EXTENDED FIRST ORDER STOCHASTIC AVERAGING METHOD FOR A CLASS

Consider a single mode system described by an equation of the type

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Abstract. An extended first order stochastic averaging procedure is proposed for a class of single-degree-of-freedom systems which can not be investigated by using the classical first order averaging method. As an illustration a system with nonlinear damping is considered.

1. INTRODUCTION

Over the past years the well-known averaging method, developed by Bogoliubov and Mitropolski [1] has proved to be a very useful tool for solving deterministic nonlinear vibration problems. The advantage of this method is that it reduces the dimension of the response coordinates. An extension of the averaging method to the field of random vibrations was originally introduced by Stratonovitch [2] and then developed by many authors.

It should be noted that although the general higher order averaging procedure was already described for deterministic differential equations [3], principally only the first order averaging has been applied in practice. It is well known, however, the effect of some nonlinear terms such as cubic stiffness is lost during the first order averaging procedure.

It implies that the first order averaging method is not sufficient to describe the effect of these nonlinear terms.

The aim of this paper is to propose an extended first order averaging procedure for a class of single-degree-of-freedom systems with random excitation. In the first paragraph, we recall some well known facts for the classical case. The main result of our extended method is given in the second paragraph. In the last paragraph, an application to systems with nonlinear damping is shown.

2. CLASSICAL FIRST ORDER STOCHASTIC AVERAGING

Consider a single mode system described by an equation of the type

$$\hat{x} + \omega^2 x = \varepsilon f_1(x, \dot{x}) + \varepsilon^2 f_2(x, \dot{x}) + \varepsilon \sigma \dot{\xi}(t), \tag{1}$$

AVERAGING METHOD FOR SECONS ASSESSED where ω is the natural frequency of the corresponding linear system ($\varepsilon = 0$), ε is a small positive parameter and σ is a constant, f_1 and f_2 are functions in x and \dot{x} . The random excitation $\xi(t)$ is assumed to be a Gaussian white noise with unit intensity, i.e. the Itô derivative of a standard Gaussian process, with

$$E(\xi(t)) = 0, \quad E(\xi(t)\xi(t+\tau)) = \delta(\tau), \tag{2}$$

where E denotes the expectation operator. The equation (1) may be considered as the following system of Itô stochastic defferential equations

to each a retained
$$dx(t)=\dot{x}dt$$
, where we are relative above self-test refer to $x\in \mathbb{R}$

$$d\dot{x}(t) = [\varepsilon f_1 + \varepsilon^2 f_2 - \omega^2 x] dt + \varepsilon \sigma dW(t), \tag{4}$$

where W(t) is a standard Wiener process:

$$E[W(t)W(t')] = \min\{t, t'\}.$$
 (5)

The solution of the linear system (1), $(\varepsilon = 0)$, has the form

where a and θ are constants. In the case where $\varepsilon \neq 0$, according to the classical averaging method, the state coordinates (x, \dot{x}) are to be transformed to the pair of (a, θ) by the change (6). Thus, the systems (3), (4) are rewritten in the following dure was already described for deterministic differential equations .([5] [sec] (mro)

The effect of these nonlinear terms.

The sum
$$K_1(a,\varphi) = \frac{\varepsilon \sigma^2}{2a\omega^2} \cos^2\varphi - \left[\frac{1}{\omega}f_1(a,\varphi) + \frac{\varepsilon}{\sigma}f_2(a,\varphi)\right] \sin\varphi$$
, and $f_1(a,\varphi) = \frac{\varepsilon}{\sigma^2} \cos^2\varphi - \frac{1}{\omega} \cos\varphi - \frac{\varepsilon}{\sigma^2} \cos\varphi + \frac{\varepsilon}{\sigma^2} \cos\varphi - \frac{\varepsilon}{\sigma^2} \cos\varphi -$

