Tạp chí Toán học Journal of Mathematics **22**(1 & 2)(1994), 104-108

A Short Communication

MINIMAX SOLUTIONS OF THE CAUCHY PROBLEMS FOR SYSTEMS OF FIRST-ORDER NONLINEAR DIFFERENTIAL EQUATIONS *

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In this note we are concerned with the Cauchy problem for systems of nonlinear first-order partial differential equations

$$\frac{\partial u_k(t,x)}{\partial t} + H_k(t,x,u(t,x),\nabla_x u_k(t,x)) = 0, \tag{1}$$

$$(t,x) \in G = (0,T) \times \mathbb{R}^n, \quad k = 1,...,m$$

$$u(T,x) = u^0(x), \quad x \in \mathbb{R}^n. \tag{2}$$

Here $u = (u_1,...,u_m) : \Omega \to \mathbb{R}^m$ represents the unknown function, $H = (H_1,...,H_m) : G \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m, u^0 = (u_1^0,...,u_m^0) : \mathbb{R}^n \to \mathbb{R}^m$ are given functions, $\nabla_x u_k = \left(\frac{\partial u_k}{\partial x_1},...,\frac{\partial u_k}{\partial x_n}\right)$.

Since a classical solution of the nonlinear problem (1), (2) can fail to exist even in the cases where H_k and u^0 are analytic functions, we need to introduce concepts of generalized solutions. In recent years a new approach has taken shape in the theory of nonlinear differential equations, based on replacement of the equation by a pair of differential inequalities (see, for example, M. G. Crandall and P. L. Lions [1], M. G. Crandall, H. Ishii and P. L. Lions [2] for the viscosity solutions and A. I. Subbotin [3], N. N. Subbotina [4] for the minimax solutions...).

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* This work is supported in part by the National Basic Research Program in Natural Sciences, Vietnam

Our primary aim here is to give a definition of minimax solutions for the problem (1)-(2) and to announce an existence and uniqueness result for these solutions. The case $m = 1$ was considered in a great detail in Subbotin [3], [4] (see, references therein).

We use the following notations. Denote

$$S := \{p \in \mathbb{R}^n, ||p||_n = 1\},$$
$$B := \{p \in \mathbb{R}^n, ||p||_n \leq 1\},$$

where $||\cdot||_n$ is the Euclidean norm in $\mathbb{R}^n$. For $r = (r_1, ..., r_m), s = (s_1, ..., s_m) \in \mathbb{R}^m$ we write $r \leq s$ if $r_k \leq s_k$ for all $k = 1, ..., m$. For functions $u, v : \overline{G} \to \mathbb{R}^m$ we also write $u \leq v$ in $\overline{G}$ if $u_k \leq v_k$ in $\overline{G}$ for all $k = 1, ..., m$.

We first assume that the function $H(t, x, r, p) = (H_1(t, x, r, p), ..., H_m(t, x, r, p))$ is continuous and positive-homogenous with respect to the variable p:

$$H_k(t, x, r, \alpha p) = \alpha H_k(t, x, r, p), \alpha \geq 0, k = 1, ..., m$$

and satisfies the following conditions:

a) for any bounded subset $D \subset G$, there exists a number $\lambda > 0$ such that for all $r \in \mathbb{R}^m, p \in S, (t, x), (t, y) \in D$:

$$||H(t, x, r, p) - H(t, y, r, p)||_m \leq \lambda ||x - y||_n.$$

b) for all $(t, x) \in G, r \in \mathbb{R}^m$ the Lipschitz condition with respect to p holds:

$$\sup\{||H(t, x, r, p) - H(t, x, r, q)||_m - ||p - q||_n.L(x) | p \in B, q \in B\} \leq 0,$$

where $L(x) = \boldsymbol{\kappa}(1 + ||x||_n), \boldsymbol{\kappa}$ is a constant;

c) the function $H(t, x, r, p)$ satisfies the *quasi-monotonicity* condition with respect to r, i.e. if $(r_1, \dots, r_m) = r \leq s = (s_1, ..., s_m)$ and $r_k = s_k$, then

