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A CHARACTERIZATION OF ARTINIAN MODULES

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Abstract. *It is shown that if every essensial submodule of a module M is a direct sum of an M-injective module and an artinian module, then M is a direct sum of a semisimple module and an artinian module. In this case, if M is finitely generated or finitely cogenerated, then M is artinian. This result considerably improves [5, Theorem 3.1, Corollary 3.2].*

Throughout this note rings R are associative with identity and all R-modules are unitary. For a module M, Soc (M) denotes the socle of M. If $M =$ Soc (M), M is called a semisimple module. For the definitions and properties of M-injective, M-projective modules we refer to [1] and [6].

By Chatters [2], a ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a projective module and a noetherian module. The module theoretical version of this result can be stated as follows

Theorem A. *A right R-module is a direct sum of an M-projective semisimple module and a noetherian module if and only if every factor module of M is a direct sum of an M-projective module and a noetherian module. In this case, if M or* Soc (M) *is finitely generated, then M is noetherian.*

Proof. The if part has been established in [4, Corollary 14.3].

Now assume that $M = S \oplus N$ where S is a semisimple M - projective module and N is noetherian. Let U be an arbitrary submodule of M. If $S \cap U = 0$, then U is embedded in N, so U is noetherian. Hence $U + N$ is noetherian. It follows the direct decomposition

$$M = T \oplus (U + N)$$

for some submodule T of S. Hence

$$M/U \cong T \oplus (U + N)/U$$

is a direct sum of an M-projective module T and a noetherian module $(U+N)/U$. If $V = S \cap U \neq 0$, then we consider $M' = M/V$. It is clear that

$$M' = S' \oplus N',$$

where S' is M-projective, semisimple and $N' \cong N$. Let U' be the image of U in M. Then $U' \cap S' = 0'$. Hence by the previous argument, M'/U' is a direct sum of an M-projective module and a noetherian module. Since $M/U \cong M'/U'$ it follows that M/U has the desired property. The last statement is clear.

Suggested by "duality" we obtain the following theorem

Theorem B. *Let R be any ring and M an R-module. Then the following statements are equivalent*

(i) M is a direct sum of an M-injective semisimple module and an artinian module.

(ii) Every submodule of M is a direct sum of an M-injective module and an artinian module.

(iii) Every essential submodule of M is a direct sum of an M-injective module and an artinian module.

In this case, if M is finitely generated or finitely cogenerated, then M is artinian.

Proof. (i) $\Rightarrow$(iii): Let $M = S \oplus A$, where S is an M-injective semisimple module and A is an artinian module. If C is an essential submodule of M, then it is easy to see that $S \subseteq C$. Therefore $C = S \oplus B$, where $B = C \cap A$ and B is an artinian module.

(iii) $\Rightarrow$(ii): Let U be a submodule of M. Then there exists a submodule X of M such that $U \oplus X$ is essential in M. By hypothesis,

$$U \oplus X = S \oplus A,$$

where S is M-injective and A is artinian. Let $T = U \cap S$, then there exists a direct summand T' of S such that T is essential in T'. Hence T' is M-injective. Let $\pi : X \oplus U \to U$ denote the canonical projection. Then $T' \cong \pi(T')$. Hence $\pi(T')$ is an M-injective submodule of M. Since $\pi(T')$ is also U-injective, we have

$$U = \pi(T') \oplus B$$

for some submodule B of U. It is easy to see that $B \cap S = 0$, and since B is a submodule of $A \oplus S, B$ is isomorphic to some submodule of A. Thus B is an artinial module, proving (ii).

(ii) $\Rightarrow$(i): Let M be an R-module such that every submodule of M is a direct sum of an M-injective module and an artinian module.

Firts we show that Soc (M) is essential in M. Let C be a submodule of M such that $C \cap \text{Soc }(M) = 0$. Then Soc $(C) = 0$. This together with the hypothesis shows that any submodule of C is M-injective and hence C-injective. It follows by [6, 16.3] that any submodule of C is a direct summand of C, showing that C is semisimple. Hence $C \subseteq \text{Soc }(M)$, therefore $C = 0$, proving that Soc (M) is essential in M. Using this we next consider two cases:

a) Soc (M) is finitely generated. Then M has a direct sum decomposition:

$$M = M_1 \oplus ... \oplus M_n ,$$

where each M_i is indecomposable. Hence, by hypothesis, each proper submodule of M_i must be artinian. It follows that each M_i is artinian. Thus M is artinian.

b) Soc (M) is infinitely generated. By hypothesis,

$$\text{Soc }(M) = S \oplus B,$$

where S is M-injective and B is artinian. It follows that

$$M = S \oplus A$$

for some submodule A of M, since S is an M-injective submodule of M. Therefore Soc $(M) = S \oplus C$, where $C = \text{Soc }(M) \cap A$, and so $C \cong B$, in particular, C is finitely generated. Moreover, it is clear that $C = \text{Soc }(A)$ and C is essential in A. Hence we may use a) to show that A is artinian.

The last statement is clear.

The proof of Theorem B is complete.

Theorem B shows in particular that the assumptions (P_1) and $J(M) \ll M$ in [5, Theorem 3.1] as well as the semi-perfectness of rings in [5, Corollary 3.2] can be removed.

Corollary. *Let $M = S \oplus A$ be a direct sum of an M-injective semisimple module S and an artinian module A. If M is quasi-projective, then S and A can be chosen to be fully invariant submodules of M.*

Proof. A submodule U of a right R-module N is called fully invariant, if for each $f \in \text{End}_R(N)$, $\quad f(U) \subseteq U$.

By hypothesis, we may assume that all minimal submodules of A are not M-injective. Hence there is no non-zero homomorphisms from S to A, this implies $f(S) \subseteq S$ for all $f \in \text{End}_R(M)$. Now, assume that M is quasi-projective. Then each submodule of S is M-projective. Let φ be a homomorphism from A to S, then $A/\text{Ker } \varphi$ is isomorphic to a submodule of S, and so $A/\text{Ker } \varphi$ is M-projective and semisimple. Moreover A is M-projective and, since A is artinian, $A/\text{Ker } \varphi$ is finitely generated. Hence by [6, 18.3] the exact sequence

$$0 \to \text{Ker } \varphi \to A \to A/\text{Ker } \varphi \to 0$$

splits, i.e. $A = \text{Ker } \varphi \oplus U$ for some submodule U of A with $U \cong A/\text{Ker } \varphi$. In particular, U is an M - injective semisimple submodule of A. But we assumed above that each simple submodule of A is not M-injective, hence $U = 0$, i.e. $\varphi(A) = 0$. From this we easily derive that for each $f \in \text{End}_R(M)$, $f(A) \subseteq A$.

The proof of Corollary is complete.

Note that by the same argument we can show that Theorem B remains true when we replace "artinian module" by "module with Krull dimension at most α" for some ordinal α.

We would like to ask the question of whether a module M is the direct sum of an M-projective semisimple module and a noetherian module if every factor module of M by its small submodule is a direct sum of an M-projective module and a noetherian module.

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