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SOME SOLVED AND UNSOLVED PROBLEMS ON GROUP RINGS

K. W. ROGGENKAMP *

Abstract. *In this note we discuss several problems in representation theory of finite groups. We first deal with problems centering around the character table : We discuss Brauer pairs, the group determinant and higher characters, and present several examples eluminating the isomorphism question. Then we turn to the table of marks and Burnside rings, and again we discuss isomorphism questions. Then we report on the state of the integral and modular isomorphism problem for group rings of finite groups and for cohomology rings. Finally we describe group rings as projective limits and introduce a kind of Čech cohomology, which gives the obstruction for the Zassenhaus conjecture on the projective limits of group rings.*

1. INTRODUCTION

The theory of groups had its origin in the work of Evariste Galois (1811–1832), Augustin Louis Cauchy (1789–1857) and Alfred Serret (1819–1885). The importance of group theory – apart from solving algebraic equations (Galois) – became apparent through the results of Sophus Lie (1842–1899) and Felix Klein (1849–1925) on continuous and discontinuous geometry.

Two independent developments lead to representation theory of finite groups: In his studies on generic discriminants Richard Dedekind (1831–1916) introduced the group determinant[1]. He proved in 1886, that the number of linear factors of the group determinant – in modern language, the number of linear characters – is equal to $|G/G'|$, where G' is the derived group. Georg Frobenius (1849–1917) – taking up the work of Dedekind, who was interested in the irreducible factors of

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[1] For details on group determinants and Frobenius k–characters we refer to [22].

the group determinant – introduced and studied k–characters[2]. Frobenius proved – without the notion of group representations the orthogonality relations for characters, and later showed that his characters were actually the traces of the irreducible representations.

Already then Frobenius has asked, which properties of the finite group G are reflected by their 'characters'.

On the other side, at that time the theory of abstract finite groups was little developed, and it was natural to study groups by their action as bijections on sets and as linear transformations on vectorspaces. Time has shown that studying linear actions of groups on vectorspaces is a much richer theory, than letting groups act on sets, since one can involve the arithmetic of the general linear groups.

In this spirit William Burnside (1852–1927) and Issai Schur (1875–1941) developed ordinary (complex) representation theory, i.e. homomorphisms

$$\phi : G \longrightarrow Gl(n, \mathbf{C})^3.$$

Such a representation is said to be irreducible, if $\mathbf{C}^n$ does not contain a proper G–invariant subspace. The traces of the irreducible representations are called 'characters'[4]. The character of an element $g \in G$ depends only on its conjugacy class $K_g := \{x \cdot g \cdot x^{-1}\}_{x \in G}$. By $CT(G)$ we denote the character table, whose rows are the irreducible characters, and the columns are the conjugacy classes of the elements in G.

More generally, let R be an integral domain, then we can consider R–representations of G; i.e. homomorphisms

$$\phi : G \longrightarrow Gl(n, R).$$

Since $Gl(n, R) \subset Mat(n, R)$, the R–algebra of all $n \times n$–matrices over R, which is a ring, the matrices $\{\phi(g)\}_{g \in G}$ generate the R–algebra

$$\Lambda_\phi := \{\sum_{g \in G} r_{\phi(g)} \cdot \phi(g)\}.$$

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[2] Frobenius called them characters because the name Carl Friedrich Gauß(1777–1855) had given to his homomorphism from classes of integral binary quadratic forms to $\mathbf{Z}/n \cdot \mathbf{Z}^\times$, the units in $\mathbf{Z}/n \cdot \mathbf{Z}$.

[3] This is to be interpreted as an action of G on the vectorspace $\mathbf{C}^n$.

[4] Note that this is not the original definition, but a theorem of Frobenius.

Now, there is a universal R-algebra, the group ring of G over R, $RG := \{\sum_{g\in G} r_g \cdot g\}$, where the addition is componentwise and the multiplication is induced from the multiplication in G. Since $\mathbf{Z}$ is the universal commutative ring, there is a natural homomorphism $\mathbf{Z}G \longrightarrow RG$ for every commutative ring R.

As mentioned above, the origin of representation theory was to get information on a finite group from its characters. This original aspect was also stressed by Richard Brauer in his Harvard Lecture on 'Modern Mathematics' [2], where he asked in 1963 among others:

What in addition to the character table determines a finite group?

There are several invariants of a finite group G arising in representation theory:

1. the character table, $CT(G)$,
2. the spectral table, $ST(G)$; i.e. the character table with the p-power maps; i.e. the map $K_g \longrightarrow K_{g^p}$ for K_g the conjugacy class of $g \in G$ and p a rational prime.
3. the 2-characters, $CT^2(G)$ – discussed by G. Frobenius in 1896 [8] – which describe the obstruction to characters being homomorphisms; i.e. the 2-character associated to an irreducible character χ is defined as $\chi^2(g,h) = \chi(g\cdot h) - \chi(g)\cdot\chi(h)$.
4. the 3-characters – more generally the k-characters, $CT^k(G)$ – which Frobenius derived from the group determinant [8]; the k-characters occur as coefficients of some monomials in the group determinant, we shall discuss this construction below in Section 2.4.
5. the group determinant $Det(G)$ [8]; i.e. $det(X_{g\cdot h^{-1}})$; for the definition we refer to Section 2.3.
6. the table of marks, $B(G)$ – the Burnside ring, $\Omega(G)$ – which tries to describe the group via its permutation representations [4], cf. Section 3.
7. the rational group algebra $\mathbf{Q}G$,
8. the group algebra $\mathbf{F}G$ for all fields,
9. the group algebra over all p–adic rings of integers,
10. the integral group ring $\mathbf{Z}G$,
11. the integral cohomology ring $H^*(G,\mathbf{Z})$, for the definition we refer to Section 4.5.
12. the *mod p* cohomology ring $H^*(G,\mathbf{F})$ for a field $\mathbf{F}$ of characteristic $p > 0$.

I shall elaborate here on some aspects of each of the problems related with the above topics. The central question here is:

Which properties of the finite group are reflected by these invariants?

To my knowledge, none of the questions has a satisfactory answer up today.

2. CHARACTERTABLES AND RELATED QUESTIONS

2.1. The character table

Remark 1. The quaternion group of order 8 and the dihedral group of order 8 have isomorphic character tables – this was known already to Burnside, Frobenius and Schur, and was presumably the reason, that the question 'What properties of a finite group are reflected by the character table' was addressed only relatively late in this century.

On the other hand, the character table reflects quite a lot of properties of the underlying group:

Theorem 1. *The character table of a finite group G determines*

1. *the length of the conjugacy classes; i.e. the index $|G : C_G(g)|$ of the centralizer of a group element in G,*
2. *the lattice of normal subgroups of G,*
3. *the character table of quotient groups,*
4. *the chief series of G; a result that was proved by W. Kimmerle and in collaboration with R. Lyons, R. Sandling and D. Teague [11], [13]; i.e. a normal series of maximal length,*
5. *whether or not for a set π of primes, G has abelian Hall π-subgroups* [5]. *If the Hall π-subgroups are abelian, then they are determined up to isomorphism. This was proved by Kimmerle and Sandling ([11], [18]) and answers another question of R. Brauer [2].*

Problem 1. Find necessary and sufficient conditions on the two groups G and H such that $CT(G) = CT(H)$.

2.2. Brauer pairs

In 1964 E. C. Dade [6] has constructed the first Brauer pair ; i.e. two non

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[5] A π-Hall subgroup H of G is a subgroup, such that all prime divisors of $|H|$ lie in π, but the index is prime to π – this generalizes Sylowsubgroups.

isomorphic groups with isomorphic spectral tables – answering another question of R. Brauer [2]. Further series with this property, which we shall discuss in detail later (cf. Section 3), are certain subgroups of the groups of semi-linear transformations of finite fields; i.e., subgroup of

$$\mathbf{F}_{p^n} \cdot F_{p^n}^{\times} \cdot \phi_n \tag{1}$$

where $\mathbf{F}_{p^n}$ is the field with p^n elements and $\mathbf{F}_{p^n}^{\times}$ is its group of units. Moreover, ϕ_n is the Frobenius automorphism, sending x to x^p. This type of example was first considered by G. Cliff and Suri. K. Sehgal [3].

Problem 2. Find necessary and sufficient conditions on the two groups G and H such that both have isomorphic spectral tables.

2.3. Group determinant

Before we come to the results of the group determinant and higher characters, let us recall the definitions:

Definition 1. Let $G = \{g_1, \ldots, g_n\}$ be a finite group, and let $\{X_{g_1} = X_1, \ldots, X_{g_n} = X_n\}$ be independent indeterminants over the field K. Let us denote by $K(X)$ the field of rational functions over K in these indeterminates. Dedekind has defined the GROUP DETERMINANT – in connection with generic discriminants – as

$$\mathcal{D}_G = \mathcal{D}_G(X) := det(X_{g_i \cdot g_j^{-1}}) \in K(X). \tag{2}$$

Then $\mathcal{D}_G$ is a homogeneous polynomial of degree $|G|$ in the variables X_g. The group determinant is independent of the numbering of the group elements, as follows from the definition of the determinant.

