A Short Communication

BOUNDARY OPERATOR METHOD FOR APPROXIMATE SOLUTION OF BIHARMONIC TYPE EQUATION *

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1. INTRODUCTION

The aim of the paper is to construct effective methods for solving the following boundary value problem

$$Lu \equiv \varepsilon \Delta^2 u - a\Delta u + bu = f(x), \quad x \in \Omega, \tag{1.1}$$

$$u\Big|_\Gamma = u_0, \tag{1.2}$$

$$\frac{\partial u}{\partial \nu}\Big|_\Gamma = u_\nu, \tag{1.3}$$

where Δ is the Laplace operator, Ω is a bounded domain in R^m with sufficiently smooth boundary Γ, ν is the outward normal to Γ, $\varepsilon > 0$, $a \geq 0$, $b \geq 0$. This problem is called the Dirichlet problem for biharmonic type equation. It meets, for $m = 2$, in the theory of plates (see [10]). Also, as well known, the solution of the stationary Navier-Stokes system may be reduced to this problem.

For solving the biharmonic equation, i.e. the equation (1.1) with $\varepsilon = 1$, $a = b = 0$, using the Dorodnhisyn's idea Palsev [7, 8] constructed an iterative method, which reduces the problem (1.1)-(1.3) to a sequence of problems for the Poisson equation and established an error estimate of order $O(1/N)$, where N is the iteration number. In [5] Glowinski et al. also proposed an iterative method for solving the biharmonic equation, but there was not obtained any estimate. In [2, 3] applying an extrapolation technique we have constructed an iterative scheme

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for the same equation, which converges with the rate of geometric progression.

Recently, for the imposed problem (1.1)-(1.3) Abramov and Ulijanova [1] proposed an iterative method, which works for any ε. Unfortunately, the convergence of the method was not proved. Besides, the method has a shortcoming in realization. That is the multiple use of the Laplace-Beltrami operator, which is difficult to be computed for an arbitrary boundary in the case $m \geq 3$.

In order to overcome the above shortcoming, in the present paper we construct an another iterative process for the problem (1.1)-(1.3), for which some results on convergence are established. Moreover, we also propose a technique for improving the convergence rate of the method up to that of geometric progression. This technique is an extrapolation of the solutions of problems containing a small parameter in boundary condition. It is efficiently used in our earlier works [2-4].

2. ITERATIVE METHOD

2.1. Reduction of the problem to a boundary operator equation

Assume that a, b are constants, $a > 0$, $b \geq 0$ and $a^2 - 4b\varepsilon \geq 0$.

As in [1] we set

$$\mu = \frac{1}{2}(a + \sqrt{a^2 - 4b\varepsilon}) \tag{2.1}$$

Furthermore we denote

$$L_2 v = \mu \Delta v - bv, \tag{2.2}$$

$$L_1 u = \frac{\varepsilon}{\mu} \Delta u - u. \tag{2.3}$$

Now let v_0 be a smooth function defined on Γ. Introduce the boundary operator B by the formula

$$Bv_0 = \frac{\partial u}{\partial \nu}\Big|_{\Gamma}, \tag{2.4}$$

where u is determined from the problems

$$L_2 v = 0, \quad x \in \Omega, \quad v|_{\Gamma} = v_0, \tag{2.5}$$

$$L_1 u = v, \quad x \in \Omega, \quad u|_{\Gamma} = 0. \tag{2.6}$$

The operator B primarily defined on smooth functions extends by continuity on whole $L_2(\Gamma)$. It is easy to prove that the opertor B is symmetric, positive and completely continuous in $L_2(\Gamma)$. Moreover, from the theory of elliptic problems

[6] it follows that B is an isomorphism of spaces $H^s(\Gamma)$ and $H^{s+1}(\Gamma)$, $s \geq 0$. Due to the compactness of the imbedding of $H^{s+1}(\Gamma)$ into $H^s(\Gamma)$ we get that B is a completely continuous mapping in $H^s(\Gamma)$, $s \geq 0$. In particular, B is completely continuous in $L_2(\Gamma) = H^0(\Gamma)$.

Now we can lead the problem (1.1)-(1.3) to the operator equation

$$Bv_0 = F, \tag{2.7}$$

where

$$F = u_\nu - \frac{\partial u_2}{\partial \nu}\Big|_\Gamma, \tag{2.8}$$

u_2 is the solution of the problem

$$L_2 v_2 = f, \quad x \in \Omega, \quad v_2\big|_\Gamma = 0, \tag{2.9}$$

$$L_1 u_2 = v_2, \quad x \in \Omega, \quad u_2\big|_\Gamma = u_0. \tag{2.10}$$

The smoothness of F depends on that of f, u_0, u_ν. Namely, using [6] it is easy to show that if $f \in H^{n-4}(\Omega)$, $u_0 \in H^{n-1/2}(\Gamma)$, $u_\nu \in H^{n-3/2}(\Gamma)$, $n \geq 4$ then $F \in H^{n-3/2}(\Gamma)$. Therefore, under the above assumptions the equation (2.7) has an unique solution $v_0 \in H^{n-5/2}(\Gamma)$.

