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A Short Communication

ON THE EXISTENCE OF POSITIVE EIGENVALUES FOR CONVEX SET-VALUED MAPS *

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This note presents a generalized version of the well-known Krein-Rutman theorem on the existence of positive eigenvalues of the adjoint linear bounded operators leaving invariant a positive polar cone in a Banach space. The results of this kind have a comparatively long history which steamed primarily from the works due to Frobenius [6] and Perron [9] on spectral properties of positive matrices. These results played an important role to many problems in mathematical programming, games theory and control theory. In particular, they were applied to study controllability and observability of control systems, see e.g. [5], [7]. During the last decade, some attention has been given to these results in the framework of set-valued analysis. Namely, some theorems on the existence of positive eigenvalues of positive set-valued maps in finite dimensional spaces were proved in Aubin and Ekeland [2] and Aubin and Frankowska [3]. The existence of eigenvectors for closed convex processes on cones with compact soles appeared in [4] for characterizing controllability of convex processes in R^n. The infinite dimensional case was treated in a recent paper due to Phat and Dieu [10] where the above mentioned Krein-Rutman theorem has been extended to closed convex processes mapping a cone with nonempty interior into itself.

In this note we shall show that a similar result holds for arbitrary set-valued maps with closed convex graphs, satisfying a weaker invariantness assumption. The proof is based on Kakutani-Ky Fan theorem on fixed points for inward set-valued maps.

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For the sake of convenience, we recall some notations. Let X, Y be Banach spaces and X^*, Y^* be their strong duals. The space X^* endowed with the weak* topogy $\sigma(X^*, X)$ will be denoted by X^*_σ. Let $M \subset X$ be a nonempty subset, then the closure and the interior of M are denoted respectively by cl M and int M; the ***positive polar*** cone M^+ is defined as $M^+ = \{x^* \in X^* : \langle x^*, x\rangle \geq 0, \forall x \in M\}$. It is well-known that if M is a closed convex cone then M^+ is convex and weak* compact. For a closed convex set $M \subset X$, the *tangent cone* to M at a point $x \in M$ is denoted by $T_M(x)$. By definition,

$$T_M(x) = \text{cl} \cup_{\lambda > 0} \lambda(M - x).$$

If M is a closed convex cone, then $T_M(x) = \text{cl}\,(Rx + M)$ (see, e.g., [3] p.143).

Let F be a set-valued map from X to Y, Dom $F := \{x \in X : F(x) \neq \emptyset\}$ and Graph $F := \{(x, y) \in X \times Y : y \in F(x)\}$. Then F is called *strict* if Dom $F = X$ and convex (resp., closed) if Graph F is convex (resp., closed). If Graph F is a cone then F is called a *process*. To every closed convex set-valued map F from X to Y such that $0 \in F(0)$ we associate the *adjoint map* $F^* : \quad Y^* \to X^*$ defined by

$$x^* \in F^*(y^*) \quad \text{iff} \quad \langle y^*, y\rangle \geq \langle x^*, x\rangle, \quad \forall (x, y) \in \text{Graph } F,$$

or, equivalently,

$$(y^*, x^*) \in \text{Graph } F^* \quad \text{iff} \quad (-x^*, y^*) \in (\text{Graph } F)^+.$$

It is easy to see that F^* is a closed convex process and $F^* = G^*$, where G is the closed convex process defined by Graph $G := T_{\text{Graph } F}(0, 0)$. The following important property of the adjoint maps was proved in [3].

Lemma 1. *Let $F : X \to Y$ be a strict closed convex set-valued map with* $(0, 0) \in$ *Graph F. Then* Dom $F^* = F(0)^+$ *and F^* is upper hemicontinuous with bounded closed convex values.*

We recall that *upper hemicontinuity* of F^* means that for each $x \in X$ the support function $y^* \to \sigma(F^*(y^*), x) := \sup_{x^* \in F^*(y^*)} \langle x^*, x\rangle$ is upper semicontinuous. It follows from the above lemma that for each $y^* \in$ Dom F^*, $F^*(y^*)$ is a convex weakly* compact subset. Moreover, it can be shown, by the same way as in [1] (Theorem 2, p. 62), that F^* is a upper semicontinuous map from Y^* to X^*_σ.