We assume now – as was done in early times – that $K = \mathbb{C}$. In modern terminology we interpret the group determinant: Let

$$\mathcal{H}_G := \sum_{g \in G} X_g \cdot g \in K(X)G$$

be the 'generic' elements. Then $\mathcal{D}_G$ is the determinant of $\mathcal{H}_G$ under the regular representation, and a decomposition of $\mathcal{D}_G$ into irreducible factors is obtained from decomposing $K(X)G$ into simple modules.

Frobenius in 1896 did not know about semi-simple algebras. He tried to find group theoretical interpretations of the following invariants:

1. The number of the distinct irreducible factors in the decomposition of the group determinant.

   We know that this is the number of conjugacy classes.

2. The degrees of these factors.

   We know that these are the degrees of the irreducible representations.

3. The multiplicity with which the different irreducible factors occur in the group determinant.

   We know that the multiplicity coincides with the degree.

Frobenius eventually gave answers to all of these questions.

The importance of the group determinant is apparent in the next surprising result:

Theorem 2 (E. Formanek, D. Sibley 1991). *The group determinant determines the group up to isomorphism ([7]).*

Formanek and Sibley proved this result using invariant theory. R. Mansfield [19] has given an easy short and direct proof of this result by Formanek and Sibley.

In the spirit of Dedekind, this result can be interpreted as: The generic discriminant determines the Galois group of a Galois extension L/K, where K is an algebraic number field.

2.4. Higher characters

Frobenius' intention was to generalize the linear characters of an abelian group – i.e. homomorphisms to $\mathbb{C}$ – to an arbitrary finite group G – as maps from G to $\mathbb{C}$. Dedekinds main goal was to generalize the decomposition of the group determinant of an abelian group into linear factors (cf. Introduction). He was mislead by the fact, that the irreducible 2–characters of $\mathbb{C}H_8$, where H_8 is the quaternion group of order 8, involved norms of elements in the quaternion algebra.

Frobenius associated to the group determinant functions – the characters – from the finite group G to $\mathbb{C}$ is as follows:

Definition 2. If Φ is an irreducible factor of $\mathcal{D}_G$, then:

1. The CHARACTER χ_Φ associated to Φ is defined as
   - $\chi_\Phi(1) = f$, where f is the degree of (Φ).
   - For $1 \neq g \in G$ the value $\chi_\Phi(g)$ is defined as the coefficient of $X_1^{f-1} \cdot X_g$

in Φ.

In Frobenius opinion, the most important property of these 'characters' is that they are constant on conjugacy classes; i.e. $\chi_\Phi(g \cdot h) = \chi_\Phi(h \cdot g)$.

2. More generally, for a natural number k, the k–character associated to Φ is defined as follows: For $k \leq f$ where f is the degree of Φ the k-character χ_Φ^k has value on a k–tuple of group elements $(\gamma_1, \gamma_2, ..., \gamma_k)$, defined as follows:
   - the coefficient of $X_1^{f-k} \cdot X_{\gamma_1} \cdot ... \cdot X_{\gamma_k}$ in Φ, provided none of the $\gamma_i = 1$.
   - If $\gamma_i = 1$, then

$$\chi_\Phi^k(\gamma_1, \gamma_2, ..., \gamma_{i-1}, 1, \gamma_{i+1}, ..., \gamma_k) = \chi_\Phi^{k-1}(\gamma_1, \gamma_2, ..., \gamma_{i-1}, \gamma_{i+1}, ..., \gamma_k). \tag{3}$$

It is then clear that for an irreducible factor Φ of the group determinant

$$\Phi = (1/f!) \cdot (\sum_{\{g_1, \gamma_2, ..., \gamma_f\}} \chi_\Phi^f(\gamma_1, \gamma_2, ..., \gamma_f) \cdot X_{\gamma_1} \cdot X_{\gamma_2} \cdot ... \cdot X_{\gamma_f}) \tag{4}$$

Moreover, already Frobenius noticed that the k–characters can be derived inductively from the ordinary characters. The k–characters can be derived from the ordinary characters, if one knows in addition to the character table $CT(G) := (\chi_i(K_j))_{1 \leq i,j \leq h}$ – K_j are the conjugacy classes – the values of $\chi_i(\gamma_1 \cdot ... \cdot \gamma_k)$ for k bounded by the maximum of the degrees of the irreducible representations; i.e. one has to know the map

$$\Psi_k : G \times G \times ... \times G \longrightarrow \cup K_j \, , \tag{5}$$

$$(\gamma_1, ..., \gamma_k) \longrightarrow K_{\gamma_1 \cdot ... \cdot \gamma_k} \, . \tag{6}$$

This apparently is not as strong as the knowledge of the multiplication table for G. (But note, that for abelian groups, Ψ_2 determines the group multiplication.) However, as we shall see later, this is a very powerful condition.

For the higher characters we have the following results:

Theorem 3.

1. *It was noted by K. W. Johnson and Surinder K. Sehgal [10] in 1991 that the groups from Equation 15 also have the same 2-characters. Their 3-characters though are different.*

2. *It was shown independently by H. J. Hoehnke in collaboration with K. W. Johnson [9] and with different techniques by Kimmerle–Roggenkamp [16][6], that the character table and the 3-characters determine a finite group up to isomorphism, and hence they also determine the group determinant.*

This is one answer to R. Brauer's question:
"What is needed in addition to the character table, to determine the finite group G?"

This result can be rephrased: The character table of G and the knowledge of the map of Equation 5

$$\Psi_3 : G \times G \times G \longrightarrow \cup K_g \,,\, (x, y, z) \longrightarrow K_{x \cdot y \cdot z}$$

determine the group G up to isomorphism.

The remaining problem here is

Problem 3. Which properties of the finite group G are determined by the character table and the 2-characters?

A partial answer is given in Proposition 1.

It should be noted, that the finite simple groups are determined by their character table, and hence the 2-characters should have some influence on extensions. We come back to this problem in Section 3

3. TABLES OF MARKS AND BURNSIDE RING

Let us recall the definition of the Burnside ring and the Burnside matrix, i.e. the table of marks: For the finite group G let $\Omega'(G)$ be the category of finite left G-sets with G-equivariant maps. Every finite G-set can be written uniquely as a disjoint union of transitive G-sets; i.e. G-sets isomorphic to $G//H$ – the left cosets of the subgroup H in G. Two such transitive G-sets are isomorphic if and only if the corresponding subgroups are conjugate in G. These isomorphism classes generate the Burnside ring $\Omega(G)$ over $\mathbf{Z}$; multiplication is given by the cartesian product and the sum is the disjoint union. A $\mathbf{Z}$-basis is given by the isomorphism classes of transitive G-sets. The **Burnside matrix** has the rows and columns indexed by the conjugacy classes of subgroups of G, and the entry corresponding to the classes of subgroups $((U), (V))$ is given by

[6] It was reported to us, that other people had made the same observation.

$$\mathrm{Hom}_{\Omega(G)}(G//U, G//V) = \{G \in G \mid {}^{g}U \subset V\}^{7}.$$

It is easy to construct two non isomorphic groups which have isomorphic Burnside matrices – i.e. table of marks (cf. Example 1, below); this means essentially that the groups have the same lattice of subgroups. It is not so easy though – and was a long open problem – to construct a Brauer pair with the same table of marks. A detailed analysis of the Brauer pair in [3] shows however, that they also have the same table of marks, as was noted by Kimmerle and the author [16].

Example 1. We consider the two groups

$$G := \langle a, b, c \mid a^7, b^{13}, c^3, [a,b],\ {}^{c}a = a^2,\ {}^{c}b = b^3\rangle \text{ and} \tag{7}$$
$$H := \langle a, b, c \mid a^7, b^{13}, c^3, [a,b],\ {}^{c}a = a^4,\ {}^{c}b = b^3\rangle\,.$$

Then these groups are not isomorphic, but it is easily seen, that they have isomorphic Burnside matrices, though the character tables of G and H are different.

A detailed analysis of the groups Equation 1 have led W. Kimmerle and the author to give partial answers to the question of when two groups

1. have isomorphic spectral tables,
2. have isomorphic Burnside matrices,
3. have isomorphic 2–characters.

A partial answer is given by

Proposition 1 (Kimmerle–Roggenkamp [16]).