$$H_k(t, x, r, p) \leq H_k(t, x, s, p), (t, x) \in G, p \in \mathbb{R}^m \tag{3}$$

d) the function $H_k(t, x, r, p) (k = 1, ..., m)$ is nonincreasing with respect to r_k, i.e. if $r_k \leq s_k$ then

$$\begin{aligned} &H_k(t, x, r_1, ..., r_{k-1}, r_k, r_{k+1}, ..., r_m, p) \\ &\geq H_k(t, x, r_1, ..., r_{k-1}, s_k, r_{k+1}, ..., r_m, p). \end{aligned} \tag{4}$$

Define the set

$$F(x) := \{f \in \mathbb{R}^n | \ ||f||_n \leq \sqrt{2}L(x)\},$$
$$F_k^B(t,x,r,q) := \{f \in F(x)| \ < f, q > \geq H_k(t,x,r,q)\},$$
$$F_k^H(t,x,r,p) := \{f \in F(x)| < f, p > \leq H_k(t,x,r,p)\}.$$

Here $(t,x) \in G, p \in P, q \in Q, P$ and Q are some nonempty sets such that $\{\alpha p \mid p \in P, \alpha \geq 0\} = \{\alpha q \mid q \in Q, \alpha \geq 0\} = \mathbb{R}^n$. For example, we can take $P = Q = S$. It is easily seen that the sets introduced above are nonempty, convex, compact, and the multivalued function $(t,x,r) \to F_k^B(t,x,r,q), (t,x,r) \to F_k^H(t,x,r,p)$ are continuous in $G \times \mathbb{R}^m$ for $p \in P, q \in Q, k = 1, ..., m$. From (3) and (4) we have

i) if $r, s \in \mathbb{R}^m, r \leq s$ and $r_k = s_k$ then

$$F_k^B(t,x,r,q) \supset F_k^B(t,x,s,q),$$
$$F_k^H(t,x,r,p) \subset F_k^H(t,x,r,p).$$

ii) if $r_k \leq s_k$, then

$$F_k^B(t,x,r_1,...,r_{k-1},r_k,r_{k+1},...,r_m,q) \subset F_k^B(t,x,r_1,...,r_{k-1},s_k,r_{k+1},...,r_m,q),$$
$$F_k^H(t,x,r_1,...,r_{k-1},r_k,r_{k+1},...,r_m,p) \supset F_k^H(t,x,r_1,...,r_{k-1},s_k,r_{k+1},...,r_m,p).$$

It can be shown that $F_k^B(t,x,r,q) \cap F_k^H(t,x,r,p) \neq \emptyset$ for all $(t,x) \in G, r \in \mathbb{R}^m, p \in P, q \in Q, k = 1, ..., m$. Then we obtain easily the equalities

$$H_k(t,x,r,w) = \sup_{q \in Q} \min_{f \in F_k^B(t,x,r,q)} < f, w >$$
$$= \inf_{p \in Q} \max_{f \in F_k^H(t,x,r,p)} < f, w >,$$

for all $(t,x) \in G, r \in \mathbb{R}^m, w \in \mathbb{R}^n$.

We say that the vector-function $u(t,x) = (u_1(t,x), ..., u_m(t,x))$ is lower (upper) semicontinuous in $\overline{G}$ if its every component $(t,x) \to u_k(t,x)$ is lower (upper) semicontinuous in $\overline{G}, k = 1, ..., m$.

