bar{l}_x(t,x) = -\frac{\frac{\partial^2 g}{\partial x \partial l}(t,x,\bar{l})}{\frac{\partial^2 g}{\partial l^2}(t,x,\bar{l})} = \frac{1}{(\alpha-1)|\bar{l}|^{\alpha-2}\left(\frac{T-t}{(1+|\bar{l}|^\alpha)^{2-\frac{1}{\alpha}}} - 1\right)}, \tag{13.a}$$

and

$$\bar{l}_t(t,x) = \frac{|\bar{l}| \text{ sgn } \bar{l}(1+|\bar{l}|^\alpha)}{(T-t)-(1+|\bar{l}|^\alpha)^{2-\frac{1}{\alpha}}}. \tag{13.b}$$

Then in $(T-1,T) \times (R \setminus \{0\})$ the partial derivatives $u_x(t,x), u_t(t,x)$ can be computed as follows

$$\begin{aligned} u_x(t,x) &= \bar{l}(t,x) + x\bar{l}_x - |\bar{l}(t,x)|^{\alpha-1}\bar{l}_x \text{ sgn } \bar{l} \\ &\quad + (T-t)(1+|\bar{l}(t,x)|^\alpha)^{\frac{1}{\alpha}-1}|\bar{l}(t,x)|^{\alpha-1}\bar{l}_x \text{ sgn } \bar{l}, \end{aligned} \tag{14.a}$$

and

$$u_t(t,x) = x\bar{l}_t - \frac{(1+|\bar{l}|^\alpha)^{\frac{1}{\alpha}-1}|\bar{l}|^\alpha[(T-t)(1+|\bar{l}|^\alpha)^{\frac{1}{\alpha}-1}-1]}{(\alpha-1)[(T-t)(1+|\bar{l}|^\alpha)^{\frac{1}{\alpha}-1}-1]} - (1+|\bar{l}|^\alpha)^{\frac{1}{\alpha}}. \tag{14.b}$$

From (9) and (10) it follows that $\bar{l}(t,x) \to 0$ when $(t,x) \to (t_0, 0)$, $\forall t_0 \in (T-1,T)$. Therefore by (13) and (14) we have

$$u_x(t,x) \to 0, \quad \text{and} \quad u_t(t,x) \to -1$$

when $(t,x) \to (t_0, 0)$.

Now let $v(t,x) = T-t$, $(t,x) \in (T-1,T) \times R$, we easily see that with $V = (T-1,T) \times R$, $F = (T-1,T) \times \{0\}$, the functions u, v satisfy the hypotheses of Lemma 3.1. Consequently $u \in C^1((T-1,T) \times R)$.

Case 2. $t \in (0, T-1)$, $x \in R$.

We have $g(t,0,l) = -\frac{|l|^\alpha}{\alpha} + (T-t)(1+|l|^\alpha)^{\frac{1}{\alpha}}$, then

$$g(t,x,l) = xl + g(t,0,l).$$

The equation $\frac{\partial^2 g}{\partial l^2} = 0$ has two roots $l_{1,2} = \pm[(T-t)^{\frac{\alpha}{2\alpha-1}} - 1]^{\frac{1}{\alpha}}$, and if $\alpha > 2$ it has one more $l_0 = 0$. The equation $\frac{\partial g}{\partial l}(t,0,l) = 0$ has three roots

$$l^*_{1,2} = \pm[(T-t)^{\frac{\alpha}{\alpha-1}} - 1]^{\frac{1}{\alpha}},$$

and $l^*_0 = 0$.

It is easily seen that $l^*_2 < l_2 < 0 < l_1 < l^*_1$, and $g(t,0,l)$ attains its maximum at l^*_1, l^*_2. Clearly, $u(t,0) = g(t,0,l^*_1) = g(t,0,l^*_2)$.