1. *Isomorphic spectral tables.*

   Let G_1 and G_2 be isomorphic groups, which act linearly on an elementary abelian p–group V such that

   (a) the groups G_i act fixed point freely on V; i.e. $Stab_{G_i}(v) = 1$ for every $0 \neq v \in V$, where $Stab_*(*)$ is the stabilizer.

   (b) For every $v \in V$ we have for the orbits

$$\mathcal{O}_{G_1}(v) = \mathcal{O}_{G_2}(v)\,; \tag{8}$$

   i.e., G_1 and G_2 have the same orbits on V.

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[7] We write ${}^{g}U := g \cdot U \cdot g^{-1}$.

Then the semi-direct products $H_i = V \rtimes G_i$[8] *have isomorphic spectral tables*[9].

2. *Isomorphic Burnside matrices.*

Under the assumptions in 1., there is a unique bijection for each $v \in V$

$$\tau_v : G_1 \longrightarrow G_2 \quad \text{defined by} \quad {}^{g}v = {}^{\tau_v(g_1)}v\,, \tag{9}$$

for $g_1 \in G_1$.

Moreover, these maps τ_v *determine maps*

$$\rho_v : G_1 \times G_1 \longrightarrow G_1 \quad \text{defined by} \quad {}^{\tau_v(g)}({}^{h}v) = {}^{\rho_v(g,h)\cdot h}v\,, \tag{10}$$

where $g, h \in G_1$. *We assume now that*

(a) *all minimal subgroups of* V *are conjugate under* G_1 *and* G_2,

(b) *there exists* $v_0 \in V$ *such that for every subgroup* $H \leq G_1$ *we have*

$$\rho_{v_0}\downarrow_{H\times H} : H \times H \longrightarrow H\,, \tag{11}$$

i.e., ρ_{v_0} *somehow preserves the subgroup structure of* G_1.

Then the above semi direct products H_1 *and* H_2 *have isomorphic Burnside matrices.*

3. *Isomorphic 2-characters.*

Assume that $\rho : G \longrightarrow H$ *is an isomorphism of finite groups and that* M *is a module for both* G *and* H. *If*

(a) *there exists a* ρ*-equivariant bijection – not necessarily a group homomorphism –*

$$\sigma : M \longrightarrow M \quad \text{with} \quad \sigma(g\cdot m) = \rho(g)\cdot\sigma(m) : g \in G, m \in M, \tag{12}$$

(b) $M \setminus \{0\}$ *consists of a single orbit for both* G *and* H,

then the semi direct products $M \rtimes G$ *and* $M \rtimes H$ *have isomorphic 2-characters.*

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[8] If M is a G-module, then the semi-direct product $M \rtimes G$ consists of pairs (m, g) with multiplication $(m, g)\cdot(n, h) = (m + g\cdot n, g\cdot h)$.

[9] The examples of Cliff and Sehgal [3] are of this type; as a matter of fact, there $V \setminus \{0\}$ consists of exactly one orbit under G_i.

However, these conditions are surely not sufficient. So we still have an open problem:

Problem 4. Find necessary and sufficient conditions for when two groups have isomorphic spectral tables and have isomorphic Burnside matrices and have isomorphic 2–characters.

Let us briefly describe the groups, which satisfy the above hypotheses:

For a prime p we denote by $\mathbf{F}_{p^n}$ the field with p^n elements. ϕ_n is the Frobenius automorphism, and

$$\Omega(p,n) = (\mathbf{F}_{p^n} \rtimes \mathbf{F}_{p^n}^{\times}) \rtimes \langle \phi_n \rangle. \tag{13}$$

Then the groups [10] are certain subgroups of $\Omega(p,n)$, which are described in detail in [16]. The smallest example is for $p = 7$ and $n = 3$. Then for these numbers

$$\mathbf{F}_{7^3}^{\times} = \langle a \rangle \cdot \langle b \rangle \cdot \langle c \rangle \tag{14}$$

is the product of cyclic groups of order 19 , 2 , 9 resp. and

$$H_1 := \mathbf{F}_{7^3} \rtimes \langle a , b , c \cdot \phi_3 \rangle \tag{15}$$

and

$$H_2 := \mathbf{F}_{7^3} \rtimes \langle a , b , c \cdot \phi_3^2 \rangle \tag{16}$$

are non isomorphic groups, which have

1. isomorphic spectral tables,
2. isomorphic 2–characters and
3. isomorphic Burnside matrices.

4. THE ISOMORPHISM PROBLEM AND COHOMOLOGY

4.1. Rational group algebras

We now turn to the various aspects centering around the isomorphism problem.

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[10] These groups are very similar to those considered by Cliff and Sehgal [3].

We first consider **rational group algebras**. Since abelian groups are determined by their primary parts, it is easily seen, that an abelian group is determined by its rational group algebra. But a much stronger statement is valid:

Let $A = \prod A_p$ be the p-primary decomposition of A. Then $A = \mathrm{lim.proj}_p A_p$ (cf. Section 5.1) is the projective limit of its p-primary parts. The group ring $\mathbf{Q}A$ though is not the projective limit of the group rings of the p-primary components. Let $\Gamma_{\mathbf{Q}A} := \mathrm{lim.proj.}_p \mathbf{Q}A_p$; then $\Gamma_{\mathbf{Q}A}$ is in general a proper epimorphic image of $\mathbf{Q}A$.

Lemma 1. *Let $\Gamma_{\mathbf{Q}A} \simeq \Gamma_{\mathbf{Q}B}$ for an abelian group A, then $A \simeq B$; i.e. the structure of the abelian group is captured already in a small part of the rational group algebra.*

In order to show the difference between $\mathbf{Q}A$ and $\Gamma_{\mathbf{Q}A}$, let p and q be different rational prime numbers, and denote by C_p and C_q resp. the cyclic groups of order p and q. For a natural number n we denote by ζ_n a primitive nth root of unity. Then

$$\mathbf{Q}G = \mathbf{Q} \times \mathbf{Q}(\zeta_p) \times \mathbf{Q}(\zeta_q) \times \mathbf{Q}(\zeta_{p\cdot q}) \text{ and }, \tag{17}$$

$$\Gamma_{\mathbf{Q}G} = \mathbf{Q} \times \mathbf{Q}(\zeta_p) \times \mathbf{Q}(\zeta_q)\ ; \text{ i.e.}$$

the faithful representations of G are missing in $\Gamma_{\mathbf{Q}G}$.

We shall turn to these projective limits later in Section 5.

Also for rational group algebras it is not known:

Problem 5. Which properties of G are reflected in $\mathbf{Q}G$?

4.2. Modular group algebras

E. C. Dade has in 1971 [5] constructed two non isomorphic groups G and H of order $p^3 \cdot q^6$ which have isomorphic group rings over every field and even over the p-adic integers $\hat{\mathbf{Z}}_p$ for every prime p. These groups though have non isomorphic integral group rings, since they are metabelian. One can construct smaller groups than Dade's with these properties [26].

One of the main problems in modular group rings is

Problem 6. Can there exist non isomorphic p–groups G and H with $\mathbf{F}_pG \simeq \mathbf{F}_pH$?

M. Wursthorn [30] has checked with the help of a computer that the answer is 'no' for 2–groups of order at most 2^6.

4.3. The isomorphism problem

We shall briefly recall some properties, which are detected by the integral group ring:

Proposition 2. *Assume, that* $\mathbf{Z}G = \mathbf{Z}H$ *as augmented algebras; i.e. the identification is compatible with the augmentation maps* [11]*. Then*

1. *the class sums* [12] *of G and those of H coincide inside* $\mathbf{Z}G$*,*
2. *G and H have isomorphic lattices of normal subgroups.*
3. *The group ring determines nilpotent Hall subgroups up to isomorphism, as well as hamiltonian Hall subgroups (cf.* [13] *W. Kimmerle [11] and R. Sandling [18]).*
4. *G and H have isomorphic spectral tables – hence abelian Hall subgroups are determined.*

Problem 7. It is an open problem, if one has $\mathbf{Z}_pG = \mathbf{Z}_pH$ as augmented algebras, where $\mathbf{Z}_p$ is the localization at p, whether one still has a class sum correspondence for p–power elements.

Under the aspect of Dade's examples with respect to $\hat{\mathbf{Z}}_p$ the most far reaching result on the isomorphism problem is the following ([27]):

Theorem 4. *Let G be a finite group with a normal p-subgroup N, such that* $C_G(N)$*, the centralizer of N is a p-group. If* $\mathbf{Z}G \simeq \mathbf{Z}H$*, then G and H are p-adically conjugate* [14]*; in particular, G and H are conjugate in* $\mathbf{Q}G$*.*

This implies, that for a solvable group G, the various quotients $G/O_{p'}(G)$ are uniquely determined by the integral group ring. The group itself is the projective limit of the projective system generated by the various $G/O_{p'}(G)$; but as mentioned above for abelian groups, the group ring $\mathbf{Z}G$ is not the projective limit of the group rings $\mathbf{Z}G/O_{p'}(G)$. We shall come back to this problem in Section 5.

