Journal of Mathematics

Tap chí Toán học 21(1 & 2)(1993), 55-61

where $u_{n+1} \in X_1$, $v_{n+1} \in X_2$.

 b ns (1) (2)

Note that the operator equation (1) has been investigated by many authors (see $[1, 2, 3, 4]$ for instance). By using the degree-theory, we can obtain existence theorems for the equation (1) (see [1, 2]). This equation may be solved by projection methods or by a special iterative method (see [3, 4]).

ON AN APPROXIMATION METHOD OWNER aw .(navig ad of FOR SOLVING QUASILINEAR mixorqqs dt-st construct the (n + OPERATOR EQUATIONS

 $\hat{A}u_{n+1} + PF'(x_n)u_{n+1} = PF(x_n)u_n - PF_{x_n}$

NGUYEN VAN KHAL

 $(i = 1, 2)$ and $[QF'(x)]_{X_2}$ a restriction of the derivative $QF'(x)$ to X_2 .

 $QF(u_{n+1}+v_{n+1})=0.$

Abstract. In this paper the author presents an approximation method for solving quasilinear operator equation $Ax + Fx = 0$ where A is a bounded linear Fredholm operator and F is a nonlinear operator. Convergence theorems and theorem of rate of convergence of an approximation method are proved. Some numerical examples are given. Let us denote $f(x)$ a restriction of the derivative $P F'(x)$ to X_i

Theorem 2.1. Let F be continuously differentiable (in the Fréchet sense) in an

beonuoddgien nego In this paper we consider the following operator equation:

$$
A \geq Ax + Fx = 0, \qquad \text{and} \qquad (1)
$$

where A is a bounded linear Fredholm operator (index zero), F is a nonlinear operator and A, $F: X \to Y$; X, Y are Banach spaces. It is well known that by the assumption of A, we have: $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$, $X_2 = \text{Ker}A$, $Y_1 =$ ImA, $\dim X_2 = \text{codim} Y_1 = m < +\infty$, Y_1 is closed in Y and the restriction \hat{A} of A to X_1 has a bounded inverse.

Let us denote by P a bounded linear projection from Y on Y_1 , $PY = Y_1$, $Q = (I - P)$, where I is a unit operator in Y. Then the equation (1) is equivalent to system: $A(p-1) > ||x+1|| + ||x+1 + c||$

then the sequence {xn}, constructed by (3) converges to a solution z^* of equation $\hat{A}u + PF(u + v) = 0,$ $QF(u + v) = 0.$

 $||x_n - x^*|| < Ra^n$.

where $u \in X_1, v \in X_2$.

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Note that the operator equation (1) has been investigated by many authors (see $\begin{bmatrix} 1, 2, 3, 4 \end{bmatrix}$ for instance). By using the degree theory, we can obtain existence theorems for the equation (1) (see $[1, 2]$). This equation may be solved by projection methods or by a special iterative method (see [3, a]).

We shall solve (1) by the following approximation method: knowing the $n-th$ approximation x_n (the 0-th approximation x_0 is supposed to be given), we construct the $(n + 1)$ -th approximation by the formulae:

$$
\hat{A}u_{n+1} + PF'(x_n)u_{n+1} = PF'(x_n)u_n - PFx_n,
$$
\n(3a)

$$
QF(u_{n+1} + v_{n+1}) = 0, \tag{3b}
$$

$$
x_{n+1} = u_{n+1} + v_{n+1} \qquad \text{MUMOM}
$$

where $u_{n+1} \in X_1$, $v_{n+1} \in X_2$.

 (1) .

TOSCIFICALLY 2. CONVERGENCE THEOREMS AND THEOREM STREAM The substant of **ON THE RATE OF CONVERGENCE**

Let us denote by $[PF'(x)]_{X_i}$ a restriction of the derivative $PF'(x)$ to X_i $(i = 1,2)$ and $[QF'(x)]_{X_2}$ a restriction of the derivative $QF'(x)$ to X_2 .

Theorem 2.1. Let F be continuously differentiable (in the Fréchet sense) in an open neighbourhood

$$
\Omega = \{x \in X : ||x - x_0|| < R\},\
$$
\n
$$
\text{and} \quad \left\| [PF'x]_{X_2} \right\| \leq \alpha, \quad \left\| QF'x \right\| \leq \beta \quad \text{for all } x \in \Omega.
$$

Assume that $[QF'x]_{X_2}$ has a uniformly bounded inverse $\left\| [QF'x]_{X_2}^{-1} \right\| \leq \gamma$ and the restriction $[A+PF'(x)]_{X_1}$ of $[A+PF'(x)]$ to X_1 has a uniformly bounded inverse $\|[A + PF'(x)]_{X_1}^{-1}\| \leq M$ for all $x \in \Omega$. X svsd sw A lo notiquuses sub

Example 1 Furthermore, assume that $||PF'x - PF'y|| \le L||x - y||$ for all $x, y \in \Omega$.

$$
N = \sqrt{H} q_0 = M \big[\alpha \beta \gamma + \frac{L}{2} (1 + \gamma \beta) \delta_0 \big] < 1, \text{ mod } 4 \text{ odd such and } (4)
$$

and solving
$$
\delta_0 = (1 + \gamma \beta) M \| A x_0 + P F x_0 \| + \gamma \| Q F x_0 \| < (1 - q_0) R, \tag{5}
$$

then the sequence $\{x_n\}$, constructed by (3) converges to a solution x^* of equation (1) , and $Au + PF(u + v) = 0,$

 $QF(u + v) = 0.$

$$
||x_n-x^*|| \leq Rq^n.
$$

On an approximation method...