For $x \neq 0$, $t \in (0, T-1)$ the equation $\frac{\partial g}{\partial l}(t,x,l) = 0$ has at most one root in each interval $(-\infty, l_2), [l_2, l_1], (l_1, +\infty)$. In accordance with the sign of $\frac{\partial g}{\partial l}(t,x,l)$ we see that the function $g(t,x,.)$ does not attain its maximum on $[l_1, l_2]$. Now, in the case $x > 0$, we have

$$\begin{aligned}\sup_{l\in(l_1,+\infty)} g(t,x,l) &\geq g(t,x,l^*_1) = g(t,0,l^*_1) + l^*_1 x = g(t,0,l^*_2) + l^*_1 x \\ &> g(t,0,l^*_2) \geq \sup_{l\in(-\infty,l_2)} (g(t,0,l) + lx) = \sup_{l\in(-\infty,l_2)} g(t,x,l).\end{aligned}$$

So $g(t,x,.)$ attains the global maximum at a unique point $l^+ \in (l_1, +\infty)$. Arguing similarly to the case 1, we have $u \in C^1((0,T-1)\times(0,+\infty))$. By such a way we also see that $u \in C^1((0,T-1)\times(-\infty,0))$. Therefore $u \in C^1((0,T-1)\times(R\setminus\{0\}))$.

On the other hand, for every sufficiently small x, ($|x| < \alpha k^2 - k + 1$, with $k = [T-t]^{\frac{\alpha}{2\alpha-1}}$), equation $\frac{\partial g}{\partial l}(t,x,l) = 0$ has exactly three roots $l^+_2 < l^+_0 < l^+_1$ and $l^+_1 \to l^*_1$, $l^+_2 \to l^*_2$ when $x \to 0$.

Since $g(t,x,l) = g(t,-x,-l)\quad \forall (t,x,l) \in G \times R$, therefore

$$u(t,x) = \max_{l\in R} g(t,x,l) = \max_{-l\in R} g(t,-x,-l) = u(t,-x).$$

This means for fixed t, $w = u(t,.)$ is an even function and as shown above, $w \in C^1(R\setminus\{0\})$.

Now we assume that w is differentiable at $x=0$. Then it is obvious that $w'(0)=0$. The Lagrange Theorem gives us

$$\frac{w(x)-w(0)}{x}=w'(\xi)=u_x(t,\xi),\quad \xi\in(0,x).$$

A direct computation similar to the case 1 shows that

$$\forall t\in(0,T-1),\ u_x(t,x)\to l_1^*\neq 0,\text{ as } x\to 0^+.$$

Consequently,

$$w'(0^+)=\lim_{x\to 0^+}\frac{w(x)-w(0)}{x}=\lim_{x\to 0^+}u_x(t,\xi)=l_1^*\neq w'(0).$$

This contradiction shows that $u_x(t,0)=w'(0)$ does not exist. Therefore $u(t,x)$ is not differentiable at $(t,0)$, $t\in[0,T-1)$.

Case 3. $T-t=1$.

In this case $\dfrac{\partial^2 g}{\partial l^2}(T-1,x,l)<0$, $l\neq 0$ and $\dfrac{\partial^2 g}{\partial l^2}(T-1,x,l)\to 0$ when $l\to 0$.

It follows that the equation $\dfrac{\partial g}{\partial l}(T-1,x,l)=0$ has a unique root $l_0\neq 0$ if $x\neq 0$ and $l_0=0$ if $x=0$. Using the argument analogous to the case 1, we conclude that $u(t,x)$ is differentiable on $(T-1,x)$, $x\neq 0$.

Theorem 2.1 has been proved completely. □

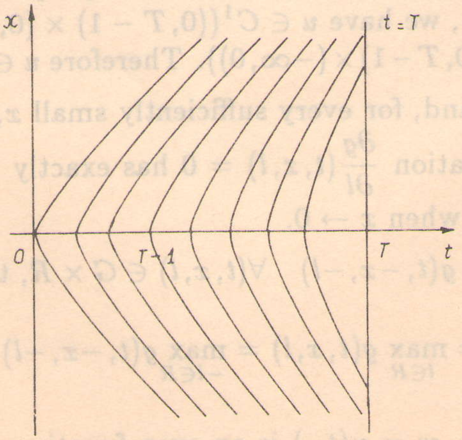

Fig. 1. The level curves of minimax solution (8)

Remark 2.2.

1. For $\alpha = 2$, the differentiability of solution $u(t, x)$ has been mentioned (without proof) in [7].
2. We notice that $u(t, x)$ is not a classical solution of (6), (7). Thus by the uniqueness of minimax solution, the problem (6), (7) does not have classical solution.
3. By the proof of case 1, we see that if $T \in (0, 1)$ then minimax solution of (6), (7) is the classical solution.
4. The function $u(t, x)$ is not a quasi-classical solution of (6),(7) since it is not differentiable on $[0, T-1) \times \{0\}$. (The definition of quasi-classical solution was introduced in [8])

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