Important consequences of this result are

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[11] The augmentation $\epsilon_G : \mathbf{Z}G \longrightarrow \mathbf{Z}$ sends $\sum_{g\in G} r_g{\cdot}g \longrightarrow \sum_{g\in G} r_g$.

[12] The class sum of an element $g \in G$ is $\sum_{x\in K_g} x \in \mathbf{Z}G$.

[13] A finite group is said to be Hamiltonian, provided every subgroup is normal. These have been classified by Dedekind.

[14] This means, that there exists a unit u in $\hat{\mathbf{Z}}_pG$ such that ${}^uG = H$.

Proposition 3.

1. *Let E be a finite group given via the exact sequence*

$$0 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 0,$$

where N is an abelian group and $G = \prod_{1\leq i\leq n} G_i$, where $O_{p_i'}(G_i) = 1$ and the various G_i have relatively prime order, then the isomorphism problem for $\mathbf{Z}G$ has a positive answer. (This was proved by L.L. Scott in collaboration with the author in case G was nilpotent and in the general situation by A. Zimmermann [28] following suggestions of the author.)

2. *Assume, that $[G,G]$ is nilpotent, then the isomorphism problem for $\mathbf{Z}G$ has a positive answer. ([12])*

The isomorphism problem is still an active area of research.

Problem 8. Let G and H be finite groups with $\mathbf{Z}G \simeq \mathbf{Z}H$. Does it then follow that G and H are isomorphic?

4.4. The Zassenhaus conjecture

We have listed above some classes of groups for which the isomorphism problem

$$\mathbf{Z}G \simeq \mathbf{Z}H \Longrightarrow G \simeq H\,?$$

has a positive answer.

This is closely related to the **Zassenhaus conjecture**:

Conjecture 1 (Zassenhaus).

$$\mathbf{Z}G = \mathbf{Z}H \Longrightarrow G = a\cdot H \cdot a^{-1} \ \textit{for some}\ a \in \mathbf{Q}H.$$

If one looks at the integral group ring, then this is a very strong condition, since one consequence is, that all possible automorphisms of $\mathbf{Z}G$ coming from automorphisms of the centre – these can be Galois automorphisms – are induced from group automorphisms.

The Zassenhaus conjecture is equivalent to the following statement [26]:

Assume that $\mathbf{Z}G = \mathbf{Z}H$ as augmented algebras. Then the class sum correspondence says, that there is a bijection [15] $\beta : G \longrightarrow H$ such that in $\mathbf{Z}G$ we

15 Note that not every bijection between groups is an isomorphism.

have

$$K_g = K_{\beta(g)} \text{ for every } g \in G.$$

Lemma 2. *The Zassenhaus conjecture is true if and only if β can be chosen, to be a group* **isomorphism**.

Yet another way of phrasing the Zassenhaus conjecture is as follows:

Lemma 3. *The Zassenhaus conjecture is true if and only if*

1. *the isomorphism problem has a positive answer for G and*
2. *for every augmented automorphism α of $\mathbf{Z}G$ there is a group automorphism ρ of G such that $\alpha \cdot \rho^{-1}$ is a* **central** *automorphism ; i.e. it is given by conjugation with a unit in $\mathbf{Q}G$.*

The above rephrasing of the Zassenhaus conjecture gives rise to an interesting modification – the Zassenhaus conjecture for p-power classes – which has mainly been considered by W. Kimmerle:

Conjecture 2 (Variation of the Zassenhaus conjecture). *Assume that $\mathbf{Z}G = \mathbf{Z}H$ as augmented algebras, then there exists an isomorphism $\beta : G \longrightarrow H$ such that $K_g = K_{\beta(g)}$ for $g \in G$ a p-power element for some prime p.*

Clearly this variation also implies, that the isomorphism problem has a positive answer.

Let me summarize the known results on the Zassenhaus conjecture and its variation.

Proposition 4.

1. *To the Zassenhaus conjecture:*

   (a) *The class of groups, for which the Zassenhaus conjecture holds is closed under direct products – the same statement holds for its variation.*

   (b) *The Zassenhaus conjecture is true for groups which have a normal p–subgroup containing its centralizer – and hence for products of those. In particular, it is true for nilpotent groups.*

   (c) *The Zassenhaus conjecture is true for symmetric groups, as was noted by G. Peterson [20] (cf. also W. Kimmerle [12]).*

   (d) *If G is solvable, and $\mathbf{Z}G = \mathbf{Z}H$ as augmented algebras, then the Sylow p-subgroups of G and H are conjugate in $\mathbf{Q}G$ [14].*

(e) *There is a metabelian group and an automorphism α of $\mathbf{Z}G$ which is a counterexample to the Zassenhaus conjecture [25], [26].*

(f) *Zassenhaus has further – after the above counterexample was known – conjectured, that the Zassenhaus conjecture is true for abelian Sylow tower groups* [16]. *There is not yet known a complete proof. However, in a first step M. Hertweck – advised by W. Kimmerle – has shown, that for metabelian groups with abelian Sylow subgroups, it central automorphisms are inner, thus verifying the variation of this conjecture for certain nilpotent extensions of these groups.*

2. *To the variation of the Zassenhaus conjecture.*

(a) *The variation surely holds for a class of groups, provided the Zassenhaus conjecture holds for this class.*

(b) *The variation of the Zassenhaus conjecture holds for G provided the commutator subgroup $[G,G]$ is nilpotent. The counterexample to the Zassenhaus conjecture mentioned above is of this form.*

Let me speculate about the isomorphism problem: It was known to L. Scott and the author, that a counterexample to the isomorphism problem would imply the existence of a counterexample to the Zassenhaus conjecture. Therefore it was necessary to construct a counterexample to the Zassenhaus conjecture. Once we had done this, we thought, that a counterexample to the isomorphism problem should now be relatively close. This was not the case though.

We did not have thought of a counterexample to the variation of the Zassenhaus conjecture, and one can show, that a counterexample to the isomorphism problem would imply the existence of a counterexample to the variation of the Zassenhaus conjecture.

Recently, the author and A. Zimmermann have constructed semilocally such a counterexample.

4.5. Cohomology rings

For a finite group G and a commutative ring we put $H^0(G,R) := R$ and for $i > 0$ we put $H^i(G,R) := Ext^i_{RG}(R,R)$, where the latter denotes the equivalence classes of long exact sequences

$$0 \longrightarrow R \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow R \longrightarrow 0\,.$$

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16 These are defined inductively, such that G has an abelian normal Sylow subgroup A and G/A is again an abelian Sylow tower group.

Addition in $Ext^i_{RG}(R,R)$ is the addition of exact sequences; i.e. the pushout along the codiagonal followed by the pullback along the diagonal. The multiplication

$$Ext^i_{RG}(R,R) \otimes_R Ext^j_{RG}(R,R) \longrightarrow Ext^{i+j}_{RG}(R,R)$$

is the composition of exact sequences. This way $H^*(G,R) := \oplus_{i=0,1,2\ldots} H^i(G,R)$ becomes a graded ring, which is commutative in the graded sense.

Let G and H be non isomorphic groups such that for every rational prime p, the groups $G/O_{p'}(G)$[17] and $H/O_{p'}(H)$ are isomorphic, then the cohomology rings

$$H^*(G,R) := \sum_{i=0,1,\ldots} H^i(G,R) \text{ and } H^*(H,R) := \sum_{i=0,1,\ldots} H^i(H,R)$$

are isomorphic for all fields R and all complete Dedekind domains of characteristic zero with finite residue field. Since $H^*(G,\mathbf{Z})$ is determined by the p–adic cohomology rings, this shows at the same time, that

$$H^*(G,\mathbf{Z}) \simeq H^*(H,\mathbf{Z})\,.$$

Examples of such groups can easily be given: the groups in Example 1.

Problem 9. Can one find necessary and sufficient conditions for two finite groups G and H to have the cohomology rings $H^*(G,\mathbf{Z})$ and $H^*(H,\mathbf{Z})$ (resp. $H^*(G,\mathbf{F}_p)$ and $H^*(H,\mathbf{F}_p)$) isomorphic?

5. ČECH COHOMOLOGY

The details of the results in this section can be found in [17].

5.1. Projective limits of groups

Let G be a finite group and let $\{N_i \,|\, 1 \leq i \leq n\}$ be a family of normal subgroups. We let $\phi_i : G \longrightarrow G/N_i := G_i$ be the natural map. $\mathcal{P}_n$ is the powerset of $\{1,\ldots,n\}$; it is partially ordered by inclusion.