The Fokker-Planck (FP) equation, written for the probability density function $p(a, \theta, t)$ of system (7) is given as follows:

$$\frac{\partial p}{\partial t} = \varepsilon \left\{ \frac{\partial}{\partial a} (K_1 p) + \frac{\partial}{\partial \theta} (K_2 p) - \frac{1}{2} \left[\frac{\partial^2}{\partial a^2} (K_{11} p) + 2 \frac{\partial^2}{\partial a \partial \theta} (K_{12} p) + \frac{\partial^2}{\partial \theta^2} (K_{22} p) \right] \right\}, (9)$$

where tituting (16) into (3) gives

(71)

Hence, in order to ignestigate the effect of the term
$$K_{11}(a,\varphi) = \frac{\varepsilon\sigma^2}{\omega^2}\sin^2\varphi$$
, $K_{12}(a,\varphi) = \frac{\varepsilon\sigma^2}{a\omega^2}\sin^2\varphi$, $K_{12}(a,\varphi) = \frac{\varepsilon\sigma^2\cos^2\varphi}{a\omega^2\cos^2\varphi}$.

$$K_{22}(a,\varphi) = \frac{\varepsilon\sigma^2\cos^2\varphi}{11a^2\omega^2\cos^2\varphi}.$$
(10)

Using the stochastic averaging method [4] the FP equation (9) is approximately replaced by the averaged FP one

$$\frac{\partial p}{\partial t} = \varepsilon \left\{ \frac{\partial}{\partial a} (\langle K_1 \rangle p) + \frac{\partial}{\partial \theta} (\langle K_2 \rangle p) - \frac{1}{2} \left[\frac{\partial^2}{\partial a^2} (\langle K_{11} \rangle p) + \frac{\partial^2}{\partial a \partial \theta} (\langle K_{12} \rangle p) + \frac{\partial^2}{\partial \theta^2} (\langle K_{22} \rangle p) \right] \right\},$$
(11)

A suitable expression of the averaging operator with respect to φ: state of the property (4.3). In order to obtain the amplitude and phase Itô differential

equations let these equations to written in the form
$$\phi$$
 and ϕ .

(12)

$$d\phi$$

$$da(t) = c(a, b) \frac{1}{2\pi i} \int_{0}^{\infty} \frac{1}{2\pi i} \int_{0}^{\infty} (a, b) d\phi$$

$$d\theta(t) = \mu(a, c) dt + \eta(a, c) dW(t) + \mu(a, c) dW(t)$$

Thus, the coefficients in (11) are obtained by averaging of the corresponding coefficients in (9). It is seen from (8) that, in the case where the function f_1 has the property

$$\langle f_1(a,\varphi)\sin\varphi\rangle = \langle f_1(a,\varphi)\cos\varphi\rangle = 0,$$
 (13)

the influence of the term $f_1(a,\varphi)$ will be lost in the averaged FP equation (11) due to the averaging. On the other hand, the expression of the solution (6) does not contain any effect of nonlinear terms.

It follows that, in the case of (13) the classical first order stochastic averaging method is not sufficient to describe the effect of the term $f_1(x, \dot{x})$. So an adequate extension is required.

Example. Consider the following system

$$\ddot{x} + \omega^2 x = \varepsilon \left(\alpha \dot{x}^{2n} x^{2m} + \varepsilon f_2(x, \dot{x}) \right) + \varepsilon \sigma \xi(t). \tag{14}$$

In this case $f_1(x,\dot{x}) = \alpha \dot{x}^{2n} x^{2m}$, $(\alpha,n,m=\text{const})$. It is an easy matter to tion $p(a, \theta, t)$ of system (7) is given as follows: show that a smale mode system described by an equation of the type

(e)
$$\left\langle f_1 \left\{ \frac{\sin \varphi}{\cos \varphi} \right\} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \alpha a^{2(n+m)} \omega^{2n} \cos^{2m} \varphi \sin^{2n} \varphi \left\{ \frac{\sin \varphi}{\cos \varphi} \right\} d\varphi = 0.$$

Hence, in order to investigate the effect of the term $f_1(x, \dot{x})$ in (14) one needs an extended averaging method.

