Journal of Mathematics · *Tạp chí Toán học* **21**(3 & 4)(1993), 85–93

EXTENDED FIRST ORDER STOCHASTIC AVERAGING METHOD FOR A CLASS OF NONLINEAR SYSTEMS

NGUYEN DONG ANH

Abstract. *An extended first order stochastic averaging procedure is proposed for a class of single-degree-of-freedom systems which can not be investigated by using the classical first order averaging method. As an illustration a system with nonlinear damping is considered.*

1. INTRODUCTION

Over the past years the well-known averaging method, developed by Bogoliubov and Mitropolski [1] has proved to be a very useful tool for solving deterministic nonlinear vibration problems. The advantage of this method is that it reduces the dimension of the response coordinates. An extension of the averaging method to the field of random vibrations was originally introduced by Stratonovitch [2] and then developed by many authors.

It should be noted that although the general higher order averaging procedure was already described for deterministic differential equations [3], principally only the first order averaging has been applied in practice. It is well known, however, the effect of some nonlinear terms such as cubic stiffness is lost during the first order averaging procedure.

It implies that the first order averaging method is not sufficient to describe the effect of these nonlinear terms.

The aim of this paper is to propose an extended first order averaging procedure for a class of single-degree-of-freedom systems with random excitation. In the first paragraph, we recall some well known facts for the classical case. The main result of our extended method is given in the second paragraph. In the last paragraph, an application to systems with nonlinear damping is shown.

2. CLASSICAL FIRST ORDER STOCHASTIC AVERAGING

Consider a single mode system described by an equation of the type

$$\ddot{x} + \omega^2 x = \varepsilon f_1(x, \dot{x}) + \varepsilon^2 f_2(x, \dot{x}) + \varepsilon\sigma\xi(t), \tag{1}$$

where ω is the natural frequency of the corresponding linear system ($\varepsilon = 0$), ε is a small positive parameter and σ is a constant, f_1 and f_2 are functions in x and $\dot{x}$. The random excitation $\xi(t)$ is assumed to be a Gaussian white noise with unit intensity, i.e. the Itô derivative of a standard Gaussian process, with

$$E\big(\xi(t)\big) = 0, \quad E\big(\xi(t)\xi(t+\tau)\big) = \delta(\tau), \tag{2}$$

where E denotes the expectation operator. The equation (1) may be considered as the following system of Itô stochastic defferential equations

$$dx(t) = \dot{x}dt, \tag{3}$$

$$d\dot{x}(t) = [\varepsilon f_1 + \varepsilon^2 f_2 - \omega^2 x]dt + \varepsilon\sigma dW(t), \tag{4}$$

where $W(t)$ is a standard Wiener process:

$$E\big[W(t)W(t')\big] = \min\{t, t'\}. \tag{5}$$

The solution of the linear system (1), ($\varepsilon = 0$), has the form

$$x(t) = a\cos\varphi, \quad \dot{x} = -a\omega\sin\varphi, \quad \varphi = \omega t + \theta, \tag{6}$$

where a and θ are constants. In the case where $\varepsilon \neq 0$, according to the classical averaging method, the state coordinates $(x, \dot{x})$ are to be transformed to the pair of (a, θ) by the change (6). Thus, the systems (3), (4) are rewritten in the following form, (see [5]).

$$\begin{aligned} da &= \varepsilon K_1(a,\varphi)dt - \frac{\varepsilon}{\omega}\sigma\sin\varphi dW(t), \\ d\theta &= \varepsilon K_2(a,\varphi)dt - \frac{\varepsilon}{a\omega}\sigma\cos\varphi dW(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned} K_1(a,\varphi) &= \frac{\varepsilon\sigma^2}{2a\omega^2}\cos^2\varphi - \Big[\frac{1}{\omega}f_1(a,\varphi) + \frac{\varepsilon}{\omega}f_2(a,\varphi)\Big]\sin\varphi, \\ K_2(a,\varphi) &= \frac{\varepsilon\sigma^2}{2a^2\omega^2}\sin 2\varphi - \frac{1}{a\omega}f_1(a,\varphi)\cos\varphi - \frac{\varepsilon}{a\omega}f_2(a,\varphi)\cos\varphi, \\ f_i(a,\varphi) &= f_i(x = a\cos\varphi, \dot{x} = -a\omega\sin\varphi), \quad i = 1, 2. \end{aligned} \tag{8}$$