For

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[17] $O_{p'}(G)$ is the largest normal subgroup of G of order prime to p.

$$S \in \mathcal{P}_n \text{ we set } G_S := G/(\prod_{i\in S} N_i) \text{ and let } \phi_S : G \longrightarrow G_S \tag{18}$$

be the natural projection. For $S \leq T$ we have a corresponding induced homomorphism

$$\phi_{S,T} : G_S \longrightarrow G_T \text{ and } \{G_S, \phi_{S,T} | S \in \mathcal{P}_n\}$$

is a projective system, and we can form the projective limit

$$\hat{G} := \text{lim.proj.}_{S\in\mathcal{P}_n}(G_S, \phi_{S,T})$$

$$= \{g_S \in G_S : \phi_{S,T}(g_S) = \phi_{S',T}(g_{S'}) \text{ for } S, S' \leq T \text{ and } g_S \in G_S, g_{S'} \in G_{S'}\}.$$

The special structure of the index set simplifies the situation considerably:

Claim 1. *Let* $\widehat{G_1} := \{(g_i)_{1\leq i\leq n} | g_i \in G_i : \phi_{i,\{i,j\}}(g_i) = \phi_{j,\{i,j\}}(g_j)\}$, *then* $\hat{G} = \widehat{G_1}$.

We put $K_{i,j} := Ker\ (G_i \longrightarrow G_{i,\{i,j\}})$.

Since the elements in G satisfy the above relations of the pullback, there is a unique map

$$\gamma : G \longrightarrow \hat{G} : g \longrightarrow (g_i := g \cdot N_i)_{1\leq i\leq n},$$

which has kernel the group $\bigcap_{1\leq i\leq n} N_i$. Thus γ is injective iff $\bigcap_{1\leq i\leq n} N_i = 1$.

The next result is of importance to check whether G is a projective limit of quotients.

Lemma 4. *Let G be a finite group, and let $\{N_i\}_{1\leq i\leq n}$ be a family of normal subgroups of G. Assume that*

1.

$$\bigcap_{1\leq i\leq n} N_i = \{1\}, \tag{19}$$

2. for every rational prime divisor p of $|G|$ there is at least one index $i := i(p)$ such that $(p, |N_{i(p)}|) = 1$.

Then G is the projective limit of $\{G/N_i\}_{1\leq i\leq n}$; i.e. γ is an isomorphism.

Remark 2.

1. These conditions are satisfied for example for G a solvable group, if $\{N_i = O_{p_i'}(G)\}_{1\leq i\leq n}$, where $\{p_i\}_{1\leq i\leq n}$ runs over all prime divisors of $|G|$. Here we take $N_{i(p)} = N_i$.

2. The above result holds more generally for G a periodic group (i.e., every element in G has finite order – for example a locally finite group) and $\{N_i\}_{i=1,2,\ldots}$ a countable set of normal subgroups.

The pullback is easily handled if the groups $G_{i,j}$ coincide:

Claim 2. *Assume, that G is the projective limit of the groups $G_i := G/N_i$, and that for each pair (i,j) with $i \neq j$ the groups $G_{i,j} := G/(N_i \cdot N_j)$ coincide. Then the projective limit consists of*

$$\{(g_i)_{1\leq i\leq n} | g_i \in G_i : \phi_{1,\{1,i\}}(g_1) = \phi_{i,\{1,i\}}(g_i), 2 \leq i \leq n\}. \qquad (20)$$

In this case a family of isomorphism $\sigma_i : H_i \longrightarrow G_i, 1 \leq i \leq n$, gives rise to an isomorphism $\text{lim.proj.}(H_i, \phi_{i,\{i,j\}}) \longrightarrow G$ *if and only if* $\sigma_1 \equiv \sigma_j \bmod G_{1,i}$.

5.2. Projective limits of group rings

We assume, that the finite group G is a projective limit of the groups $G_i := G/N_i$, $1 \leq i \leq n$. We use the notation of Section 5.1. The group homomorphisms

$$\phi_{S,T} : G_S \longrightarrow G_T \text{ for } S \subset T \qquad (21)$$

induce augmented homomorphisms

$$\phi_{S,T} : \mathbf{Z}G_S \longrightarrow \mathbf{Z}G_T{}^{18} \qquad (22)$$

Though G is the projective limit of $\{G_S\ ,\ \phi_{S,T}\}$, the group ring $\mathbf{Z}G$ is by no means the projective limit of $\{\mathbf{Z}G_S, \phi_{S,T}\}$ (cf. Equation 17). As a matter of fact

$$\Gamma(G) := \text{lim.proj.}_{S\in\mathcal{P}}(\mathbf{Z}G_S, \phi_{S,T}) \qquad (23)$$

is in general rationally a proper quotient of $\mathbf{Z}G$. The induced ring homomorphism $\phi : \mathbf{Z}G \longrightarrow \Gamma(G)$ has kernel

$$\text{Ker}\ (\phi) = \cap_{1\leq i\leq n} I(G, N_i), \qquad (24)$$

where $I(G, N_i)$ is the kernel of the natural map $\mathbf{Z}G \longrightarrow \mathbf{Z}G_i$.

As for groups one shows

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[18] It should not cause any confusion that we use the same name as for the group homomorphism.

$$\Gamma(G) = \{(x_i) \mid x_i \in \mathbf{Z}G_i : \phi_{i,\{i,j\}}(x_i) = \phi_{j,\{i,j\}}(x_j)\}\,. \tag{25}$$

Definition 3. Let G be a solvable group and put $N_i := O_{p_i'}(G)$. Then we have seen above, that G is the projective limit of the groups G/N_i. In this case we shall write $\Gamma_0(G)$ for the projective limit of the group rings $\mathbf{Z}G/O_{p_i'}(G)$.

Remark 3.

1. In case the projective limit is a pullback; i.e. there are only two factors G_1, G_2, then the natural map

$$\phi : \mathbf{Z}G \longrightarrow \Gamma_0(G)$$

is surjective.

2. In general, I do not know, whether this map is always surjective.

Claim 3. *Let*

$$\Lambda_i \xrightarrow{\phi_{i,j}} \Lambda_{i,j} \xleftarrow{\phi_{j,i}} \Lambda_j \tag{26}$$

be a projective system of rings or groups [19].

Assume, that there exists an index i_0 such that for every $j \neq i_0$ the map $\phi_{j,i_0} : \Lambda_j \longrightarrow \Lambda_{j,i_0}$ is the identity. Then the natural map

$$\Lambda_{i_0} \longrightarrow \text{lim.proj.}(\{\Lambda_i\}) \tag{27}$$

$$x \longrightarrow (\phi_{i_0,j}(x), x)_{j \neq i_0} \tag{28}$$

is an isomorphism.

Thanks to Theorem 4 we may assume, that for every prime divisor p of $|G|$ the groups $O_{p'}(G) \neq 1$. We shall always assume that.

Definition 4. Let $\{p_i\}_{1\leq i\leq n}$ be the different prime divisors of $|G|$. We denote by B_{p_i} the principal block [20] in $\mathbf{Z}_{p_i}G \cap \mathbf{Q}G$ [21]. e_{p_i} denotes the corresponding central idempotent in $\hat{\mathbf{Z}}_{p_i}G \cap \mathbf{Q}G$ [22]. We now let ϵ be the smallest central idempotent in

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[19] For the sake of simplicity we shall assume, that the rings $\Lambda_{i,j}$ are quotients of Λ_i and Λ_j resp. and that the maps $\{\phi_{i,j}\}$ are the associated quotient maps.

[20] The blocks are the indecomposable ring direct summands, and the principal block is the unique block containing the trivial module.

[21] $\hat{\mathbf{Z}}_p$ is the localization of $\mathbf{Z}$ at p.

[22] Note that – our group is solvable – the idempotents do lie in $\mathbf{Q}G$.

QG such that $\epsilon \cdot e_{p_i} = e_{p_i}$ for every i.

We note that $\Gamma(G)$ is an augmented **Z**–order in the separable Q–algebra $QG \cdot \epsilon$. However, as remarked above (cf. Remark 3), I do not know, whether $\Gamma(G) = \mathbf{Z}G \cdot \epsilon$.

Example 2. Let $G := \prod_{1 \le i \le n} P_i$ be a nilpotent group with P_i a Sylow p_i–subgroup. Since products are special cases of projective limits, $G = \text{lim.proj.}_{1 \le i \le n}(P_i)$ is a projective limit, and $\Gamma_0(G)$ is the product of $\{\mathbf{Z}P_i\}$ in the category of **Z**–augmented algebras [23]; i.e., if

$$\epsilon_G : \mathbf{Z}G \longrightarrow \mathbf{Z}$$

is the augmentation, then

$$\Gamma_0(G) = \{(x_i)_{1 \le i \le n} : x_i \in \mathbf{Z}P_i, \epsilon_{P_i}(x_i) = \epsilon_{P_j}(x_j)\}\,; \tag{29}$$

i.e., rationally, $\Gamma_0(G)$ consists of those irreducible modules, where at most one of the groups P_i acts non trivially.