3. EXTENDED FIRST ORDER STOCHASTIC AVERAGING

Suppose that the nonlinear system (1) exhibits the property (13). instead of (6) we consider the following change, purposely,

$$x(t) = a\cos\varphi + \varepsilon u(a,\varphi), \qquad + (a\langle A \rangle) = \begin{cases} a & \text{(15)} \end{cases}$$

$$x(t) = a\cos\varphi + \varepsilon u(a,\varphi), \qquad (15)$$

$$\dot{x}(t) = -a\omega\sin\varphi + \varepsilon\frac{\partial u}{\partial t}, \quad \varphi = \omega t + \theta. \qquad (16)$$

A suitable expression of the function $u(a, \varphi)$ will be determined later using the property (13). In order to obtain the amplitude and phase Itô differential equations let these equations be written in the form

$$da(t) = \alpha(a, \varphi)dt + \beta(a, \varphi)dW(t),$$

$$d\theta(t) = \mu(a, \varphi)dt + \gamma(a, \varphi)dW(t),$$
(17)

where $\alpha, \beta, \mu, \gamma$ are unknown functions of a and φ . For a given function $F(t, a, \varphi)$ the Itô differential rule [5] gives the property

(E1)
$$dF = \begin{bmatrix} \frac{\partial F}{\partial t} + (\ell_1 + \ell_2)F \end{bmatrix} dt + \ell_3 F dW(t), \tag{18}$$
(11) noting the term of the local solution and the second of the second of

due to the averaging. On the other hand, the expression of the solution and where the operators ℓ_1 , ℓ_2 , ℓ_3 denote nonlinear terms.

It follows that, in the case of (13) the
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Differentiating (15) with respect to the time and using (18) yields

$$dx(t) = \left[-a\omega \sin \varphi + \varepsilon \frac{\partial u}{\partial t} + (\ell_1 + \ell_2)(a\cos \varphi + \varepsilon u) \right] dt \quad (20)$$

$$\left\{ \left[\cos \omega \cos \varphi + \varepsilon u \right] dW(t) \right\} \quad (20)$$

Substituting (16) into (3) gives

Hence, the system (23) can be rewritten in two separable systems
$$dx(t) = \left[-a\omega \sin \varphi + \varepsilon \frac{\partial u}{\partial t} \right] dt. \tag{21}$$

Further, differentiating (16) with respect to t and using (18) yields

$$d\dot{x}(t) = \left[-a\omega^2 \sin\varphi + \left(\varepsilon \frac{\partial^2 u}{\partial t^2} + (\ell_1 + \ell_2) \left(a\cos\varphi + \varepsilon \frac{\partial u}{\partial t}\right) \right] dt$$
(82)
$$+ \ell_3 \left(-a\omega \sin\varphi + \varepsilon \frac{\partial u}{\partial t} \right) dW(t). \qquad (22)$$

Comparing now (20) with (21), and (22) with (4), and noting that $\varphi = \omega t + \theta$, we obtain the following relations

$$(\ell_1 + \ell_2)(a\cos\varphi + \varepsilon u) = 0, \quad \ell_3(a\cos\varphi + \varepsilon u) = 0,$$

$$(\ell_1 + \ell_2)\left(-a\omega\sin\varphi + \varepsilon\frac{\partial u}{\partial t}\right) = \varepsilon\left[f_1(a,\varphi) - \omega^2\left(u + \frac{\partial^2 u}{\partial \varphi^2}\right)\right]$$

$$+ \varepsilon^2 F_2(a,\varphi) + \varepsilon^3 ...,$$

$$\ell_3\left(-a\omega\sin\varphi + \varepsilon\frac{\partial u}{\partial t}\right) = \varepsilon\sigma,$$

$$(82) \text{ and } (91), (98) \text{ and } (91)$$

where

$$F_2(a,\varphi) = f_2(a,\varphi) + \left(\frac{\partial f_1}{\partial x}u + \frac{\partial f_1}{\partial \dot{x}}\omega \frac{\partial u}{\partial \varphi}\right)\Big|_{\substack{x = a\cos\varphi\\ \dot{x} = -a\omega\sin\varphi}} \tag{24}$$

