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DISTRIBUTION AND VARIANCE OF STEREOLOGICAL ESTIMATORS OF VOLUME AND SURFACE AREA

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Abstract. *An analytical expression of the stereological estimators of volume and surface area is derived. By using this expression we will give some formulae for density functions, the variances of the stereological estimators and upper bounds of the variances for a case of balls.*

1. INTRODUCTION

A new set of fundamental stereological formulae based on isotropically orientated probes through fixed points has been derived by Jensen E. B. et al. (cf. [1], [2], [3]). These formulae are special cases of a generalized version of an integral geometric formula of Blaschke - Petkantschin type (cf. [4], [5]).

The following stereological formulae for volume and surface area in three dimensions are of practical importance:

Let $\mathbf{Z}$ be a bounded, convex subset in $\mathbb{R}^3$. Select a fixed arbitrary point $X \in \mathbf{Z}$. Suppose $L_2(X)$ is an isotropic random plane through X, generating the random section $\mathbf{Z} \cap L_2(X)$.

Let

$$\hat{V}(\mathbf{Z}) = 2\int_{\mathbf{Z}\cap L_2(X)} d(X,Y)\nu_2(dY), \tag{1}$$

$$\hat{S}(\mathbf{Z}) = 2\int_{\mathbf{Z}\cap L_2(X)} d(X,Y)/\sin\alpha(Y)\nu_1(dY), \tag{2}$$

where

$d(X,Y)$ is the distance between the two points X and Y,

$\alpha(Y)$ is the angle between the tangent plane to the boundary surface $\partial\mathbf{Z}$ at $Y \in \partial\mathbf{Z}$ and $L_2(X)$,

$\nu_1(dY)$ is the differential element of i- dimensional volume measure in $\mathbb{R}^3$, $i = 1, 2, 3$.

Then $\hat{V}(\mathbf{Z})$ and $\hat{S}(\mathbf{Z})$ are unbiased stereological estimators of the corresponding volume and surface area of the subset $\mathbf{Z}$:

$$E\hat{V}(\mathbf{Z}) = V(\mathbf{Z}), \tag{3}$$

$$E\hat{S}(\mathbf{Z}) = S(\mathbf{Z}). \tag{4}$$

In particular, the mean values do not depend on the choice of the point $X \in \mathbf{Z}$.

In the present paper, we will study the variances and probability densities of these estimators in the special case where $\mathbf{Z}$ is a sphere. Upper bounds of the variances are given. Direct proof of the unbiasedness is also given in spherical case.

2. AN ANALYTICAL EXPRESSION OF THE STEREOLOGICAL ESTIMATORS OF VOLUME AND SURFACE AREA

Without loss of generality, we can suppose that the sphere under consideration is centered at the origin O. Coordinate axes are denoted by n_1, n_2, n_3. Because of the summetry of the sphere, we need only to consider the case where the fixed point X lies on the n_3-axis, i.e. $X = (0, 0, X_3)$, $\quad 0 \leq X_3 \leq R$.

It is obvious that $\mathbf{Z} \cap L_2(X)$ is a circle of center I where OI coincides with the normal of $L_2(X)$. The radius r of the circle depends on the orientation of $L_2(X)$. Now, let us consider the integrands of (1) and (2), in particular, the function $d(X, Y)$ and let us give a geometrical interpretation of the estimators of $\hat{V}(\mathbf{Z})$ and $\hat{S}(\mathbf{Z})$.

The integrand $d(X, Y)$ forms a cone with the top at X, its symmetrical axis through X is perpendicular to section plane $\mathbf{Z} \cap L_2(X)$ and its top angle is right (cf. Fig. 1).

Therefore,

$$\hat{V}(\mathbf{Z}) = 2 \int_{\mathbf{Z} \cap L_2(X)} d(X, Y)\nu_2(dY)$$

is equal to 2 times of the volume of the bounded set enclosed by upper surface: $d(X, Y)$ (the cone), lower surface: $\mathbf{Z} \cap L_2(X)$ and lateral surface: cylinder with