One can easily determine a **Z**–basis of $\Gamma_0(G)$.

Definition 5. Let

$$*_i : \mathbf{Z}G_i \longrightarrow \mathbf{Z}G_i, \sum_{x \in G_i} z_x \cdot x \longrightarrow \sum_{x \in G_i} z_x \cdot x^{-1} \tag{30}$$

be the involution of $\mathbf{Z}G_i$. Then $(*_i)_{1 \le i \le n}$ induces an involution $*_G$ on $\Gamma_0(G)$, which is induced from the involution on **Z**G.

Lemma 5. *Let G and H be solvable groups with $\Gamma_0(G) = \Gamma_0(H)$ as augmented algebras. If the involutions $*_G$ and $*_H$ coincide on $\Gamma_0(G)$, then $G = H$ in $\Gamma_0(G)$* [24 25].

Let us list some more properties of $\Gamma_0(G)$ or more generally of $\Gamma(G)$ (cf. Equation 23), provided G is a projective limit.

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[23] This shows, that $\Gamma_0(G)$ is a very natural construction. The corresponding group ring is the tensor product, which is the product in the category of **Z**–algebras.

[24] The result holds for arbitrary projective limits.

[25] For **Z**G this is a result of Banachevski [1].

Lemma 6.

1. *$\Gamma_0(G)$ has no non trivial idempotents.*
2. *Let $u := (u_i)_{1 \leq i \leq n}$ in $V(\Gamma_0(G))$, the group of units of augmentation 1 in $\Gamma_0(G)$, such that $u^n = 1$ for n a divisor of $|G|$, then either*
   (a) *the coefficient of 1 in u_i is zero for all $1 \leq i \leq n$ or*
   (b) *there is an index i_0 with $u_{i_0} = 1$, but then*
$$u_j \in 1 + \; Ker\,(\mathbf{Z}G_j \longrightarrow \mathbf{Z}G_{i_0,j})\,. \tag{31}$$

Lemma 7. *Let $H \leq V(\mathbf{Z}G)$, then $\phi(H) \leq V(\Gamma_0(G))$* [26] *is isomorphic to H provided G is nilpotent.*

The next result can be deduced from theorems of A. Weiss [29].

Lemma 8. *Let G be nilpotent, and assume that either $H \leq V(\mathbf{Z}G)$ or $H \leq V(\Gamma_0(G))$ is a finite group, then H is isomorphic to a subgroup of G and moreover, $\phi(H)$ in $\Gamma_0(G)$ rationally conjugate to a subgroup of G; i.e. the Zassenhaus conjecture for finite groups holds for $\Gamma_0(G)$.*

We point out one important property of $\Gamma_0(G)$, which follows easily from the proof of the isomorphism problem for nilpotent groups [24]:

Theorem 5. *Assume that G is nilpotent. If $\Gamma_0(G) = \Gamma_0(H)$ - with the usual modification (cf. [26]) we may assume, that this is an equality of augmented algebras - then $G \simeq H$; moreover, even the Zassenhaus conjecture is true for Γ_0; i.e. there exists a unit $a \in \mathbf{Q}\Gamma(G)$ with $a \cdot G \cdot a^{-1} = H$ in $\Gamma(G)$.*

Claim 4. *Assume that $A = \prod_{1 \leq i \leq n} P_i$ is an abelian group written as a product of its Sylow p-subgroups, and let us denote by $V_f(\Gamma_0(A))$ the augmented units of finite order in $\Gamma_0(A)$* [27]. *Then $V_f(\Gamma_0(A)) = V_f(\mathbf{Z}A) = A$; i.e. the group of units of finite order is the same in the integral group rings and in this very small quotient, and it is just A.*

Remark 4. Let G be a solvable group and put $N_i = O_{p_i'}(G)$ for the prime divisors

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[26] Recall that $\phi : \mathbf{Z}G \longrightarrow \Gamma_0(G)$ is the natural homomorphism.

[27] i.e., units of augmentation one. Note that in case of a commutative ring, $V_f(\Gamma_0(A))$ is a group.

p_i of $|G|$. It is well known, that $\hat{\mathbf{Z}}_{p_i}G_i$ [28] – with $G_i = G/N_i$ – is the principal block $B_0(p_i)$ of $\hat{\mathbf{Z}}_{p_i}G$. Let e_i be the central idempotent in $\mathbf{Q}G$ such that $B_0(p_i) = \hat{\mathbf{Z}}_{p_i}G \cdot e_i$. If we put $e = \sum e_i$, then $\mathbf{Q}\Gamma_0(G) = \mathbf{Q}G \cdot e$.

This shows also, that $\mathbf{Z}G$ and Γ_0 do have the same cohomology rings, since the cohomology rings are defined p–adically and live in the principal blocks.

The next example was pointed out to me by Wolfgang Kimmerle:

Example 3. Let $G = Psl(2,q)$, then $\mathbf{Z}G = \Gamma_0(G)$, quite contrary to the situation of a solvable group. The reason is, that for every p, the p–adic group ring is of the form $\widehat{\mathbf{Z}_p G} = B_0(p) \times M(p)$, where $B_0(p)$ is the principal block and $M(p)$ is a block of defect zero [29].

5.3. Čech cohomology

These results were essentially noted by L. L. Scott in collaboration with the author.

Definition 6. For the finite group G we denote by $Aut(G)$, $Aut_c(G)$ and $Aut_p(G)$ the group of automorphisms of G, and of automorphisms $\gamma \in Aut(G)$ resp. such that for every $g \in G$ the elements g and $\gamma(g)$ are conjugate in G and of automorphisms $\delta \in Aut(G)$ resp. such that for every p–power element $g \in G$ (i.e. it has order a power of p) the element g and $\delta(g)$ are conjugate in G for all p resp.

Claim 5. *Let $\rho \in Aut(G)$, and let M be a G-module. For the split extension*

$$\mathcal{E} : 1 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1 \tag{32}$$

representing an element in $H^2(G,M)$ - notation $\mathcal{E} \in H^2(G,M)$ - the automorphism ρ extends to an automorphism *ρ_0 of E if the G-modules M and ${}^{\rho}M$ are isomorphic – where ${}^{\rho}M$ is M but the G-action is twisted by ρ.*

We would like to stress, that the pullback along ρ gives always rise to an **isomorphism of groups** but this is in general not an **automorphism**. There is no hope, that the modules M and ${}^{\rho}M$ are isomorphic. As a matter of fact, the automorphism ρ induces an auto equivalence of the category of $\mathbf{Z}G$–modules. This equivalence is trivial iff ρ induces an inner automorphism on $\mathbf{Z}G$.

---

[28] $\hat{\mathbf{Z}}_p$ is the p–adic completion of $\mathbf{Z}$.

[29] This is a block which is a full matrix ring over an unramified extension of $\hat{\mathbf{Z}}_p$.

The following result was essentially noted by L. L. Scott and the author.

Lemma 9. *Let $\gamma \in Aut_c(G)$ be an automorphism such that g and $\gamma(g)$ for every $g \in G$ are conjugate, and let*

$$\mathcal{E} : 1 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1 \tag{33}$$

represent 0 in $H^2(G, M)$ for a finite G-module $M = \sum_{1 \leq i \leq n} M_i$, where M_i are the various p_i-primary components of M. If M_i is a characteristic section in a finitely generated projective $\hat{\mathbf{Z}}_{p_i}G$-module - i.e. there is a finitely generated projective $\hat{\mathbf{Z}}_{p_i}G$-module P and characteristic submodules $L_1 \subset L_2$ such that $M_i \simeq L_2/L_1$ - then γ extends to an automorphism γ_0 of $\mathcal{E}$, in particular it extends to a group automorphism of E.

Remark 5.

1. The above conditions are satisfied for a semi-simple finite G-module M. In fact, such a module decomposes into a direct sum of simple modules, and they are the radical quotients of p-adic indecomposable projective modules.
2. The above conditions are satisfied, if the modules M_i have order prime to $|G|$. In fact, we can assume, that $M = M_i$ is an indecomposable $\hat{\mathbf{Z}}_pG$-module. Since p does not divide $|G|$, the ring $\hat{\mathbf{Z}}_pG$ is a direct sum of matrix rings $(R_i)_{n_i}$, where R_i is an unramified extension of $\hat{\mathbf{Z}}_p$. Since M is indecomposable, it is a module for $(R_i)_{n_i}$ for some i. Then $M \simeq L/p^n \cdot L$ for the projective indecomposable $(R_i)_{n_i}$-module L. This shows, that M is the epimorphic image of a projective module modulo a characteristic ideal.

Definition 7.