The function $u(a,\varphi)$ will be chosen such as $\omega = 0$

$$f_1(a,\varphi) - \omega^2 \left(u + \frac{\partial^2 u}{\partial \varphi^2}\right) = 0.$$
 (25)

Assuming that we have (13) and expanding $f_1(a,\varphi)$ into a Fourier series

$$f_{1}(a,\varphi) = \langle f_{1}(a,\varphi) \rangle \text{ if } (\varphi,\varphi) \cdot \overline{\varphi} = \langle \varphi, \varphi \rangle \circ \overline{\varphi} = \langle \varphi, \varphi \rangle \circ \overline{\varphi} \circ \overline{\varphi}$$

we get from (25) the following expression for $u(a,\varphi)$

$$u(a,\varphi) = \frac{1}{\omega^2} \left\{ \langle f_1(a,\varphi) \rangle + \frac{1}{16} \rangle + \frac{1}{16} \rangle + 2 \sum_{n=2}^{\infty} \frac{1}{1 - n^2} \left[\langle f_1(a,\varphi) \sin n\varphi \rangle \sin n\varphi + \langle f_1(a,\varphi) \cos n\varphi \rangle \cos n\varphi \right] \right\}.$$
(26)

Hence, the system (23) can be rewritten in two separable systems (14) one agents

$$\ell_3(a\cos\varphi + \varepsilon u) = 0, \quad \ell_3\left(-a\omega\sin\varphi + \varepsilon\omega\frac{\partial u}{\partial\varphi}\right) = \varepsilon\sigma$$

$$\text{split}(81) \text{ gatau bas to through the sum } (31) \text{ gatau bas to through the sum } (31) \text{ gatau bas to through the sum } (31) \text{ gatau bas to through the sum } (31) \text{ gatau bas to through } (31) \text{ gatau bas to through } (31)$$

and

$$\ell_1(a\cos\varphi + \varepsilon u) = -\ell_2(a\cos\varphi + \varepsilon u),$$

$$\ell_1\left(-a\omega\sin\varphi + \varepsilon\omega\frac{\partial u}{\partial\varphi}\right) = -\ell_2(-a\omega\sin\varphi + \varepsilon\omega\frac{\partial u}{\partial\varphi}) + \varepsilon^2F_2(a,\varphi) + \varepsilon^3..$$
(28)

Applying (19) to (27) we have

$$(27) \text{ we have } (19) \text{ to } (27) \text{ we have } (19) \text{ drive } (27) \text{ bind } (13) \text{ drive } (27) \text{ won gainsquiso} (28) \text{ drive } (29) \text{ won gainsquiso} (29) \text{ drive } (29) \text{ driv$$

Hence,

$$\beta(a,\varphi) = -\frac{\varepsilon}{\omega}\sigma\sin\varphi + \varepsilon^2.., \quad \gamma(a,\varphi) = -\frac{\varepsilon}{a\omega}\sigma\cos\varphi + \varepsilon^2..$$
(30)
g (30), (19) into (28) yields

Substituting (30), (19) into (28) yields

$$\left(\cos\varphi + \varepsilon\frac{\partial u}{\partial a}\right)\alpha + \left(-a\sin\varphi + \varepsilon\frac{\partial u}{\partial\varphi}\right)\mu = \frac{\varepsilon^2\sigma^2}{2a\omega^2}\cos\varphi(1+\sin^2\varphi) + \varepsilon^3..$$

$$\left(-\omega\sin\varphi + \varepsilon\omega\frac{\partial^2 u}{\partial a^2}\right)\alpha + \left(-a\omega\cos\varphi + \varepsilon\frac{\partial^2 u}{\partial\varphi^2}\right)\mu$$

$$= \frac{\varepsilon^2\sigma^2}{2a\omega}\sin\varphi\cos^2\varphi + \varepsilon^2f_2(a,\varphi) + \varepsilon^3..$$
(31)