Thus, the problem (1.1)-(1.3) has been reduced to the operator equation (2.7) in the Hilbert space $L_2(\Gamma)$.

Remark 2.1. In the case $a = b = 0$ we put at once $\varepsilon \Delta u = v$ as done in [2, 3], while the method of [1] does not work.

2.2. Iterative method

In order to construct an iterative method for solving the problem (1.1)-(1.3) we use the two-layer iterative scheme for the operator equation (2.7), to which the problem was reduced. The iterative process is defined as follows

$$\frac{v_0^{(k+1)} - v_0^{(k)}}{\tau} + Bv_0^{(k)} = F, \quad k = \overline{0, M-1}, \tag{2.11}$$

$$v_0^{(0)} \in L_2(\Gamma).$$

Theorem 2.1. *The sequence* $\{v_0^{(k)}\}$ *given by (2.11) converges to the solution* v_0 *of the equation (2.7) if*

$$0 < \tau < \frac{2}{\|B\|}. \tag{2.12}$$

Theorem 2.2. *The iterative process (2.11) can be realized by the following*

Algorithm 2.1.

Step 1. Given a starting approximation $v_0^{(0)} \in L_2(\Gamma)$.

Step 2. Knowing $v_0^{(k)}$, $(k = 0, 1, \ldots)$, *solve the problems*

$$L_2 v^{(k)} = f, \quad x \in \Omega, \quad v^{(k)}\big|_\Gamma = v_0^{(k)},$$
$$L_1 u^{(k)} = v^{(k)}, \quad x \in \Omega, \quad u^{(k)}\big|_\Gamma = u_0.$$

Step 3. Compute $\dfrac{\partial u^{(k)}}{\partial \nu}\Big|_\Gamma$.

Step 4. Compute the approximation

$$v_0^{(k+1)} = v_0^{(k)} - \tau\Big(\frac{\partial u^{(k)}}{\partial \nu}\Big|_\Gamma - u_\nu\Big).$$

Corollary 2.1. *For the sequence* $\{u^{(k)}\}$ *generated by Algorithm 2.1 we have*

$$\|u^{(k)} - u\|_{H^{5/2}(\Omega)} \to 0$$

as $k \to \infty$, *where* u *is the solution of the original problem (1.1)-(1.3).*

3. ACCELERATED ITERATIVE METHOD

In order to construct a faster iterative method for solving the problem (1.1)-(1.3), following to the extrapolation technique in [2-4] we consider the corresponding perturbed problem

$$Lu_\delta \equiv \varepsilon\Delta^2 u_\delta - a\Delta u_\delta + bu_\delta = f(x), \quad x \in \Omega, \tag{3.1}$$

$$u_\delta\big|_\Gamma = u_0, \tag{3.2}$$

$$\delta\Big(\frac{\varepsilon}{\mu}\Delta u_\delta - u_\delta\Big)\Big|_\Gamma + \frac{\partial u_\delta}{\partial \nu}\Big|_\Gamma = u_\nu, \tag{3.3}$$

where δ is a small parameter, $\delta > 0$.

This problem may be reduced to the following operator equation

$$(B + \delta I)v_{\delta 0} = F, \tag{3.4}$$

where $v_{\delta 0} = \left(\frac{\varepsilon}{\mu}\Delta u_\delta - u_\delta\right)\big|_\Gamma$, and B, F are defined in Section 2.

Theorem 3.1. *Let $f \in H^{n-4}(\Omega)$, $u_0 \in H^{n-1/2}(\Gamma)$, $u_\nu \in H^{n-3/2}(\Gamma)$, $n \geq 4$. Then for the solution of the problem (3.1)-(3.3) there holds the asymptotic expansion*

$$u_\delta = u + \sum_{i=1}^{N} \delta^i w_i + \delta^{N+1} z_\delta, \quad x \in \Omega, \quad 0 \leq N \leq n - 5/2,$$

where u is the solution of (1.1)-(1.3), w_i $(i = \overline{1,N})$ are functions independent of δ, $w_i \in H^{n-i}(\Omega)$, $z_\delta \in H^{n-N}(\Omega)$ and

$$\|z_\delta\|_{H^{5/2}(\Omega)} \leq C_1,$$

C_1 being independent of δ.