1. Let G be the projective limit with respect to normal subgroups $\{N_i\}_{1 \leq i \leq n}$ (cf. Section 5.1). We require, that the normal subgroups N_i are characteristic. We write
$$G_i := G/N_i \text{ and } G_{i,j} := G/(N_i \cdot N_j)$$
with natural homomorphisms
$$\phi_i : G \longrightarrow G_i \text{ and } \phi_{i,j} : G_i \longrightarrow G_{i,j}\,, \tag{34}$$

We use the notation $\underline{G}$ for G, if we want to stress, that we view G as a projective limit.

2. We define the cocycles

$$Z(\underline{G}, \underline{Aut_*(G)}) := \{(\rho_{i,j})_{1\le i,j\le n} : \rho_{i,j} \in Aut_*(G_{i,j})^{30} : \tag{35}$$

$$\rho_{i,i} = id\,,\ \rho_{i,j}^{-1} = \rho_{j,i}\}\,, \tag{36}$$

where $Aut_*(-)$ stands for $Aut(-)$ or $Aut_c(-)$ or for $Aut_p(-)$ [31]. Then this is in general not a group with multiplication componentwise – for this one needs $\sigma_{i,j}, \cdot \rho_{i,j} = \rho_{i,j} \cdot \sigma_{i,j}$.

We next define an equivalence relation on $Z(\underline{G}, \underline{Aut_*(G)})$:

$$(\rho_{i,j}) \equiv (\sigma_{i,j}) \text{ iff } \rho_i \cdot \rho_{i,j} \cdot \rho_j^{-1} = \sigma_{i,j} \text{ for } \rho_i \in Aut_*(G_i),\ 1 \le i \le n\,. \tag{37}$$

This is easily seen to be an equivalence relation. With $\rho_{i,j}$ also the family $\rho_i \cdot \rho_{i,j} \cdot \rho_j^{-1}$ is a cocycle for $\rho_i \in Aut_*(G_i)$, $1 \le i \le n$.

The equivalence classes form a pointed set, denoted by $\check{H}(\underline{G}, \underline{Aut_*(G)})$, the Čech cohomology set. The class of the identity is the point and consists of the coboundaries

$$B(\underline{G}, \underline{Aut_*(G)}) := \tag{38}$$

$$\{(\rho_{i,j}) \in Z(\underline{G}, \underline{Aut_*(G)}) \mid \rho_{i,j} = \rho_i \cdot \rho_j^{-1} \text{ for } \rho_i \in Aut_*(G_i)\}\,, \tag{39}$$

which is easily seen to be a subset of the cocycles. This is a Čech style cohomology set [32], and if we consider homomorphisms of such sets, then these should be morphisms in the category of pointed sets.

Remark 6. G is the projective limit induced from $\{G_i\}_{1\le i\le n}$. Hence, given

$$(\rho_i)_{1\le i\le n}\,,\ \rho_i \in Aut_*(G_i)\,, \tag{40}$$

there exist $\rho \in Aut_*(G)$ which induces ρ_i on G_i [33] iff $\rho_i \cdot \rho_j^{-1} = 1$.

The importance of this cohomology is apparent, if one deals with the question of isomorphisms of projective limits.

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30 observe that $G_{i,j} = G_{j,i}$.

31 instead of requiring, that N_i is characteristic, it is often enough to require that N_i is $*$–invariant; i.e. invariant with respect to $Aut_*(G)$.

32 our projective limit should be compared to the covering of a topological space by n open sets.

33 This is where we need, that the normal subgroups are $*$–invariant.

Assume in addition, that H is the projective limit of $H_i := H/M_i, 1 \leq i \leq n$, where M_i are $*$–invariant. Assume that we are given isomorphisms

$$\sigma_i : G_i \longrightarrow H_i \in Iso_*(G_i, H_i)^{34}. \tag{41}$$

The obstruction, to when G and H are isomorphic via an isomorphism in $Iso_*(G, H)$ lies in the cocycle

$$\sigma_{i,j} := \sigma_i \cdot \sigma_j^{-1} : H_{i,j} \longrightarrow H_{i,j},\ 1 \leq i, j \leq n,$$

$$(\sigma_{i,j}) \in Z(\underline{H}, \underline{Aut_*(H)}). \tag{42}$$

In fact, we have

Lemma 10. Let $\sigma(i,j)$ be defined as in Equation 42. Then there exists

$$\sigma \in Iso_*(G, H) \text{ iff the cocycle } (\sigma_{i,j}) \text{ lies in } B(\underline{H}, \underline{Aut_*(H)})\,; \tag{43}$$

i.e. there exist $\rho_i \in Aut_*(H_i)$ with $\sigma_{i,j} = \rho_i \cdot \rho_j^{-1}$.

The situation becomes quite simple, if we assume, that the groups $H_{i,j}$ coincide for all i, j.

Claim 6. *Assume that $M_i \cdot M_j = M_0$ is the same for all i, j with $i \neq j$. Let $\sigma_{i,j}$ be defined as in Equation 42. Then there exists $\sigma \in Iso_*(G, H)$ iff there exist $\rho_i \in Aut_*(H_i)$ with $\sigma_{1,i} = \rho_i$ modulo M_0 for $2 \leq i \leq n$; i.e. if the maps $\sigma_{1,i}$ lift to ρ_i in $Aut_*(H_i)$ for $2 \leq i \leq n$.*

The importance of $\check{H}(G, Aut_*(G))$ lies also in the construction of the various modifications of projective limits.

Lemma 11. *Let $G = \text{lim. proj.}(G_S)$ be the projective limit.*

1. *Given a cocycle $\rho = (\rho_{ij}) \in Z(G, Aut_*(G))$(cf. Definition 7). Then the definition shows, that*

$$G(\rho) = \{(g_i) \in \prod_{1 \leq i \leq n} G_i : \rho_{ij} \cdot \phi_j(g_j) = \phi_i(g_i)\}$$

is a group.

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34 This notation should be selfexplanatory.

2. *$G(\rho) \simeq G$ with an isomorphism in $Iso_*(H(\rho), G)$ as "projective limit" if and only if $\rho \in B(\underline{G}, \underline{Aut_*(G)})$ is a coboundary.*

5.4. The Zassenhaus conjecture revisited

We assume from now on, that G is a finite solvable group, and hence Γ_0 (cf. Definition 3) is defined. Moreover, we put $N_i = O_{p_i'}(G)$, $1 \leq i \leq n$, for all prime divisors p_i of $|G|$, and $G = G/N_i$. Then $\Gamma_0(G)$ is the projective limit of the group rings $\mathbf{Z}G_i$. We denote by

$$\phi : \mathbf{Z}G \longrightarrow \Gamma_0(G) \tag{44}$$

the projection map, which is an augmented homomorphism. Let $I(G, N_i) \subset \mathbf{Z}G$ be the augmentation ideal of N_i; we denote by $\Gamma_0(G, N_i) = \phi(I(G, N_i))$ its image in $\Gamma_0(G)$. The group homomorphism $\phi_i : G \longrightarrow G_i$ induces an augmented homomorphism of $\mathbf{Z}$–orders $\Gamma_0(G) \longrightarrow \mathbf{Z}G_i$, which we also denote by ϕ_i. Its kernel is $\Gamma_0(G, N_i)$. This kernel is characteristic in $\Gamma_0(G)$, since p–adically the quotient modulo $\Gamma_0(G, N_i)$ is the principal block of $\Gamma_0(G)$. We shall keep this notation for the rest of this section

Assume, that $\Gamma_0 := \Gamma_0(G) = \Gamma_0(H)$ as augmented algebras. Since Ker (ϕ_i) is characteristic in Γ_0, we get an equality

$$\mathbf{Z}G_i = \mathbf{Z}H_i \text{ and so } a_i \cdot G_i \cdot a_i^{-1} = H_i, 1 \leq i \leq n \tag{45}$$

for units a_i in $\mathbf{Q}G_i$. We denote this homomorphism by $\sigma_i : G_i \longrightarrow H_i$, $g \longrightarrow a_i \cdot g \cdot a_i^{-1}$. We have the induced homomorphisms $\phi_{i,\{i,j\}} : \mathbf{Z}G_i \longrightarrow \mathbf{Z}G_{i,j}$ and so conjugation with a_i will act on $\mathbf{Z}G_{i,j} = \mathbf{Z}H_{i,j}$. We now consider the map

$$\sigma_{i,j} := \sigma_i \cdot \sigma_j^{-1} : H_{i,j} \longrightarrow H_{i,j}. \tag{46}$$

Then $\sigma := (\sigma_{i,j})$ is a cocycle in $Z(\underline{H}, \underline{Aut_c(H)})$, since $\sigma_{i,j}$ is a central automorphism.

When we now apply Lemma 10, then we obtain the following result, which probably is also known to L. L. Scott.