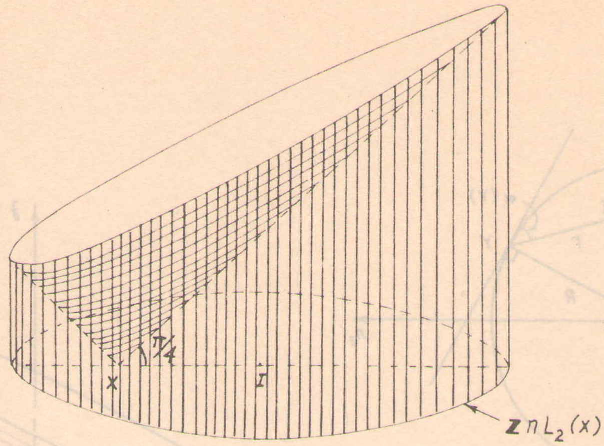

Fig. 1

lower base is the circle of center I and of radius r (cf. Fig. 1). This is also true for any bounded, convex subset.

When $\mathbf{Z}$ is a sphere, the tangent plane to the surface $\partial\mathbf{Z}$ at $Y \in \partial\mathbf{Z} \cap L_2(X)$ is always perpendicular to radius vector OY (cf. Fig. 2).

Therefore

$$\sin\alpha(Y) = \cos(\widehat{OYI}) = \frac{r}{R}$$

and analogously,

$$\hat{S}(\mathbf{Z}) = 2\int_{\partial\mathbf{Z}\cap L_2(X)} d(X,Y)/\sin\alpha(Y) \quad \nu_1(dY) = \frac{2R}{r}\int_{\partial\mathbf{Z}\cap L_2(X)} d(X,Y)\nu_1(dY)$$

is equal to $\frac{2R}{r}$ times of the surface area of the lateral surface of the cylinder described above (Fig. 1).

Now, we reduce the calculation of stereological estimators $\hat{V}(\mathbf{Z}), \hat{S}(\mathbf{Z})$ determined by (1) and (2), to that of the volume and surface area by double integrals. We determine a system of coordinates (x, y, z) with I as the origin, $L_2(X)$ as (x, y)- plane and with the normal vector OI as the $\mathbf{Z}$-axis (cf. Fig. 3).

Then, the cone equation is

$$z = \sqrt{(x-b)^2 + y^2}$$

and the equation of the circle section is

$$x^2 + y^2 = r^2.$$

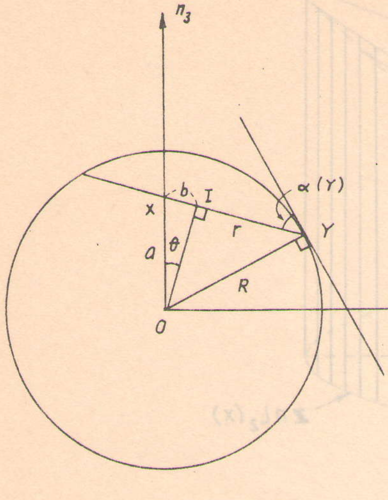

Fig. 2

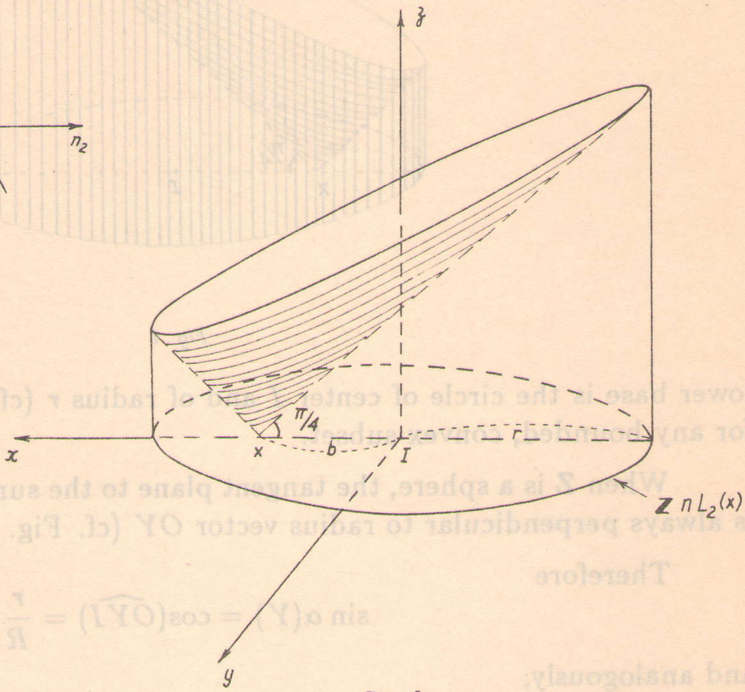

Fig. 3

Therefore,

$$\hat{V}(\mathbf{Z}) = 2 \iint_{\{x^2+y^2 \leq z^2\}} \sqrt{(x-b)^2 + y^2}\, dx dy. \tag{5}$$