Now we construct an approximate solution U^E of the problem (1.1)-(1.3) by the formula

$$U^E = \sum_{i=1}^{N+1} \gamma_i u_{\delta/i},$$

where

$$\gamma_i = \frac{(-1)^{N+1-i} i^{N+1}}{i!(N+1-i)!},$$

$u_{\delta/i}$ is the solution of (3.1)-(3.3) with the parameter δ/i $(i = \overline{1, N+1})$.

Theorem 3.2. *Under the assumptions of Theorem 3.1 we have the estimate*

$$\|U^E - u\|_{H^{5/2}(\Omega)} \leq C_2 \delta^{N+1},$$

where C_2 is a constant independent of δ.

For solving the operator equation (3.4), as is usual, we use the two-layer scheme

$$\frac{v_{\delta 0}^{(k+1)} - v_{\delta 0}^{(k)}}{\tau_\delta^{(k+1)}} + (\delta I + B) v_{\delta 0}^{(k)} = F, \quad k = \overline{0, M-1} \tag{3.5}$$

$$v_{\delta 0}^{(0)} \in L_2(\Gamma),$$

where $\{\tau_\delta^{(k+1)}\}$ is the Chebyshev collection of parameters [9], defined by the formulae

$$\tau_\delta^{(0)} = \frac{2}{\gamma_\delta^{(1)} + \gamma_\delta^{(2)}}, \quad \tau_\delta^{(k)} = \frac{\tau_\delta^{(0)}}{\rho_\delta t_k + 1}, \quad t_k = \cos\frac{2k-1}{2M}\pi,$$
$$\rho_\delta = \frac{1-\xi_\delta}{1+\xi_\delta}, \quad \xi_\delta = \frac{\gamma_\delta^{(1)}}{\gamma_\delta^{(2)}}, \quad \gamma_\delta^{(1)} = \delta, \quad \gamma_\delta^{(2)} = \delta + \|B\|. \tag{3.6}$$

In the case of the simple iteration $\tau_\delta^{(k)} = \tau_\delta, k = 1, 2, \ldots$ we get

$$\|v_{\delta 0}^{(k)} - v_{\delta 0}\| \leq (\rho_\delta)^k \|v_{\delta 0}^{(0)} - v_{\delta 0}\|. \tag{3.7}$$

The iterative process (3.5) can be realized by the following

Algorithm 3.1.

Step 1. Given a starting approximation $v_{\delta_0}^{(0)} \in L_2(\Gamma)$.

Step 2. Knowing $v_{\delta 0}^{(k)}$, $(k = 0, 1, \ldots)$ *solve the problems*

$$L_2 v_\delta^{(k)} = f, \quad x \in \Omega,$$
$$v_\delta^{(k)}\big|_\Gamma = v_{\delta 0}^{(k)},$$
$$L_1 u_\delta^{(k)} = v_\delta^{(k)}, \quad x \in \Omega,$$
$$u_\delta^{(k)}\big|_\Gamma = u_0.$$

Step 3. Compute $\dfrac{\partial u_\delta^{(k)}}{\partial \nu}\Big|_\Gamma$.

Step 4. Compute the approximation

$$v_{\delta 0}^{(k+1)} = v_{\delta 0}^{(k)} - \tau_\delta\Big(\frac{\partial u_\delta^{(k)}}{\partial \nu}\Big|_\Gamma - u_\nu\Big).$$

For $u_\delta^{(k)}$ we obtain the error estimate

$$\|u_\delta^{(k)} - u_\delta\|_{H^{5/2}(\Omega)} \leq C(\rho_\delta)^k, \quad k = 1, 2, \ldots$$

where $C = \text{const}$ and ρ_δ is given by (3.6).

Therefore, in order to obtain an approximate solution of the original problem (1.1)-(1.3) with the given accuracy δ^* we have to choose δ, such that $\delta^{N+1} = \delta^*$, where N is defined in Theorem 3.1 and then solve $N+1$ problems (3.1)-(3.3) with parameter δ/i $(i = \overline{1, N+1})$ by Algorithm 3.1 with the accuracy δ.

4. NUMERICAL EXPERIMENTS

For the equation (1.10) with $a = 1$, by experimental way we chose the iterative parameter $\tau = 2\sqrt{\varepsilon}$, ensuring the convergence of Algorithm 2.1. When fixing ε with this selection of τ, the numerical results show that the number of iterations slightly depends on b. For the accelerated method the computation time on sequential computer required by Algorithm 3.1 usually does not exceed 90% of the computation time required by Algorithm 2.1. The Algorithm 3.1 will show its advantage over Algorithm 2.1 if each perturbed problem is solved independently on its individual processor of parallel computer.

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