For fixed $(t_0, x_0) \in \overline{G}$ denote

$$u(t,x,t_0,x_0,k) := (u_1(t,x), ..., u_{k-1}(t,x), u_k(t_0,x_0), u_{k+1}(t,x), ..., u_m(t,x)).$$

Further, for a function $v : \overline{G} \to \mathbb{R}^1$ we define

$$v^*(z) := \lim_{\epsilon \downarrow 0} \sup\{v(y)| \ ||z-y||_{n+1} < \epsilon, y \in \overline{G}\}$$

and

$$v_*(z) := \liminf_{\epsilon \downarrow 0} \{v(y) \mid ||z-y||_{n+1} < \epsilon, y \in \overline{G}\},$$

for $z \in \overline{G}$. We note that v^* and v_* are upper and lower semicontinuous functions, respectively, on $\overline{G}$ with values in $\mathbb{R}^1 \cup \{\pm\infty\}$ and $v_* \leq v \leq v^*$ in $\overline{G}$.

For a function $u : \overline{G} \to \mathbb{R}^m$ we write $u^* = (u_1^*, ..., u_m^*)$ and $u_* = (u_{1*}, ..., u_{m*})$, and u^*, u_* are upper and lower semicontinuous, respectively, on $\overline{G}$.

Let $u = (u_1, ..., u_m) : \overline{G} \to \mathbb{R}^n$ be a locally bounded function and $(t_0, x_0) \in \overline{G}$. Denote by $X_k^B(t_0, x_0, u, q)$ and $X_k^H(t_0, x_0, u, p)$ the sets of absolutely continuous functions $[0,T] \ni t \mapsto x(t) \in \mathbb{R}^n$, satisfying for almost all $t \in [0,T]$ the differential inclusions

$$\frac{dx}{dt}(t) \in F_k^B(t, x(t), u_*(t, x(t), t_0, x_0, k, q))$$

and

$$\frac{dx}{dt}(t) \in F_k^H(t, x(t), u^*(t, x(t), t_0, x_0, k, p))$$

respectively and also the condition $x(t_0) = x_0$. It is known that by the properties of the multivalued functions F_k^B, F_k^H and in virtue of Theorem Π 3 in [3], $X_k^B(t_0, x_0, u, q)$ and $X_k^H(t_0, x_0, u, p)$ are nonempty compact sets in $C([0,T], \mathbb{R}^n)$.

Definition 1. Let $u : \overline{G} \to \mathbb{R}^m$ be a locally bounded function. We call u a minimax supersolution of Problem (1), (2) if for all $t \in [0,T), \tau \in (t,T], x \in \mathbb{R}^n$ and $k = 1, ..., m$, then

$$\sup_{q \in Q} \min_{X_k^B(t,x,u,q)} \{u_k(\tau, x(\tau)) - u_k(t,x)\} \leq 0,$$

$$u(T,x) \geq u^0(x), x \in \mathbb{R}^n.$$

Definition 2. Let $u : \overline{G} \to \mathbb{R}^m$ be a locally bounded function. We call u a minimax subsolution of Problem (1), (2) if for all $t \in [0,T), \tau \in (t,T], x \in \mathbb{R}^n$ and $k = 1, ..., m$, then

$$\inf_{p \in P} \max_{X_k^H(t,x,u,p)} \{u_k(\tau, x(\tau)) - u_k(t,x)\} \geq 0,$$

$$u(T,x) \leq u^0(x), x \in \mathbb{R}^n.$$

The sets of supersolutions and subsolutions will be denoted by Sol_B and Sol_H respectively.

Definition 3. Let $u : \overline{G} \to \mathbb{R}^m$ be a locally bounded function. We call u a minimax solution of Problem (1), (2) if u is simultaneously a minimax supersolution and a minimax subsolution of the same problem.

The basic results of this note are the following

Theorem 1. *Suppose that $u \in C^1(G, \mathbb{R}^m) \cap C(\overline{G}, \mathbb{R}^m)$ is a global classical solution of Problem (1), (2). Then u is also a minimax solution of the same problem.*

Theorem 2. *The minimax solution u of Problem (1), (2) satisfies the equation (1) at each point (t, x) where u is differentiable.*

Theorem 3. *Suppose that the conditions a) - d) are satisfied and $u^0(x)$ is continuous. Then there exists a unique minimax solution for the Cauchy problem (1), (2).*

Proofs of Theorems 1-3 will be published elsewhere.

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Received February 1, 1994