Using the polar coordinates

$$x = b + u \cos t$$
$$y = u \sin t\,, \quad 0 \leq t \leq 2\pi$$

we get, after some calculations,

$$\begin{aligned} \hat{V}(\mathbf{Z}) &= \frac{4}{3} \int_0^{\pi} (4b^2 \cos^2 t + r^2 - b^2) \sqrt{b^2 \cos^2 t + r^2 - b^2}\; dt \\ &= \frac{8}{3} \int_0^{\pi/2} (4b^2 \cos^2 t + r^2 - b^2) \sqrt{b^2 \cos^2 t + r^2 - b^2}\; dt. \end{aligned} \tag{6}$$

Similarly, for $\hat{S}(\mathbf{Z})$, we can use the equations of the lateral surface $y = \pm\sqrt{r^2 - x^2}$,

$$\hat{S}(\mathbf{Z}) = \frac{4R}{r} \iint_{(D)} \sqrt{1 + y_x'^2 + y_z'^2}\, dx dz$$

where

$$(D) = \begin{cases} x^2 + y^2 = r^2 \\ (x-b)^2 + y^2 = z^2 \ , \ z = 0 \ , \ -r \leq x \leq r \end{cases}$$

$$= \{z^2 = r^2 + b^2 - 2bx \ , \ z = 0 \ , \ -r \leq x \leq r\}$$

to get finally

$$\hat{S}(\mathbf{Z}) = 4R \int_0^{\pi} \sqrt{r^2 + b^2 - 2rb\cos t}\, dt \, . \tag{7}$$

Note that (7) can be obtained more directly as follows: In the (x, y)- plane , by the law of cosines, the integrand $d(X, Y)$ is

$$d(X,Y) = \sqrt{IX^2 + IY^2 - 2IX.IY.\cos(\widehat{XIY})} = \sqrt{b^2 + r^2 - 2br\cos t}$$

and

$$\nu_1(dY) = rdt.$$

The radius r of the circle section and the length b of segment IX depend on the radius R of the sphere and on the position of the point X (i.e. the length of a segment OX) as well as on the random plane $L_2(X)$ passing by X.

As we noted previously, the vector OI coincides with the normal of a random plane $L_2(X)$. A determination of a random plane $L_2(X)$ passing by X is equivalent to the determination of its normal OI. The orientation of the normal OI is determined by (φ, θ)-angles in the spherical coordinates. Fortunately, a section $\mathbf{Z} \cap L_2(X)$ is insensitive to changes of the angle φ. Thus, after a rotation $L_2(X)$ with a fixed angle θ, and varying angle φ, the radius r and length b of the circle section do not change. Therefore $\hat{V}(\mathbf{Z})$ and $\hat{S}(\mathbf{Z})$ do not change when φ varying.

Angle θ varies from O to $\frac{\pi}{2}$ with the probability density $p(\theta) = \sin\theta$.

We have (cf. Fig. 2)

$$b = a\sin\theta, \tag{8}$$

$$r = \sqrt{R^2 - a^2\cos^2\theta}\, . \tag{9}$$

In two special cases, $\hat{V}(\mathbf{Z})$ and $\hat{S}(\mathbf{Z})$ can be given in a closed form. If point X coincides with the origin $(X \equiv O)$, then $a = 0, b = 0, r = R$. From (6) and (7) we have

$$\hat{V}(\mathbf{Z}) = \frac{4}{3}\pi R^3 = V(\mathbf{Z}), \qquad \hat{S}(\mathbf{Z}) = 4\pi R^2 = S(\mathbf{Z}). \tag{10}$$

If point X lies on the sphere, then $a = R, b = r = R\sin\theta$. From (6) and (7) we get

$$\hat{V}(\mathbf{Z}) = \frac{64}{9}R^3 \sin^3\theta, \tag{11}$$

$$\hat{S}(\mathbf{Z}) = 16R^2 \sin\theta. \tag{12}$$

The unbiasedness of estimators $\hat{V}(\mathbf{Z})$ and $\hat{S}(\mathbf{Z})$ can be proved as follows.