Now, let K be a closed convex cone in Y and $F : X \to Y$ be a strict closed convex set-valued map with $(0, 0) \in$ Graph F. We say that K is *invariant* by F if

$$\forall x \in K, \quad F(x) \subset T_K(x). \tag{1}$$

Lemma 2. *If K is invariant by F, then*

$$\forall y^* \in K^+, \quad F^*(y^*) \cap T_{K^+}(y^*) \neq \emptyset, \tag{2}$$

where $T_{K^+}(y^*)$ is the tangent cone of K^+ in X^*_σ at y^*.

The proof is similar to that of Proposition 1.12 in [3]. Suppose that (2) fails to hold, then by the Hahn-Banach theorem, there exist constants c and ε, $\varepsilon > 0$, and a nonzero $x_0 \in X$ such that

$$\sup_{x^* \in F^*(y^*)} \langle x^*, x_0 \rangle \leq c - \varepsilon < c \leq \inf_{z^* \in T_{K^+}(y^*)} \langle z^*, x_0 \rangle \tag{3}$$

for some $y^* \in K^+$. Since $T_{K^+}(y^*) = \mathrm{cl}_\sigma\,(\mathbb{R}y^* + K^+)$, this implies $c = 0$, $\langle y^*, x_0 \rangle = 0$ and $x_0 \in K^{++} = K$. Since $T_K(x_0) = \mathrm{cl}\,(\mathbb{R}x_0 + K)$ and $y^* \in K^+$, it follows that $y^* \in (T_K(x_0))^+$. On the other hand, for the closed convex process $G : X \to Y$, defined by

$$\mathrm{Graph}\; G = T_{\mathrm{Graph}\; F}(0,0),$$

it is easy to verify that $F^* = G^*$ and

$$\forall x \in K, \quad G(x) \subset T_K(x).$$

Therefore, $(T_K(x_0))^+ \subset (G(x_0))^+$, and thus we have $y^* \in (G(x_0))^+$, or equivalently, $\inf_{y \in G(x_0)} \langle y^*, y \rangle \geq 0$. Then, in view of (3) and Proposition 2.6.4 in [3] we can write

$$0 \leq \inf_{y \in G(x_0)} \langle y^*, y \rangle = \sup_{x^* \in G^*(y^*)} \langle x^*, x_0 \rangle = \sup_{x^* \in F^*(y^*)} \langle x^*, x_0 \rangle \leq -\varepsilon,$$

a contradiction.

We are now in a position to prove the main result of this note.

Theorem. *Let F be a strict closed convex set-valued map from a Banach space X into itself such that $0 \in F(0)$. Let $K \subset X$ be a closed convex cone such that $K \neq X$ and $\mathrm{int}\, K \neq \emptyset$. If K is invariant by F then the adjoint set-valued map F^* has an eigenvector $x_0^* \in K^+$, $x_0^* \neq 0$, associated with nonnegative eigenvalue $\lambda \geq 0$, i.e.*

$$\lambda x_0^* \in F^*(x_0^*).$$

Proof. Since $K \neq X$, $K^+ \neq \{0\}$. It follows from (1) that $F(0) \subset T_K(0) = K$ and hence $K^+ \subset F(0)^+ = \mathrm{Dom}\, F^*$. Thus, for each $y^* \in K^+, F^*(y^*)$ is a nonempty convex weakly* compact set, by Lemma 1. Taking $x_0 \in \operatorname{int} K$ and letting $\alpha > 0$ be so small that $x_0 - \alpha e \in K$ for all e in the unit ball of X, we readily verify that

$$\langle x^*, x_0\rangle \geq \alpha ||x^*||, \quad \forall x^* \in K^+. \tag{4}$$

Define

$$H = \{x^* \in K^+ : \langle x^*, x_0\rangle = \alpha\}.$$

Then H is a nonempty convex and weakly* closed set, and by (4), $||x^*|| \leq 1, \quad \forall x^* \in H$. Therefore, by Alaoglu's theorem, H is weakly* compact.