The Fokker-Planck (FP) equation, written for the probability density function $p(a, \theta, t)$ of system (7) is given as follows:

$$\frac{\partial p}{\partial t} = \varepsilon\Big\{\frac{\partial}{\partial a}(K_1 p)+\frac{\partial}{\partial \theta}(K_2 p)-\frac{1}{2}\Big[\frac{\partial^2}{\partial a^2}(K_{11}p)+2\frac{\partial^2}{\partial a\partial\theta}(K_{12}p)+\frac{\partial^2}{\partial\theta^2}(K_{22}p)\Big]\Big\}, \quad (9)$$

where

$$K_{11}(a,\varphi) = \frac{\varepsilon\sigma^2}{\omega^2}\sin^2\varphi, \quad K_{12}(a,\varphi) = \frac{\varepsilon\sigma^2}{a\omega^2}\sin\varphi\cos\varphi,$$

$$K_{22}(a,\varphi) = \frac{\varepsilon\sigma^2\cos^2\varphi}{a^2\omega^2} \quad (10)$$

Using the stochastic averaging method [4] the FP equation (9) is approximately replaced by the averaged FP one

$$\frac{\partial p}{\partial t} = \varepsilon\Big\{\frac{\partial}{\partial a}(\langle K_1\rangle p)+\frac{\partial}{\partial \theta}(\langle K_2\rangle p)$$

$$-\frac{1}{2}\Big[\frac{\partial^2}{\partial a^2}(\langle K_{11}\rangle p)+2\frac{\partial^2}{\partial a\partial\theta}(\langle K_{12}\rangle p)+\frac{\partial^2}{\partial\theta^2}(\langle K_{22}\rangle p)\Big]\Big\}, \quad (11)$$

where $\langle\ \rangle$ is the averaging operator with respect to φ:

$$\langle\ \rangle = \frac{1}{2\pi}\int_0^{2\pi}(\ )d\varphi. \quad (12)$$

Thus, the coefficients in (11) are obtained by averaging of the corresponding coefficients in (9). It is seen from (8) that, in the case where the function f_1 has the property

$$\langle f_1(a,\varphi)\sin\varphi\rangle = \langle f_1(a,\varphi)\cos\varphi\rangle = 0, \quad (13)$$

the influence of the term $f_1(a,\varphi)$ will be lost in the averaged FP equation (11) due to the averaging. On the other hand, the expression of the solution (6) does not contain any effect of nonlinear terms.

It follows that, in the case of (13) the classical first order stochastic averaging method is not sufficient to describe the effect of the term $f_1(x,\dot{x})$. So an adequate extension is required.

Example. Consider the following system

$$\ddot{x}+\omega^2 x = \varepsilon\big(\alpha\dot{x}^{2n}x^{2m} + \varepsilon f_2(x,\dot{x})\big) + \varepsilon\sigma\xi(t). \quad (14)$$

In this case $f_1(x,\dot{x}) = \alpha\dot{x}^{2n}x^{2m}$, $(\alpha, n, m = \text{const})$. It is an easy matter to show that

$$\left\langle f_1\begin{Bmatrix}\sin\varphi\\ \cos\varphi\end{Bmatrix}\right\rangle = \frac{1}{2\pi}\int_0^{2\pi}\alpha a^{2(n+m)}\omega^{2n}\cos^{2m}\varphi\sin^{2n}\varphi\begin{Bmatrix}\sin\varphi\\ \cos\varphi\end{Bmatrix}d\varphi = 0.$$

Hence, in order to investigate the effect of the term $f_1(x,\dot{x})$ in (14) one needs an extended averaging method.

3. EXTENDED FIRST ORDER STOCHASTIC AVERAGING

Suppose that the nonlinear system (1) exhibits the property (13). Now, instead of (6) we consider the following change, purposely,

$$x(t) = a\cos\varphi + \varepsilon u(a,\varphi), \tag{15}$$

$$\dot{x}(t) = -a\omega\sin\varphi + \varepsilon\frac{\partial u}{\partial t}\,, \quad \varphi = \omega t + \theta. \tag{16}$$