From (6), (8) and (9) we have

$$E\hat{V}(\mathbf{Z}) = \frac{8}{3}\int_0^{\frac{\pi}{2}} \sin\theta d\theta \int_0^{\frac{\pi}{2}} (4b^2\cos^2 t + r^2 - b^2)\sqrt{b^2\cos^2 t + r^2 - b^2}\,dt$$
$$= \frac{8}{3}\int_0^{\frac{\pi}{2}} dt \int_0^{\frac{\pi}{2}} (4a^2\cos^2 t\sin^2\theta + R^2 - a^2)\sqrt{a^2\cos^2 t\sin^2\theta + R^2 - a^2}\,\sin\theta d\theta.$$

Leaving out most of the detailed calculations, we get

$$E\hat{V}(\mathbf{Z}) = \frac{8}{3}\int_0^{\frac{\pi}{2}}\Big\{(R^2 - a^2)^{\frac{3}{2}} + \frac{3}{2}a^2\sqrt{R^2 - a^2}\cos^2 t +$$
$$\frac{3a}{2}\cos t(R^2 - a^2 + a^2\cos^2 t)\ \arcsin\frac{a\cos t}{\sqrt{R^2 - a^2 + a^2\cos^2 t}}\Big\}dt$$
$$= \frac{4}{3}\pi(R^2 - a^2)^{\frac{3}{2}} + \pi a^2\sqrt{R^2 - a^2} + 4a\int_0^1\Big\{\frac{u}{\sqrt{1-u^2}}(R^2 - a^2 + a^2u^2)$$
$$\arcsin\frac{au}{\sqrt{R^2 - a^2 + a^2u^2}}\Big\}du.$$

We need to calculate the last integral

$$I = 4a\int_0^1 \frac{u}{\sqrt{1-u^2}}(R^2 - a^2 + a^2u^2)\arcsin\frac{au}{\sqrt{R^2 - a^2 + a^2u^2}}du.$$

Putting $\arcsin \frac{au}{\sqrt{R^2-a^2+a^2u^2}} = t$ we get $u = \frac{\sqrt{R^2-a^2}}{a}\mathrm{tg}t$ and hence

$$I = 4a \int_0^{t_0} \frac{\frac{\sqrt{R^2-a^2}}{a}\mathrm{tg}t}{\sqrt{1-\frac{R^2-a^2}{a^2}\mathrm{tg}^2 t}}(R^2-a^2)(1+\mathrm{tg}^2 t)t.d\left(\frac{\sqrt{R^2-a^2}}{a}\mathrm{tg}t\right)$$

$$= \frac{4a}{3}(3R^2-a^2)\int_0^{t_0}\sqrt{1-\frac{R^2-a^2}{a^2}\mathrm{tg}^2 t\,dt}$$

$$+ \frac{4a}{3}(R^2-a^2)\int_0^{t_0}\sqrt{1-\frac{R^2-a^2}{a^2}\mathrm{tg}^2 t}\ \mathrm{tg}^2 t\,dt.$$

Futher, putting $\frac{\sqrt{R^2-a^2}}{a}\mathrm{tg}t = \cos v$ yields

$$I = \frac{8}{3}R^2\sqrt{R^2-a^2}\int_0^{\frac{\pi}{2}}\frac{\sin^2 v}{\frac{R^2-a^2}{a^2}+\cos^2 v}dv + \frac{4a^2}{3}\sqrt{R^2-a^2}\int_0^{\frac{\pi}{2}}\sin^2 v dv$$

$$= \frac{4}{3}\pi R^3 - \frac{4}{3}\pi(R^2-a^2)^{\frac{3}{2}} - a^2\pi\sqrt{R^2-a^2}.$$

Consequently,

$$E\hat{V}(\mathbf{Z}) = \frac{4}{3}\pi R^3 = V(\mathbf{Z}).$$

Thus equation (3) is proved.

Now from (7), (8) and (9) we have

$$E\hat{S}(\mathbf{Z}) = 4R\int_0^{\frac{\pi}{2}}\sin\theta d\theta\int_0^{\pi}\sqrt{R^2-a^2\cos 2\theta - 2a\sin\theta\sqrt{R^2-a^2\cos^2\theta}\cos t}\,dt. \tag{13}$$

If the point X lies on the sphere, from (12) we get

$$E\hat{S}(\mathbf{Z}) = 16R^2\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta = 4\pi R^2 = S(\mathbf{Z}).$$

If the point X is inside the sphere $(0 < a < R)$, then by the numerical integration method (with $R = 1$) we can see that the right-hand side of (13) is equal to $S(\mathbf{Z})$.

