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# ON SOLUTION OF BOUNDARY-VALUE PROBLEMS FOR NONLINEAR HYPERBOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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**Abstract.** *In this paper, we study a boundary-value problem for nonlinear partial differential equations of hyperbolic type with discontinuous coefficients. The existence and uniqueness of solution of the problem is proved. This paper can be considered as an extension of the results in [1], [2].*

## 1. INTRODUCTION

In [1] we considered the following boundary-value problem (BVP):

$$A(t,x,y,D_t^iD_x^jD_y^\beta)U(t,x,y)=F(t,x,y), \tag{1.1}$$

$$D_t^iU(t,x,y)\big|_{t=0}=\psi_i(x,y),\quad i=1,2,\dots,m, \tag{1.2a}$$

$$\alpha'_{ij\beta}D_t^iD_x^jD_y^\beta U(t,x,y)\big|_{x=-0}=\alpha''_{ij\beta}D_t^iD_x^jD_y^\beta U(t,x,y)\big|_{x=+0}, \tag{1.3}$$

with $\quad i+j+|\beta|=m,\quad j=1,2,\dots,2\Big[\dfrac{m+1}{2}\Big]-1$

where $(t,x,y)\in G:=(0,T)\times\Omega\subset R^{n-1}$,

$$D_t^i:=\frac{\partial^i}{\partial t^i}\,,\quad D_x^j:=\frac{\partial^j}{\partial x^j}\,,\quad D_y^\beta:=D_{y_1}^{\beta_1}D_{y_2}^{\beta_2}\dots D_{y_{n-2}}^{\beta_{n-2}}\quad\text{and}$$

$$A(t,x,y,D_t^iD_x^jD_y^\beta):=\sum_{i+j+|\beta|\le m+1}a_{ij\beta(t,x,y)}D_t^iD_x^jD_y^\beta$$

is a partial differential operator of hyperbolic type with respect to $t$ whose coefficients are bounded for $i+j+|\beta|\le m+1$, piecewise sufficiently smooth for

$i + j + |\beta| = m + 1$, and $F(t,x,y) \in L^2(G)$. The coefficients $a_{ij\beta(t,x,y)}$ (with $i + j + |\beta| = m + 1$) are supposed to possess discontinuous points only on hyperplane $\{x = 0\}$, tend to 0 while $x$ tends to $\pm 0$ for the odd $j$ and satisfy the following conditions:

$$\frac{a''_{ij\beta}}{a'_{ij\beta}} = \frac{a''_{(m+1)00}}{a'_{(m+1)00}} \left\{ \frac{a''_{i2[(m+1)/2]\beta'}}{a'_{i2[(m+1)/2]\beta'}} \right\}^{j/2[(m+1)/2]} \tag{1.4}$$

for the even $j = 0, 2, 4, \ldots 2[(m+1)/2]$ where $a''_{ij\beta} = \lim a_{ij\beta(t,x,y)}$ and $a'_{ij\beta} = \lim a_{ij\beta(t,x,y)}$ while $x$ tends to $\pm 0$ respectively. In the case when $m$ is even we require that $a_{1m0} = a_{0m1}$. The coefficients $\alpha'_{ij\beta}$, $\alpha''_{ij\beta}$ in the condition (1.3) satisfy the relations below:

$$\frac{\alpha''_{ij\beta}}{\alpha'_{ij\beta}} = \left\{ \frac{a''_{(m+1)00}}{a'_{(m+1)00}} \right\}^{(-1)^j} \left\{ \frac{\alpha''_{m00}}{\alpha'_{m00}} \right\}^{(-1)^j} \frac{a''_{i2[(j+1)/2]\beta}}{a'_{i2[(j+1)/2]\beta}}. \tag{1.5}$$

In the particular case when the initial conditions (1.2a) is replaced by

$$D_t^i U(t,x,y)\big|_{t=0} = 0 \quad i = 1, 2, \ldots, m \tag{1.2b}$$

we proved the following energy inequality:

$$\big\|U(t,x,y)\big\|_{H^m(G)} \le C \big\|AU(t,x,y)\big\|_{L^2(G)} \tag{1.6}$$

In [2] for the problem (1.1), (1.2b), (1.3) (with conditions (1.4), (1.5)) we also proved the dual inequality. The energy inequality (1.6) and dual inequality ensure the existence and the uniqueness of solution of this problem for any right hand side $F(t,x,y) \in L^2(G)$ (for details about this linear problem we refer to [1], [2]).