A suitable expression of the function $u(a,\varphi)$ will be determined later using the property (13). In order to obtain the amplitude and phase Itô differential equations let these equations be written in the form

$$\begin{aligned} da(t) &= \alpha(a,\varphi)dt + \beta(a,\varphi)dW(t),\\ d\theta(t) &= \mu(a,\varphi)dt + \gamma(a,\varphi)dW(t), \end{aligned} \tag{17}$$

where α, β, μ, γ are unknown functions of a and φ. For a given function $F(t,a,\varphi)$ the Itô differential rule [5] gives

$$dF = \left[\frac{\partial F}{\partial t} + (\ell_1 + \ell_2)F\right]dt + \ell_3 F\,dW(t), \tag{18}$$

where the operators ℓ_1, ℓ_2, ℓ_3 denote

$$\begin{aligned} \ell_1 &= \alpha\frac{\partial}{\partial a} + \mu\frac{\partial}{\partial\varphi},\\ \ell_2 &= \frac{1}{2}\beta^2\frac{\partial^2}{\partial a^2} + \beta\gamma\frac{\partial^2}{\partial a\partial\varphi} + \frac{1}{2}\gamma^2\frac{\partial^2}{\partial\varphi^2},\\ \ell_3 &= \beta\frac{\partial}{\partial a} + \gamma\frac{\partial}{\partial\varphi}. \end{aligned} \tag{19}$$

Differentiating (15) with respect to the time and using (18) yields

$$d\dot{x}(t) = \Big[-a\omega\sin\varphi + \varepsilon\frac{\partial u}{\partial t} + (\ell_1+\ell_2)(a\cos\varphi+\varepsilon u)\Big]dt + \ell_3(a\cos\varphi+\varepsilon u)dW(t). \tag{20}$$

Substituting (16) into (3) gives

$$dx(t) = \Big[-a\omega\sin\varphi + \varepsilon\frac{\partial u}{\partial t}\Big]dt. \tag{21}$$

Further, differentiating (16) with respect to t and using (18) yields

$$d\dot{x}(t) = \Big[-a\omega^2\sin\varphi + \varepsilon\frac{\partial^2 u}{\partial t^2} + (\ell_1+\ell_2)\Big(a\cos\varphi+\varepsilon\frac{\partial u}{\partial t}\Big)\Big]dt + \ell_3\Big(-a\omega\sin\varphi+\varepsilon\frac{\partial u}{\partial t}\Big)dW(t). \tag{22}$$

Comparing now (20) with (21), and (22) with (4), and noting that $\varphi = \omega t + \theta$, we obtain the following relations

$$\begin{aligned}
&(\ell_1+\ell_2)(a\cos\varphi+\varepsilon u) = 0, \quad \ell_3(a\cos\varphi+\varepsilon u) = 0,\\
&(\ell_1+\ell_2)\Big(-a\omega\sin\varphi+\varepsilon\frac{\partial u}{\partial t}\Big) = \varepsilon\Big[f_1(a,\varphi) - \omega^2\Big(u+\frac{\partial^2 u}{\partial\varphi^2}\Big)\Big]\\
&\qquad + \varepsilon^2 F_2(a,\varphi) + \varepsilon^3 \ldots,\\
&\ell_3\Big(-a\omega\sin\varphi+\varepsilon\frac{\partial u}{\partial t}\Big) = \varepsilon\sigma,
\end{aligned} \tag{23}$$

where

$$F_2(a,\varphi) = f_2(a,\varphi) + \Big(\frac{\partial f_1}{\partial x}u + \frac{\partial f_1}{\partial \dot{x}}\omega\frac{\partial u}{\partial\varphi}\Big)\Big|_{\substack{x=a\cos\varphi\\ \dot{x}=-a\omega\sin\varphi}} \tag{24}$$

The function $u(a,\varphi)$ will be chosen such as

$$f_1(a,\varphi) - \omega^2\Big(u + \frac{\partial^2 u}{\partial\varphi^2}\Big) = 0. \tag{25}$$

Assuming that we have (13) and expanding $f_1(a,\varphi)$ into a Fourier series

$$f_1(a,\varphi) = \langle f_1(a,\varphi)\rangle + 2\sum_{n=2}\Big\{\langle f_1(a,\varphi)\sin n\varphi\rangle\sin n\varphi + \langle f_1(a,\varphi)\cos n\varphi\rangle\cos n\varphi\Big\}$$

we get from (25) the following expression for $u(a,\varphi)$

$$u(a,\varphi)=\frac{1}{\omega^2}\Big\{\langle f_1(a,\varphi)\rangle$$
$$+2\sum_{n=2}\frac{1}{1-n^2}\Big[\langle f_1(a,\varphi)\sin n\varphi\rangle\sin n\varphi+\langle f_1(a,\varphi)\cos n\varphi\rangle\cos n\varphi\Big]\Big\}. \tag{26}$$