For each $x^* \in H$, we define the affine operator $f(x^*)$ by putting

$$f(x^*)g = x^* + g - (\langle g, x_0\rangle/\alpha)x^*, \tag{5}$$

for $g \in X^*$. Then, since $H \subset K^+ \subset \operatorname{Dom} F^*$, by virtue of Lemma 1, the set-valued map $Q(x^*) := f(x^*)F^*(x^*)$ is upper hemicontinuous with nonempty convex weakly* compact values for $x^* \in H$. We shall show that Q has a fixed point in H, by using the generalized Kakutani-Ky Fan theorem for inward set-valued maps (see, e.g. [3], Theorem 3.2.5, p.87). To this end, let us calculate, for $x^* \in H$, the tangent cone $T_H(x^*)$, H being regarded as a subset in X^*_σ. Since, by definition, $H = K^+ \cap \{y^* \in X^* : \langle y^*, x_0\rangle = \alpha\}$ we have

$$T_H(x^*) \subset T_{K^+}(x^*) \cap \{y^* \in X^* : \langle y^*, x_0\rangle = 0\}.$$

On the other hand, if $\langle y^*, x_0\rangle = 0$ and $y^* \in Rx^* + K^+$, then either $y^* = 0$ or $y^* = -(\langle g, x_0\rangle/\alpha)x^* + g$ for some nonzero $g \in K^+$. From the latter case it follows that $y^* \in \cup_{\lambda>0}\lambda(H - x^*)$. Therefore $T_H(x^*) \supset T_{K^+}(x^*) \cap \{y^* : \langle y^*, x_0\rangle = 0\}$ and, consequently,

$$T_H(x^*) = T_{K^+}(x^*) \cap \{y^* \in X^* : \langle y^*, x_0\rangle = 0\}. \tag{6}$$

Now, for each $x^* \in H$ and $y^* \in Rx^* + K^+$, we have obviously that $\langle f(x^*)y^* - x^*, x_0\rangle = 0$ and $f(x^*)y^* - x^* \in Rx^* + K^+$, and hence, by (6), $f(x^*)T_{K^+}(x^*) \subset T_H(x^*) + x^*$. Since $f(x^*)(F^*(x^*) \cap T_{K^+}(x^*)) \subset f(x^*)F^*(x^*) \cap f(x^*)T_{K^+}(x^*) \subset Q(x^*) \cap (T_H(x^*) + x^*)$, it follows from Lemma 2 that

$$Q(x^*) \cap (T_H(x^*) + x^*) \neq \emptyset, \quad \forall x^* \in H.$$

Thus $Q: \quad H \to X^*$ is an inward upper hemicontinuous map from convex weakly* compact subsets of H. By the mentioned Kakutani-Ky Fan theorem, there exists $x_0^* \in H$ such that $x_0^* \in Q(x_0^*)$. By the definition of Q, it follows that there exists $y_0^* \in F^*(x_0^*)$ such that

$$x_0^* = x_0^* + y_0^* - (\langle y_0^*, x_0\rangle/\alpha)x_0^*,$$

which implies $y_0^* = (\langle y_0^*, x_0\rangle/\alpha)x_0^* \in F^*(x_0^*)$. Thus x_0^* is a nonzero eigenvector of F^* with eigenvalue $\frac{1}{\alpha}\langle y_0, x_0\rangle \geq 0$ and $x_0^* \in K^+$. The proof is complete.

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