## 2. BOUNDARY-VALUE PROBLEM FOR NONLINEAR EQUATIONS

In this section we consider a new problem by replacing equation (1.1) in the above problem (1.1), (1.2b), (1.3) (with the conditions (1.4), (1.5)) by the nonlinear equation below:

$$A(t,x,y,D_t^i D_x^j D_y^\beta)U(t,x,y) = F(t,x,y,D_t^i D_x^j D_y^\beta U). \tag{2.1}$$

For this nonlinear equation, the following theorem holds.

**Theorem 2.1.** *If $F(t,x,y,D_t^iD_x^jD_y^\beta U)$ is a real nonlinear bounded function of variables $t$, $x$, $y$, $D_t^iD_x^jD_y^\beta U$ (with $i+j+|\beta|\leq m-1$) and $F(t,x,y,D_t^iD_x^jD_y^\beta U)$ has first and second bounded derivatives with respect to variable $D_t^iD_x^jD_y^\beta U$ in $G$, then the solution of the BVP (2.1), (1.2b), (1.3) (with the conditions (1.4), (1.5)) exists and is unique.*

*Proof of Theorem 2.1.* For each $\tau\in[0,1)$ let us define the differential equation

$$\Phi_\tau(t,x,y,U)=A(t,x,y,D_t^iD_x^jD_y^\beta)U(t,x,y)-\tau F(t,x,y,D_t^iD_x^jD_y^\beta U)=0. \quad (2.2)$$

Consider the iterated Newton's type methods

$$\Big[A-\tau\Big(\sum_{i+j+|\beta|\leq m-1}F'_{ij\beta}(U_k)\Big)D_t^iD_x^jD_y^\beta\Big](U_{k+1}-U_k)=-\Phi_\tau(U_k), \quad (2.3a)$$

or

$$\Big[A-\tau\Big(\sum_{i+j+|\beta|\leq m-1}F'_{ij\beta}(U_k)\Big)D_t^iD_x^jD_y^\beta\Big](U_{k+1}-U_k)=-[AU_k-\tau F(U_k)], \quad (2.3b)$$

where

$$F'_{ij\beta}(U_k):=\frac{\partial F}{\partial D_t^iD_x^jD_y^\beta U}(U_k).$$

The relation (2.3b) is equivalent to

$$\Big[A-\tau\Big(\sum_{i+j+|\beta|\leq m-1}F'_{ij\beta}(U_k)\Big)D_t^iD_x^jD_y^\beta\Big]U_{k+1}$$
$$=\tau F(U_k)-\tau\Big(\sum_{i+j+|\beta|\leq m-1}F'_{ij\beta}(U_k)\Big)D_t^iD_x^jD_y^\beta U_k. \quad (2.3c)$$

For completing the proof of Theorem 2.1, we need the following lemma.

**Lemma 2.1.** *Assume the conditions of Theorem 2.1 and let suppose that there exists $\tau_0\in[0,1)$ such that the problem (2.2), (1.2b), (1.3) possesses a solution $U_0$, then the equation $\Phi_\tau(t,x,y,U)=0$ possesses an unique solution in $H^m(G)$ for some $\tau\in(\tau_0,1)$.*

*Proof of Lemma 2.1.* Replacing $k$ by $k-1$ in (2.3c) we get

$$AU_k=\tau\Big(\sum_{i+j+|\beta|\leq m-1}F'_{ij\beta}(U_{k-1})\Big)D_t^iD_x^jD_y^\beta U_k+\tau F(U_{k-1})$$
$$-\tau\Big(\sum_{i+j+|\beta|\leq m-1}F'_{ij\beta}(U_{k-1})\Big)D_t^iD_x^jD_y^\beta U_{k-1}. \quad (2.4)$$

Substituting $AU_k$ from (2.4) in (2.3b) yields

$$\Big[A-\tau\Big(\sum_{i+j+|\beta|\le m-1} F'_{ij\beta}(U_k)\Big)D_t^iD_x^jD_y^\beta\Big](U_{k+1}-U_k)=\tau\big(F(U_k)-F(U_{k-1})\big)$$
$$-\tau\Big(\sum_{i+j+|\beta|\le m-1} F'_{ij\beta}(U_{k-1})\Big)D_t^iD_x^jD_y^\beta(U_k-U_{k-1}). \qquad (2.5)$$