Hence, the system (23) can be rewritten in two separable systems

$$\ell_3(a\cos\varphi+\varepsilon u)=0,\quad \ell_3\Big(-a\omega\sin\varphi+\varepsilon\omega\frac{\partial u}{\partial\varphi}\Big)=\varepsilon\sigma \tag{27}$$

and

$$\ell_1(a\cos\varphi+\varepsilon u)=-\ell_2(a\cos\varphi+\varepsilon u),$$
$$\ell_1\Big(-a\omega\sin\varphi+\varepsilon\omega\frac{\partial u}{\partial\varphi}\Big)=-\ell_2\Big(-a\omega\sin\varphi+\varepsilon\omega\frac{\partial u}{\partial\varphi}\Big)+\varepsilon^2F_2(a,\varphi)+\varepsilon^3.. \tag{28}$$

Applying (19) to (27) we have

$$\Big(\cos\varphi+\varepsilon\frac{\partial u}{\partial a}\Big)\beta+\Big(-a\sin\varphi+\varepsilon\frac{\partial u}{\partial\varphi}\Big)\gamma=0,$$
$$\Big(-\omega\sin\varphi+\varepsilon\omega\frac{\partial^2u}{\partial\varphi\partial a}\Big)\beta+\Big(-a\omega\cos\varphi+\varepsilon\frac{\partial^2u}{\partial\varphi^2}\Big)\gamma=\varepsilon\sigma. \tag{29}$$

Hence,

$$\beta(a,\varphi)=-\frac{\varepsilon}{\omega}\sigma\sin\varphi+\varepsilon^2..,\quad \gamma(a,\varphi)=-\frac{\varepsilon}{a\omega}\sigma\cos\varphi+\varepsilon^2.. \tag{30}$$

Substituting (30), (19) into (28) yields

$$\Big(\cos\varphi+\varepsilon\frac{\partial u}{\partial a}\Big)\alpha+\Big(-a\sin\varphi+\varepsilon\frac{\partial u}{\partial\varphi}\Big)\mu=\frac{\varepsilon^2\sigma^2}{2a\omega^2}\cos\varphi(1+\sin^2\varphi)+\varepsilon^3..$$
$$\Big(-\omega\sin\varphi+\varepsilon\omega\frac{\partial^2u}{\partial a^2}\Big)\alpha+\Big(-a\omega\cos\varphi+\varepsilon\frac{\partial^2u}{\partial\varphi^2}\Big)\mu \tag{31}$$
$$=\frac{\varepsilon^2\sigma^2}{2a\omega}\sin\varphi\cos^2\varphi+\varepsilon^2f_2(a,\varphi)+\varepsilon^3..$$

From (31) one gets

$$\alpha(a,\varphi)=\frac{\varepsilon^2\sigma^2}{2a\omega^2}\cos^2\varphi-\frac{\varepsilon^2}{\omega}F_2(a,\varphi)\sin\varphi+\varepsilon^3..,$$
$$\mu(a,\varphi)=-\frac{\varepsilon^2\sigma^2}{2a^2\omega^2}\sin2\varphi-\frac{\varepsilon^2}{a\omega}F_2(a,\varphi)\cos\varphi+\varepsilon^3.. \tag{32}$$

Hence, the amplitude and phase differential equations (17) are ready defined where $\alpha(a,\varphi)$, $\beta(a,\varphi)$, $\mu(a,\varphi)$, $\gamma(a,\varphi)$ are given in (32) and (30). The averaged FP equation, written for probability density function $p(a,\theta,t)$ of system (17) takes the form

$$\frac{\partial p}{\partial t} = \varepsilon^2\Big\{\frac{\partial}{\partial a}(\langle\alpha\rangle p) + \frac{\partial}{\partial\theta}(\langle\mu\rangle p) - \frac{1}{2}\Big[\frac{\partial^2}{\partial a^2}(\langle\beta^2\rangle p) + \frac{2\partial^2}{\partial a\partial\varphi}(\langle\beta\gamma\rangle p) + \frac{\partial^2}{\partial\theta^2}(\langle\gamma^2\rangle p)\Big]\Big\} + \varepsilon^3\ldots \tag{33}$$

It is seen from (24) and (32) that the coefficients $\alpha(a,\varphi)$, $\mu(a,\varphi)$ contain the derivatives of the function $f_1(x,\dot{x})$. So, the effect of the nonlinear term $f_1(x,\dot{x})$ may be investigated by using the averaged FP equation (33) and also the expression of solution (15), (16).

