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A Short Communication

SOME RESULTS ON *SI*-RINGS

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We give a brief report on the main results of our paper [4] which has been accepted for publication in the Journal of Algebra. The following results on a ring R are obtained : (i) R is the ring direct sum of a semiprimary SI-ring and a right CS right SI-ring with zero right socle if and only if every cyclic semiprimitive right R-module is a direct sum of a projective module and an injective module; (ii) R is the ring direct sum of a semiprimary SI-ring and a right and left SI-ring with zero right (and left) socle if and only if every finitely (or 2-) generated semiprimitive right R-module is a direct sum of a projective module and an injective module; (iii) R is the ring direct sum of a semisimple ring and a right SI-domain if and only if every cyclic semiprimitive right R-module is projective or injective, as a consequence, R is semisimple if and only if every 2-generated semiprimitive right R-module is projective or injective.

All rings discussed here are associative rings with identity and all modules are unitary. A right R-module M is called semiprimitive if the Jacobson radical of M is zero, i.e. if the intersection of all maximal submodules of M is zero. Let M be a right R-module, where R is a ring. Then M is defined to be a CS-module if each submodule of M is contained essentially in a direct summand of M.. A ring R is called right CS if R is a CS-module as a right R-module. Recently, CS-modules have been extensively studied, and the number of papers devoted to them is so large that we are unable to quote them here. Therefore we only refer to Dung-Huynh-Smith-Wisbauer [1] for basic properties of CS-modules as well as their application to the structure of rings.

Right (resp. left) SI-rings, i.e. rings for which all singular right (resp. left) modules are injective, have been introduced and investigated by Goodearl [2] and the structure of right SI-rings was obtained by him in Theorem 3.11 of [2]. In [5, Corollary 5], Osofsky and Smith showed that a ring R is right SI if every cyclic singular right R-module is injective. This enables us to show that a ring R is right SI, if and only if every cyclic semiprimitive singular right R-module is injective. In particular, if the singular submodule $Z(C)$ of every cyclic semiprimitive right module C over a ring R is injective, then R is right SI. The complement B of $Z(C)$ in C is then a non-singular direct summand of C which is not projective in general. However, if for example R is the ring direct sum of a semiprimary SI-ring and finitely many right SI-domains, then such a submodule B is always projective. Therefore it is natural to ask the following question

(*) *Which rings R can be characterized by the property that every cyclic semiprimitive right R-module is a direct sum of a projective module and an injective module?*

On the other hand, rings each of whose cyclic (resp. finitely generated) right modules is a direct sum of a projective module and an injective module (briefly, right $CDPI$-rings (resp. right $FGPI$-rings)) have been introduced and investigated by Smith [6] (resp. [7]). In [5, Proposition 2] it was shown that right $CDPI$-rings are right noetherian and right SI. However, as shown in [6, Example 4.12], there are artinian SI-rings which are not right $CDPI$. In connecting this with (*) we show that a ring R is the ring direct sum of a semiprimary SI-ring and right CS right SI-ring if and only if every cyclic semiprimitive right R-module is a direct sum of a projective module and an injective module. One direction of this statement is clear. Assume conversely that every cyclic semiprimitive right R-module is a direct sum of a projective module and an injective module. Then R is right SI and it splits into a ring direct sum of a ring A and a ring B such that $A/\mathrm{Soc}(A_A)$ is semisimple and $\mathrm{Soc}(B_B) = 0$. For showing that A is semiprimary and B is right CS it requires much work and this is the main part of the paper.

If we strengthen the hypothesis on a ring R by assuming the same decomposition property for finitely (or -2-) generated semiprimitive right R-modules, then R is exactly the ring direct sum of a semiprimary SI-ring and a right and left SI-ring with zero (right or left) socle. In particular, a right $FGPI$-ring is the ring direct sum of a right artinian SI-ring and a semiprime right and left noetherian, right and left SI-ring.

Finally we consider the property that every cyclic semiprimitive right module over a ring R is injective or projective and show that R is then exactly the ring direct sum of a semisimple ring and a right SI-domain. This improves the main result of [3] and [6, Theorem 2.12]. As a consequence we obtain that a ring R is semisimple if and only if every 2- generated semiprimitive right R-module is projective or injective.

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