3. THE VARIANCE AND PROBABILITY DENSITY OF ESTIMATORS

In this section we consider some special cases of the above result.

a) If the point X is coincides with the origin, from (10) we have

$$\operatorname{Var} \hat{V}(\mathbf{Z}) = 0,$$

$$\operatorname{Var} \hat{S}(\mathbf{Z}) = 0,$$

where Var denotes the variance of an estimator.

b) If the point X lies on the sphere, from (11) we get

$$E\{\hat{V}(\mathbf{Z})\}^2 = \frac{64^2}{9^2} R^2 \int_0^{\frac{\pi}{2}} \sin^7 \theta d\theta = \frac{64^2}{9^2} \cdot \frac{16}{35} R^6,$$

$$\operatorname{Var} \hat{V}(\mathbf{Z}) = \frac{16}{9} R^6 \left(\frac{64^2}{9.35} - \pi^2 \right) \approx \frac{16}{9} R^6 .3,14 \approx \frac{\{E\hat{V}(\mathbf{Z})\}^2}{\pi},$$

$$\sigma = \sqrt{\operatorname{Var} \hat{V}(\mathbf{Z})} \approx \frac{E\hat{V}(\mathbf{Z})}{1,77}.$$

If $\quad 0 < X \leq \frac{64}{9} R^3 = c$ then

$$\text{Prob } \{\hat{V}(\mathbf{Z}) < x\} = \text{ Prob } \left\{ \theta < \arcsin \sqrt[3]{\frac{x}{c}} \right\}.$$

The probability density of $\hat{V}(\mathbf{Z})$ is

$$p_{\hat{V}}(x) = \frac{1}{3\left(\frac{x}{c}\right)^{1/3} \sqrt{1 - \left(\frac{x}{c}\right)^{2/3}}}, \quad 0 < x \leq c = \frac{64}{9} R^3.$$

From (12) we get

$$E\{\hat{S}(\mathbf{Z})\}^2 = 16^2 R^4 \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta = \frac{512}{3} R^4,$$

$$\operatorname{Var} \hat{S}(\mathbf{Z}) = 16 R^4 \left(\frac{32}{3} - \pi^2 \right) \approx 12,8 R^4,$$

$$\sigma = \sqrt{\operatorname{Var} \hat{S}(\mathbf{Z})} \approx \frac{\sqrt{12,8}}{4\pi} E\hat{S}(\mathbf{Z}) \approx 0,28 E\hat{S}(\mathbf{Z}).$$

If $0 < x \leq 16R^2$, the probability density of $\hat{S}(\mathbf{Z})$ is

$$p_{\hat{S}}(x) = \frac{x}{16R^2\sqrt{256R^4 - x^2}}.$$

c) If the point X is inside the sphere, then

$$E\{\hat{V}(\mathbf{Z})\}^2 =$$

$$\frac{64}{9}\int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}}(4a^2\sin^2\theta\cos^2 t+R^2-a^2)\sqrt{a^2\sin^2\theta\cos^2 t+R^2-a^2dt}\right\}^2\sin\theta d\theta,$$

$$E\{\hat{S}(\mathbf{Z})\}^2 =$$

$$16R^2\int_0^{\frac{\pi}{2}}\left\{\int_0^{\pi}\sqrt{R^2-a^2\cos 2\theta-2a\sin\theta\sqrt{R^2-a^2\cos^2\theta}\cos t dt}\right\}^2\sin\theta d\theta.$$

By the numerical integration method (with $R=1$) we note that $E\{\hat{V}(\mathbf{Z})\}^2$ and $E\{\hat{S}(\mathbf{Z})\}^2$ decrease when a decreases from R to 0, i.e. $\operatorname{Var}\hat{V}(\mathbf{Z})$ and $\operatorname{Var}\hat{S}(\mathbf{Z})$ are maximal when X lies on the sphere and tend to 0 when x tends to the origin. We have

$$\operatorname{Var}\hat{V}(\mathbf{Z}) \le 5,6R^6,$$
$$\operatorname{Var}\hat{S}(\mathbf{Z}) \le 12,8R^4.$$

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