By setting $W_k=U_{k+1}-U_k$ and $L_k=\Big[A-\tau\Big(\sum_{i+j+|\beta|\le m-1} F'_{ij\beta}(U_k)\Big)D_t^iD_x^jD_y^\beta\Big]$, the relation (2.5) takes the form:

$$L_kW_k=\tau g_k \qquad (2.6)$$

with

$$g_k=F(U_k)-F(U_{k-1})-\Big(\sum_{i+j+|\beta|\le m-1} F'_{ij\beta}(U_{k-1})\Big)D_t^iD_x^jD_y^\beta(U_k-U_{k-1}).$$

It is evident that the differential operator $L_k$ defined as above is a linear operator of hyperbolic type whose coefficients satisfy all the conditions as required for the coefficients of $A$ (in particular the conditions (1.4), (1.5)). Therefore the results reported in Section 1 for the problem (1.1), (1.2b), (1.3) can be apply to the problem (2.6), (1.2b), (1.3). Using the energy inequality (1.6), we obtain

$$\|W_k\|_{H^m(G)}\le \tau C\|g_k\|_{L^2(G)}. \qquad (2.7)$$

Since

$$\|g_k\|_{L^2(G)}=\Big\|F(U_k)-F(U_{k-1})$$
$$-\Big(\sum_{i+j+|\beta|\le m-1} F'_{ij\beta}(U_{k-1})\Big)D_t^iD_x^jD^\beta(U_k-U_{k-1})\Big\|_{L^2(G)}$$
$$\le\frac{1}{2}\Gamma\|W_{k-1}\|^2_{H^{m-1}(G)}\le\frac{1}{2}\Gamma\|W_{k-1}\|^2_{H^m(G)},$$

we get from (2.7) the following inequality

$$\|W_k\|_{H^m(G)}\le\frac{\Gamma\tau C}{2}\|W_{k-1}\|^2_{H^m(G)}. \qquad (2.8a)$$

(Here we recall that the function $F$ and its first and second derivatives are supposed to be bounded).

We rewrite (2.8) in another form:

$$\frac{\Gamma\tau C}{2}\|W_k\|_{H^m(G)} \le \left[\frac{\Gamma\tau C}{2}\|W_{k-1}\|_{H^m(G)}\right]^2. \tag{2.8b}$$

Thus

$$\frac{\Gamma\tau C}{2}\|W_n\|_{H^m(G)} \le \left[\frac{\Gamma\tau C}{2}\|W_0\|_{H^m(G)}\right]^{2^n}. \tag{2.9}$$

Now we estimate $\|W_0\|_{H^m(G)}$. In the equality (2.3b) by setting $k=0$, we get

$$\left[A-\tau\Big(\sum_{i+j+|\beta|\le m-1} F'_{ij\beta}(U_0)\Big)D_t^iD_x^jD_y^\beta\right](U_1-U_0) = -[AU_0-\tau F(U_0)]. \tag{2.3d}$$

On the other hand,

$$\begin{aligned}\|W_0\|_{H^m(G)} &\le C\|AU_0-\tau F(U_0)\|_{L^2(G)}\\ &\le C\|AU_0-\tau_0F(U_0)+(\tau_0-\tau)F(U_0)\|_{L^2(G)},\end{aligned} \tag{2.10}$$

and hence, since $AU_0-\tau_0F(U_0)=0$, we have

$$\|W_0\|_{H^m(G)} \le C(\tau-\tau_0)\|F(U_0)\|_{L^2(G)}. \tag{2.11}$$

From the inequalities (2.11) and (2.9) it follows

$$\frac{\Gamma\tau C}{2}\|W_n\|_{H^m(G)} \le \left[\frac{\Gamma C^2}{2}(\tau-\tau_0)\|F(U_0)\|_{L^2(G)}\right]^{2^n}. \tag{2.12}$$

By choosing $\tau$ closed enough to $\tau_0$ one have

$$\frac{\Gamma C^2}{2}(\tau-\tau_0)\|F(U_0)\|_{L^2(G)} \le 1. \tag{2.13}$$

The inequalities (2.12) and (2.13) prove Lemma 2.1. □

Finally, observing that the problems (2.2), (1.2b), (1.3) possesses the solution for $\tau=0$ (see [1], [2]) and applying parameter continuation method, Theorem 2.1 easily follows. □

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