4. SYSTEM WITH NONLINEAR DAMPING

The proposed extended first order stochastic averaging method will be used for investigating a system with nonlinear damping. We note also that since the first order averaging is used the terms of ε^n $(n \geq 3)$ will be neglected in the FP equation. Consider the nonlinear system described by the following equation

$$\ddot{x} + \omega^2 x = -\varepsilon\beta\dot{x}^2 - 2\varepsilon^2 h\dot{x} + \varepsilon\sigma\xi(t), \tag{34}$$

where $h, \beta > 0$. So one has the equation (1) in which

$$f_1(x,\dot{x}) = -\beta\dot{x}^2, \quad f_2(x,\dot{x}) = -2h\dot{x}. \tag{35}$$

It can be shown that the condition (13) is satisfied. Substituting (35) into (25) yields

$$u(a,\varphi) = -\frac{\beta}{2}a^2 - \frac{\beta}{6}a^2\cos 2\varphi. \tag{36}$$

Hence, using (15), we see that the solution of equation (34) takes the form

$$x = a\cos\varphi - \varepsilon\frac{\beta a^2}{2}\Big(1 + \frac{1}{3}\cos 2\varphi\Big). \tag{37}$$

Using (35), (36) one gets from (24)

$$F_2(a,\varphi) = -2ha\omega\sin\varphi + \frac{4}{3}\beta^2 a^3\omega^2\sin^2\varphi\cos\varphi. \tag{38}$$

Substituting (38) into (32) gives

$$\alpha(a,\varphi) = \varepsilon^2\Big\{\frac{\sigma^2}{2a\omega^2}\cos^2\varphi - 2ha\sin^2\varphi - \frac{2}{3}\beta^2 a^3\omega\sin^2\varphi\sin 2\varphi\Big\}. \quad (39)$$

So, one gets from (39) and (30)

$$\langle\alpha(a,\varphi)\rangle = \varepsilon^2\Big\{\frac{\sigma^2}{4a\omega^2} - ha\Big\}, \quad \langle\beta^2(a,\varphi)\rangle = \frac{\varepsilon^2\sigma^2}{2\omega^2}. \quad (40)$$

Substituting (40) into (33) yields

$$p(a) = \frac{4h\omega^2}{\sigma^2} a\exp\Big\{-\frac{2h\omega^2}{\sigma^2}a^2\Big\}. \quad (41)$$

From (41) one gets

$$E(a^2) = \frac{\sigma^2}{2h\omega^2}. \quad (42)$$

Using (37) and (42) yields the following mean value of the displacement

$$E(x) = -\varepsilon\frac{\beta}{2}E(a^2) + \varepsilon^2 .. = -\frac{\varepsilon\beta\sigma^2}{4h\omega^2} + \varepsilon^2 .. \quad (43)$$

In the absence of the nonlinear damping $-\varepsilon\beta\dot{x}^2$ (i.e. $\beta = 0$), one has $E(x) = 0$. So, the expression (43) implies that the nonlinear damping $-\varepsilon\beta\dot{x}^2$ reduces the mean value of the displacement.

Conclusion. The well-known stochastic averaging method has been extended to study a class of nonlinear systems, which can not be investigated by using the classical first order stochastic averaging.

Acknowledgement. Support from the fundamental research project in natural sciences is gratefully acknowledged.

REFERENCES

1. N. N. Bogoliubov and Iu. A. Mitropolskii, *Asymptotic methods in the theory of nonlinear oscillations*, Nauka, Moscow, 1961 (Russian).
2. R. Stratonovitch, *Topics in the theory of random noise*, v. 1, Gordon and Breach, New York, 1963.

3. Iu. A. Mitropolskii, *Averaging method in nonlinear mechanics*, Naukova-Dumka, Kiev, 1971 (Russian).

4. R. Khasminskii, *Averaging principle for the parabolic and eliptic differential equations and Markovian processes with small diffusion*, Theory Probability Appl., 8 (1963), 1-21.

5. Iu. A. Mitropolskii, N. V. Dao and N.D. Anh, *Nonlinear oscillations in systems of arbitrary order*, Naykova-Dumka, Kiev, 1992 (Russian).

Institute of Mechanics
Doi Can Str. 224
Hanoi, Vietnam

Received April 2, 1993