From (31) one gets
$$\alpha(a,\varphi) = \frac{\varepsilon^2 \sigma^2}{2a\omega^2} \cos^2 \varphi - \frac{\varepsilon^2}{\omega} F_2(a,\varphi) \sin \varphi + \varepsilon^3, \quad (32)$$

$$\mu(a,\varphi) = \frac{\varepsilon^2 \sigma^2}{2a^2\omega^2} \sin 2\varphi - \frac{\varepsilon^2}{a\omega} F_2(a,\varphi) \cos \varphi + \varepsilon^3.$$

From (41) one gets

Hence, the amplitude and phase differential equations (17) are ready defined where $\alpha(a,\varphi), \beta(a,\varphi), \mu(a,\varphi), \gamma(a,\varphi)$ are given in (32) and (30). The averaged FP equation, written for probability density function $p(a, \theta, t)$ of system (17) takes the form

$$\frac{\partial p}{\partial t} = \varepsilon^{2} \left\{ \frac{\partial}{\partial a} (\langle \alpha \rangle p) + \frac{\partial}{\partial \theta} (\langle \mu \rangle p) - \frac{1}{2} \left[\frac{\partial^{2}}{\partial a^{2}} (\langle \beta^{2} \rangle p) \right] + \frac{2\partial^{2}}{\partial a \partial \varphi} (\langle \beta \gamma \rangle p) + \frac{\partial^{2}}{\partial \theta^{2}} (\langle \gamma^{2} \rangle p) \right] \right\} + \varepsilon^{3}.$$
(33)

It is seen from (24) and (32) that the coefficients $\alpha(a,\varphi)$, $\mu(a,\varphi)$ contain the derivatives of the function $f_1(x,\dot{x})$. So, the effect of the nonlinear term $f_1(x,\dot{x})$ may be investigated by using the averaged FP equation (33) and also the expression of solution (15), (16).

4. SYSTEM WITH NONLINEAR DAMPING

The proposed extended first order stochastic averaging method will be used for investigating a system with nonlinear damping. We note also that since the first order averaging is used the terms of ε^n $(n \geq 3)$ will be neglected in the FP equation. Consider the nonlinear system described by the following equation

$$\ddot{x} + \omega^2 x = -\varepsilon \beta \dot{x}^2 - 2\varepsilon^2 h \dot{x} + \varepsilon \sigma \xi(t), \tag{34}$$

where $h, \beta > 0$. So one has the equation (1) in which split the substantial split the split split in which split the split split split in which split spli

It can be shown that the condition (13) is satisfied. Substituting (35) into (25) yields

(36) nowledgement. Support
$$\varphi \cos 2\phi = \frac{\beta}{6} \dot{a}^2 = \pm \frac{\beta}{2} \dot{a}^2 = \pm (\varphi, a) \mathbf{w}$$
 ch project in natural sciences is gratefully acknowledged.

Hence, using (15), we see that the solution of equation (34) takes the form

$$x = a\cos\varphi - \varepsilon \frac{\beta a^2}{2} \left(1 + \frac{1}{3}\cos 2\varphi\right). \tag{37}$$

tions, Nauka, Moscow, 1961 (Bussian)

Using (35), (36) one gets from (24)

$$F_2(a,\varphi)=-2ha\omega\sin\varphi+rac{\pi}{3}eta^2a^3\omega^2\sin^2\varphi\cos\varphi.$$

