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ON MULTI-PARAMETER ASYMPTOTIC ERROR EXPANSIONS FOR MULTI-DIMENSIONAL DIFFUSION-CONVECTION EQUATIONS *

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Abstract. *This paper is concerned with the Dirichlet boundary value problem for multi-dimensional diffusion-convection equations. An unconditionally monotone difference scheme is investigated and the existence of a multi-parameter error expansion is established.*

1. INTRODUCTION

In finite difference methods when there exists an asymptotic error expansion with respect to stepsize, the Richardson extrapolation can be used for accelerating the rate of convergence of the method (see [2, 4, 7, 8, 9, 10] and the references therein). It reduces the necessary number of algebraic equations to be solved and thereby provides an efficient algorithm with respect to both computing time and storage requirements. For a multidimensional problem it is worth noting that a multi-parameter error expansion is more efficient than a one-parameter expansion.

In [9] a general multi-parameter error expansion for operator equations is investigated and later in [10] it is established for the Dirichlet self-adjoint elliptic multi-dimensional boundary-value problem. In this paper we consider the Dirichlet boundary-value problem for stationary multidimensional diffusion-convection equations. For this problem, a difference scheme which satisfies the maximum principle for any grid stepsize is investigated in [5], but the asymptotic error expansion has not been considered yet. We propose a scheme which not only satisfies

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the maximum principle but admits a same multiparameter error expansion as in [10].

2. THE DIFFERENTIAL PROBLEM

We shall use the same notations as in [10]. Let Ω be an open bounded domain in R^n and Γ its boundary. Let functions of n variables $x_1, \dots x_n$: $p_i(x)$, $s_i(x)$, $q(x)$, $f(x)$ on $\overline{\Omega}$ and $g(x)$ on Γ be given.

Consider the differential operator

$$Lu = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(p_i \frac{\partial u}{\partial x_i}\Big) + s_i \frac{\partial u}{\partial x_i} - qu, \quad x \in \Omega. \tag{2.1}$$

The differential problem is

$$Lu = f, \qquad x \in \Omega, \tag{2.2}$$

$$u = g, \qquad x \in \Gamma. \tag{2.3}$$

Assume that there exist a real number λ $(0 < \lambda < 1)$ and a positive integer m so that

$$p_i \in C^{2m+1+\lambda}(\overline{\Omega}); \quad s_i,\ q,\ f \in C^{2m+\lambda}(\overline{\Omega}), \tag{2.4}$$

$$\Gamma \in C^{2m+2+\lambda}, \ g \in C^{2m+2+\lambda}(\Gamma). \tag{2.5}$$

Assume also

$$p_i \geq \text{const} > 0, \qquad q \geq 0. \tag{2.6}$$

3. THE DISCRETE PROLLEM (DIFFERENCE SCHEME)

3.1. Grid and grid function

We use the same definitions and notations concerning grid, grid point, interior grid point, regular and irregular interior grid point as in [10], p.17. So, we have

$$\Omega_h = \Omega_{h,r} \cup \Omega_{h,ir},$$

where $\Omega_{h,r}$ and $\Omega_{h,ir}$ denote the sets of regular and irregular interior grid points.

Concerning the convection term, let us denote

$$s_i^+ = 0.5(s_i + |s_i|), \quad s_i^- = 0.5(s_i - |s_i|)$$

so that

$$s_i = s_i^+ + s_i^-, \quad |s_i| = s_i^+ - s_i^-.$$

3.2. Approximation at $P \in \Omega_{h,r}$

To approximate the given functions $p_i(x)$ we consider the discrete functions a_i defined in [10], p.18 and for $s_i(x)$ we consider at $P \in \Omega_{h,r}$ the following discrete functions

$$\begin{aligned} b_i^+ &= s_i^+(P)/p_i(P), \quad b_i^- = s_i^-(P)/p_i(P), \\ R_i &= R_i(P) = 0.5h_i|s_i(P)|/p_i(P), \\ \alpha_i &= \alpha_i(P) = 1 - R_i(P) + (R_i(P))^2. \end{aligned}$$

Now at $P \in \Omega_{h,r}$, we consider the discrete operator

$$L_h(v) = \sum_{i=1}^{n} \alpha_i (a_i v_{\bar{x}_i})_{x_i} + b_i^+ a_i^{(+1)} v_{x_i} + b_i^- a_i v_{\bar{x}_i} - qv. \tag{3.1}$$

3.3. Approximation at $P \in \Omega_{h,ir}$

Now let $P \in \Omega_{h,ir}$. As in [10], p.19, 20 we consider Lagrange's interpolating polynomial.