Theorem 6. *Let G be a solvable group and assume, that $\Gamma_0(G) = \Gamma_0(H)$ as augmented algebras, and assume that the central cocycle $(\sigma_{i,j})$ is defined as in Equation 42.*

1. *The groups G and H are isomorphic if and only if $(\sigma_{i,j}) \in B(\underline{H}, \underline{Aut(H)})$.*
2. *The Zassenhaus conjecture (cf. Conjecture 1) is true for $\Gamma_0(G) = \Gamma_0(H)$ for the groups G and H if and only if $(\sigma_{i,j}) \in B(\underline{H}, \underline{Aut_c(H)})$.*

3. *The p-version of the Zassenhaus conjecture (cf. Conjecture 2) is true for* $\Gamma_0(G) = \Gamma_0(H)$ *for the groups G and H if and only if* $(\sigma_{i,j}) \in B(\underline{H}, \underline{Aut_p(H)})$.

An immediate consequence is the following result:

Proposition 5. *Assume that G is a solvable group and* $\Gamma_0(G) = \Gamma_0(H)$. *If the groups* $H_{i,j}$ *are abelian, then the Zassenhaus conjecture holds for* $\Gamma_0(G)$.

Remark 7. Assume that in Theorem 6 all the groups $H_{i,j}$ are the same, then the conclusion of the theorem are valid, provided, the maps $\sigma_{1,i}$, $2 \leq i \leq n$ can be lifted to elements in $Aut_*(H_i)$ (cf. Claim 7). If we now invoke Claim 6 and Lemma 9 we obtain:

Theorem 7. *Assume that G is a solvable group and* $\Gamma_0(G) = \Gamma_0(H)$. *If the groups* $H_{i,j} = G_0$ *are the same for all pairs* $\{i, j\}$ *and if the groups* G_i *are extensions*

$$\mathcal{E}_i : 1 \longrightarrow M_i \longrightarrow G_i \longrightarrow G_0 \longrightarrow 1 \tag{47}$$

with finite G_0*-modules* $M_i = \sum_{1 \leq i \leq n} M_i^k$, *where* M_i^k *are the various* p_k*-primary components. If* M_i^k *is a characteristic section in a finitely generated projective* $\hat{\mathbf{Z}}_{p_k} G$*-module, then* $G \simeq H$.

The hypotheses of the last theorem are satisfied in particular, if M_i are semi-simple G_0-modules or $char(M_i)$ is prime to $|G|$.

In case $n = 2$ we get the following result, since the hypothesis $H_{i,j} = H_0$ is then automatic.

Corollary 1. *Assume that G is a pullback*

$$\begin{array}{ccc} G_1 & \longrightarrow & G_0 \\ \uparrow & & \uparrow \\ G & \longrightarrow & G_2 \end{array}$$

with $G_i = G/N_i$.

1. *If the Zassenhaus conjecture holds for* $\mathbf{Z}G_i$, *then the isomorphism problem holds for G, provided for every central isomorphism* γ *of* G_0, *there exist* $\rho_i \in Aut(G_i)$ *such that* $\rho_1 \cdot \rho_2^{-1} = \gamma$. *This latter condition is satisfied, in case* $M_1 :=$ Ker $(G_1 \longrightarrow G_0)$ *is abelian and semi-simple as* $\mathbf{Z}G_0$*-module or the characteristic of* M_1 *is prime to* $|G|$[35].

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[35] It suffices to assume that M_1 satisfies the hypothesis of M_1 in Theorem 7.

2. *(a) Assume, that the p-version of the Zassenhaus conjecture holds for* $\mathbf{Z}G_i$*, and assume*

   (b) that every p-central automorphism γ *of* G_0 *can be written as*

$$\rho_1 \cdot \rho_2^{-1} = \gamma \text{ for } \rho_i \in Aut(G_i)\,. \tag{48}$$

   Then the isomorphism problem has a positive answer for $\mathbf{Z}G$.

Example 4. In the paper [25] an example of a group ring $\mathbf{Z}G$ and an augmented group basis H was given such that for these two group bases G and H the Zassenhaus conjecture is not valid. However, in the projective limit with respect to $\{O_{p_i'}\}$, all the groups $H_{i,j}$ are abelian, and hence by Proposition 6, the Zassenhaus conjecture holds for $\Gamma_0(G) = \Gamma_0(H)$.

Assume now again that $\Gamma_0(G) = \Gamma_0(H)$. The main result now describes G in terms of H and the cocycle σ from Equation 46:

Theorem 8. *Assume, that* $\Gamma_0 := \Gamma_0(G) = \Gamma_0(H)$ *as augmented algebras, and let the cocycle* $\sigma \in Z(\underline{H}, \underline{Aut_c(H)})$ *be defined as in Equation 46.*

Then $G \simeq H(\sigma)$*, where* $H(\sigma) = \{(h_i)_{1\le i\le n} \mid h_i \in H_i : \sigma_{i,j}(h_j) = h_i\}$.

Remark 8.

1. Assume that $\mathbf{Z}G \simeq \mathbf{Z}H$ as augmented algebras, for G a finite solvable group, then also H is solvable, and we have $\Gamma_0(G) \simeq \Gamma_0(H)$, and so the conclusion of Theorem 8 says $G \simeq H(\sigma)$ for the associated cocycle σ.
2. Given a central cocycle $\sigma := (\sigma_{i,j}) \in Z(\underline{H}, \underline{Aut_c(H)})$, we can interpret σ also as an element $\sigma_{\mathbf{Z}} = \sigma \in Z(\underline{\Gamma_0(H)}, \underline{Aut_c(\Gamma_0(H))})$[36]. We can then form the group $H(\sigma)$ and the ring $\Gamma_0(H)(\sigma_{\mathbf{Z}}) \simeq \Gamma_0(H(\sigma))$. Then $H(\sigma) \simeq H$ if and only if $\sigma \in B(\underline{H}, \underline{Aut(H)})$; i.e. σ is a coboundary with respect to all automorphisms of H. Similarly, $\Gamma_0(H)(\sigma_{\mathbf{Z}}) \simeq H$ if and only if $\sigma \in B(\Gamma_0(H), Aut(\Gamma_0(H)))$; i.e. $\sigma_{\mathbf{Z}}$ is a coboundary with respect to all automorphisms of $\Gamma_0(H)$.
3. In order to find two non isomorphic solvable groups G and H with $\Gamma_0(G) \simeq \Gamma_0(H)$ it is thus necessary and sufficient, to find a group H and $\sigma \in Z(\underline{H}, \underline{Aut_c(H)})$, such that $1 \neq [\sigma] \in \check{H}(\underline{H}, \underline{Aut_c(H)})$ but $1 = [\sigma] \in \check{H}(\underline{\Gamma_0(H)}, \underline{Aut(\Gamma_0(H))})$. We just point out, that it is necessary to have such an example

[36] Here $Aut_c(R)$ are the ring automorphisms of the ring R, which leave the centre of the ring R elementwise fixed.

if one wants to construct a counterexample to the isomorphism problem. Indeed we have found such an example.

A special case needs some attention:

Proposition 6. *Let H be a solvable group and write it as the projective limit with respect to $\{O_{p'}(G)\}_{1\le i\le n}$. Assume, that $H_0 := H_{i,j}$ is the same for all pairs $\{i,j\}$, $i \neq j$. For the kernels $K_i :=$ Ker $(H_i \longrightarrow H_0)$ we require that K_i is a Sylow p_i subgroup of H_i. If*

$$\sigma \in Z(\underline{H}, \underline{Aut_c(H)}) \tag{49}$$

such that $1 \neq [\sigma] \in \check{H}(\underline{H}, \underline{Aut_c(H)})$, then there is a group G not isomorphic to H with

$$\mathbf{Z}_\pi \otimes_{\mathbf{Z}} \Gamma_0(G) \simeq \mathbf{Z}_\pi \otimes_{\mathbf{Z}} \Gamma_0(H), \tag{50}$$

where $\mathbf{Z}_\pi$ is the semilocalisation of $\mathbf{Z}$ at all the prime divisors of $|G|$.

Note 1. The above conditions just mean, that we have central automorphisms $\sigma_{1,i}$ of H_0, $2 \le i \le n$, such that there can not be found automorphisms $\sigma_i : H_i \longrightarrow H_i$ with $\sigma_{i,j} = \sigma_1 \cdot \sigma_j^{-1}$.

Claim 7. *Given an exact sequence of groups with K a p-group*

$$1 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 1. \tag{51}$$

If $(|K|, |H|) = 1$, then a central automorphism $\sigma_0 \in Aut_c(\mathbf{Z}_q H)$ can be lifted to a central automorphism $\sigma \in Aut_c(\mathbf{Z}_q G)$.

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Mathematisches Institut B
Universität Stuttgart
Germany

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