Substituting (38) into (32) gives appared to the safe base but il que est, and il

$$\alpha(a,\varphi) = \varepsilon^2 \left\{ \frac{\sigma^2}{2a\omega^2} \cos^2 \varphi - 2ha \sin^2 \varphi - \frac{2}{3}\beta^2 a^3 \omega \sin^2 \varphi \sin 2\varphi \right\}. \tag{39}$$

ce(a, a), Muny 1, w(a, p), r(a, p) are given in (32) and (30). The averaged FP

So, one gets from (39) and (30) (q(u)) $\frac{1}{100}$ (q(u)) $\frac{1}{100}$ $\frac{1}{100}$ $\frac{1}{100}$ $\frac{1}{100}$ $\frac{1}{100}$ $\frac{1}{100}$ $\frac{1}{100}$

$$\langle \alpha(a,\varphi) \rangle = \varepsilon^2 \left\{ \frac{\sigma^2}{4a\omega^2} - ha \right\}, \quad \langle \beta^2(a,\varphi) \rangle = \frac{\varepsilon^2 \sigma^2}{2\omega^2}.$$
 (40)

It is seen from (24) and (32) that the coefficiable (33) yields coefficiable (40) gaithstable derivatives of the function $f_1(x,x)$. So, the effect of the nonlinear term $f_1(x,x)$

may be investigated by us us
$$\frac{2h\omega^2}{\sigma^2}$$
 erged if $\frac{2h\omega^2}{\sigma^2}$ and also the expression $p(a) = \frac{2h\omega^2}{\sigma^2}a\exp\left\{-\frac{2h\omega^2}{\sigma^2}a^2\right\}$. (15), (15)

From (41) one gets

$$E(a^2) = \frac{\sigma^2}{2h\omega^2} \cdot \text{mad indo in indo } 2$$
 (42)

Using (37) and (42) yields the following mean value of the displacement

and the balance additive (E
$$_{2}$$
) is to an rest and $E(x) = -\varepsilon \frac{\beta}{2} E(a^{2}) + \varepsilon^{2}$. $= -\frac{\varepsilon \beta \sigma^{2}}{4h\omega^{2}} + \varepsilon^{2}$. (43)

In the absence of the nonlinear damping $-\varepsilon\beta\dot{x}^2$ (i.e. $\beta=0$), one has E(x)=0. So, the expression (43) implies that the nonlinear damping $-\varepsilon\beta\dot{x}^2$ reduces the mean value of the displacement.

Conclusion. The well-known stochastic averaging method has been extended to study a class of nonlinear systems, which can not be investigated by using the classical first order stochastic averaging.

Acknowledgement. Support from the fundamental research project in natural sciences is gratefully acknowledged.

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Received April 2, 1993

Abstract. In this note we consider the midual influence of properties of algebras and their enburgation of topic transmission. For an intent almemoral algebra W and finite con statement and to be the show R as directions of anti-chilly if A is semiffront with the 1 St. M. St. March M. March and S. St. St. St. St. St. St. St. March 2 St. March 2 St. March 2 St. M. St. M with alcolor 1 K 2 2 and that K is a first discussion of sensitivine ring and I C A 2 and

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If A is semiorance with firtuil distression, then It is a Golde ring

salt. A subalgeors A of an algebra R over a field P is suff to have finite country sion if the vector space (E/A, +) is finite dimensional," ... & least of a line of the space of a range of the content of the semilarly a sucring A of a ring P is said to have finite on the abelian

reperture of braile codinguistional left ideals were considered in [1] and [3]

Subrings of finite index were investigated in 2, subalgebras of finite codinension were studied in [4] Loud A. L. Maicev shows that if an algebra R over the field

I has a left ideal H of finite confinencion and H is representable by matrices over some field then R is also representable by inatrices over a field. in [3] S A. Against and LAV. Small show that in an affine Plangbra he finite confinencional

property of a left ideal is equivalent to the algebra R being algebraic over the left

The finite codimensional property of an algebra has connections with properties of representability and fluite dimensional (fighte) approximability of alrebras (eings).