Let $w(t)$ be a smooth enough function on $\overline{\Omega}$, then for $P \in \Omega_{h,ir}$ we define a point $Q \in \Gamma$ as in [10], p.19. Then,

$$w(P) = J_d w(P) + \Lambda_d . w(Q) + R(0), \tag{3.2}$$

where

$$J_d w(P) = \sum_{k=1}^{2m} (-1)^{k-1} \frac{(2m)!}{k!(2m-k)!} \cdot \frac{d}{d+k} \cdot w(-kH), \tag{3.3}$$

$$\Lambda_d = \prod_{k=1}^{2m} \frac{k}{d+k}. \tag{3.4}$$

Note that in [10] there was an error consisting of writing $\sum$ instead of Π as in (3.4) here.

Concerning the remaining term $R(0)$ see Lemma 4 below.

3.4. The discrete problem (difference scheme)

Now we consider the discrete problem

$$L_h v(P) = f(P), \quad P \in \Omega_{h,r}, \tag{3.5}$$

$$v(P) = J_d v(P) + \Lambda_d \cdot v(Q), \quad P \in \Omega_{h,ir} \tag{3.6}$$

$$v(P) = g(P), \quad P \in \Gamma, \tag{3.7}$$

where the operators L_h, J_d and the number Λ_d are defined in (3.1), (3.3), (3.4).

4. THE RESULTS

4.1. Preliminary lemmas

We have the analogous lemmas as in [10].

Lemma 1. *The problem (2.2)-(2.6) has a unique solution*

$$u \in C^{2m+2+\lambda}(\overline{\Omega}).$$

For the proof see [3].

Lemma 2. *With any grid stepsize the discrete operator L_h satisfies the maximum principle on $\Omega_{h,r}$:*

1/ If $v \neq$ const, $L_h(v) \geq 0$ at all $P \in \Omega_{h,r}$ then v cannot attain its positive maximum value at $P \in \Omega_{h,r}$;

2/ If $v \neq$ const, $L_h(v) \leq 0$ at all $P \in \Omega_{h,r}$ then v cannot attain its negative minimum value at $P \in \Omega_{h,r}$.

Proof. At first we note that

$$\alpha_i(P) \geq 3/4 \quad \text{with any} \quad h_i.$$

Now at $P \in \Omega_{h,r}$ the discrete operator can be written in the form

$$L_h(v) = \sum_{i=1}^{n} \left[(\alpha_i/h_i + b_i^+) a_i^{(+1)} v_{x_i} - (\alpha_i/h_i - b_i^-) a_i v_{\overline{x}_i} \right] - qv.$$

Then, with any i and any P, we have

$$(\alpha_i/h_i + b_i^+)a_i^{(+1)} > 0, \quad (\alpha_i/h_i - b_i^-)a_i > 0.$$

From that the lemma follows (see [6], p.239).

Now by applying Taylor's formula we obtain

Lemma 3. *For any function* $w \in C^{2\ell+2+\lambda}(\overline{\Omega})$ *we have*

$$L_h w = Lw + \sum_{i=1}^{n}\sum_{k=1}^{\ell} h_i^{2k} F_{ik}(w) + r_1, \quad \ell \geq 1,$$
$$L_h w = Lw + r_1, \quad \ell = 0,$$

where $F_{ik}(w)$ *depends only on* w *and on the derivatives of* w *up to order* $2k+2$ *and* $|r_1| \leq \text{const.}|h|^{2\ell+\lambda}$.

It should be noticed that in the formulation of Lemma 6 in [10] there is an inexactness. Therefore, below we shall restate this lemma and include its detailed proof.

Lemma 4. *If* $w(t) \in C^{M+1}[-2mH, dH]$, $M \leq 2m$, *then*

$$|R(0)| \leq H^{M+1}.c(M,m) \max_{t\in[-2mH,dH]} \left|w^{(M+1)}(t)\right|. \tag{4.1}$$

Proof. Let us denote

$$t_0 = dH, \quad t_i = -iH, \quad i = 1,\ldots,2m$$

and by $P_M(t)$ denote the interpolating polynomial of degree M at the nodes $t_0, t_1, \ldots, t_M$, so that

$$P_M(t_i) = w(t_i), \quad i = 0,1,\ldots,M. \tag{4.2}$$

For the remaining term

$$r_M(t) = w(t) - P_M(t)$$

we have (see [1])

$$\left|r_M(t)\right| \leq \frac{|\omega_M(t)|}{(M+1)!} \max_{t\in[-2mH,dH]} \left|w^{(M+1)}(t)\right|,$$

where

$$\omega_M(t) = (t - t_0)(t - t_1)\dots(t - t_M).$$

It is easy to get

$$\left|r_M(0)\right| \le H^{M+1} \frac{d}{M+1} \max_{t \in [-2mH, dH]} \left|w^{(M+1)}(t)\right|. \tag{4.3}$$

Hence, in the case $M = 2m$ the estimate (4.1) follows immediately.

Now we consider the important case when $M < 2m$. We shall estimate the remainder from the interpolation at $2m+1$ nodes of the function $w(t) \in C^{M+1}[-2mH, dH]$.

We have

$$r_{2m}(t) = w(t) - P_{2m}(t) = P_M(t) + r_M(t) - P_{2m}(t). \tag{4.4}$$

Denote

$$\Phi_i(t) = \prod_{\substack{j=0 \\ j \ne i}}^{2m} \frac{t - t_j}{t_i - t_j}, \quad i = 0, 1, \dots, 2m.$$

Then we have

$$P_{2m}(t) = \sum_{i=0}^{2m} w(t_i)\Phi_i(t),$$

$$P_M(t) = \sum_{i=0}^{2m} P_M(t_i)\Phi_i(t).$$

Substituting these expressions into (4.4) we obtain

$$r_{2m}(t) = \sum_{i=0}^{2m} (P_M(t_i) - w(t_i))\Phi_i(t) + r_M(t).$$

Furthermore, in view of (4.2) we have

$$r_{2m}(t) = \sum_{i=M+1}^{2m} \big(P_M(t_i) - w(t_i)\big)\Phi_i(t) + r_M(t)$$

$$= -\sum_{i=M+1}^{2m} r_M(t_i)\Phi_i(t) + r_M(t)$$

Consequently,

$$\left|r_{2m}(0)\right| \le \left|r_M(0)\right| + \sum_{i=M+1}^{2m} \left|r_M(t_i)\right| \left|\Phi_i(0)\right|.$$

It is easy to verify that

$$\left|r_M(t_i)\right| \le c_1(M,m)\; H^{M+1} \max_{t\in[-2mH,dH]} \left|w^{(M+1)}(t)\right|,$$
$$\left|\Phi_i(0)\right| \le c_2(m), \quad i = M+1,\dots,2m.$$

Taking into account that $R(0) = r_{2m}(0)$, from the above estimates and (4.3) we obtain (4.1). Thus, the lemma is proved.

4.2. Monotony

A scheme is called monotone if the discrete operator L_h satisfies the maximum principle. It is called unconditionally monotone if it is monotone with any grid stepsizes h. So by Lemma 2 we have

Proposition 1. *The scheme (3.5)-(3.7) is unconditionally monotone.*

4.3. Solution of the discrete problem

Using Lemmas 1-4 and Lemmas 4, 5, 7 in [10], we obtain the following results.

Theorem 1. *The discrete problem (3.5)-(3.7) has an unique solution which is the limit of $v^{(\nu)}$, calculated by the iterations*

$$L_h v^{(\nu)}(P) = f(P), \quad P \in \Omega_{h,r}$$
$$v^{(\nu)}(P) = J_d v^{(\nu-1)}(P) + \Lambda_d \cdot v^{(\nu-1)}(Q), \quad P \in \Omega_{h,ir}$$
$$v^{(\nu)}(P) = g(P), \quad P \in \Gamma.$$

Proof. Similar to that of Theorem 1 in [10], p.21.

4.4. Asymptotic error expansion

Theorem 2. *There exist functions*

$$w_{[j]} = w_{j_1 j_2 \dots j_n} \in C^{2m-2|j|+2+\lambda}(\overline{\Omega}), \quad j \in J,$$

$|j| = k$, $k = 1, \ldots, m$, independent of h_i, so that we have the asymptotic error expansion

$$v(P) = u(P) + S_m(P) + O(|h|^{2m+\lambda}), \quad P \in \Omega_h,$$

where v and u are solutions of the discrete and differential problems respectively, and

$$S_m = \sum_{k=1}^{m} \sum_{|j|=k} h_1^{2j_1} h_2^{2j_2} \ldots h_n^{2j_n} w_{[j]}.$$

Proof. Similar to that of Theorem 2 in [10].

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