Proof.

 $\overline{6}$

 (15)

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We may choose $t > 1$ such that $\pi^{(1)} = + ||_{1-\pi^{(1)}} ||_{2} \times ||_{\pi^{(1)}}$

$$
q = M\big[\alpha t \gamma \beta + \frac{L}{2}(1 + t \gamma \beta)\delta\big] < 1, \qquad 0 \leq s \leq 10 \text{ and } \text{ and } \tag{7}
$$

$$
\delta = M(1 + t\gamma\beta) ||Ax_0 + PFx_0|| + t\gamma ||QFx_0|| < (1 - q)R.
$$
 (8)

For $n \geq 0$ let us define $x_n = u_n + v_n$, $\overline{x}_n = u_{n+1} + v_n$, and $0 < ||(\sqrt{x})||$

$$
\Omega_n = \{x \in X : ||x - \overline{x}_n|| \le r_n\}, \quad S_n = \{v \in X_2 : ||v - v_n|| \le r_n\},\newline \lambda_n = u_{n+1} - u_n; \quad \mu_n = v_{n+1} - v_n; \quad r_n = t\gamma||QF\overline{x}_n||;
$$
\n
$$
G(v) = QF(u_{n+1} + v). \Box ||v| = ||\Box u - ||\Box u||
$$

We first observe that $u_{n+1} + S_n \subset \Omega_n$ and v_{n+1} and v_{n+1} and v_{n+1} is true for $\|\mu_n\| = r_n = 0$.

$$
G'_n(v) = [QF'(u_{n+1} + v)]_{X_2}
$$

so that u is not a non-
sond, $l = n$ not out or
the will show by mathematical induction for *n* the following relations:
the solution is:

$$
x_{n} \in \Omega, \quad n \ge 0,
$$
\n
$$
\|\lambda_{0}\| \le M\|Ax_{0} + PFx_{0}\|,
$$
\n
$$
\|\lambda_{n}\| \le M\alpha\|\mu_{n-1}\| + \frac{L}{2}(\|\lambda_{n-1}\| + \|\mu_{n-1}\|)\|\lambda_{n-1}\|, \quad n \ge 1,
$$
\n
$$
\|\lambda_{n}\| \le q\|\lambda_{n-1}\|, \quad n \ge 2,
$$
\n
$$
\overline{x}_{n} \in \Omega, \quad n \ge 0,
$$
\n
$$
\|QF\overline{x}_{0}\| \le \beta\|\lambda_{0}\| + \|QFx_{0}\|, \quad 1 \ge 1,
$$
\n
$$
\|\alpha_{n} \in \Omega, \quad n \ge 0,
$$
\n
$$
\|QF\overline{x}_{n}\| \le \beta\|\lambda_{n}\|, \quad n \ge 1,
$$
\n
$$
\Omega_{n} \subset \Omega, \quad n \ge 0.
$$
\n(14)

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It can easily be seen that x^* is a solution of the equation (1) . The estimation (6) is true.

Indeed, assume that for $n \geq 1$ such that $x_n \in \Omega$, then

omuses notitibbs $n! ||\lambda_n|| \equiv ||u_{n+1} - u_n|| \leq M ||(A + PF)x_n||$ with $0 = \frac{1}{\infty} [(\pi)^{n+1}]$ test not only $|u|| \leq ||x||$ or the politicial considers λ or u and $\lambda \ni u$ and 0 and $\lambda \ni u$ and 0 and $\lambda \ni u$ and 0 and $\lambda \ni u$ and constructed by (3), converges to x^{*} at quadratic rate:

$$
\|\lambda_n\| \leq M \Big\|\int\limits_0^1 PF' \big[x_{n-1} + t(x_n - x_{n-1})\big] (x_n - x_{n-1}) dt - PF' x_{n-1} (u_n - u_{n-1}) \Big\|,
$$

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hence

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Proof.

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hence

 (T)

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$$
|\lambda_n|| \leq M [\alpha ||\mu_{n-1}|| + \frac{L}{2} (||\lambda_{n-1}|| + ||\mu_{n-1}||) ||\lambda_{n-1}||] \cdot_{(S \cap \mathbb{Z})}
$$

Assume that for a $n \geq 0$, (9)-(14) are true. Then $(u_{n+1} + v) \in \Omega_n \subset \Omega$ and

$$
\mathbb{E}(\mathbb{Q} - 1) > \|G'_n(v)^{-1}\| \leq \gamma \quad \text{for all } v \in S_n, \, i + 1\}.
$$

If
$$
||QF(\overline{x}_n)|| > 0
$$
 then $r_n = t\gamma ||QF\overline{x}_n|| > \gamma ||G_n(v_n)||$ for $t > 1$.

It follows readily from the theorem of Hadamard of local topological isomorphism (see [5], p.139) that there exists $v_{n+1} \in S_n$ so that $G_n(v_{n+1}) = 0$, $\|u_{n+1} - u_n\|$ $\mu_n = v_{n+1} - v_n$; $r_n = t\gamma \|QF\|$ and¹

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 $||\lambda_n|| \leq q||\lambda_{n-1}||, \quad n \geq 2,$

 $\bar{x}_n \in \Omega$, $n \geq 0$.

 $||QF\bar{x}_n|| < \beta ||\lambda_n||$

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If $QF(\overline{x}_n) = 0$, we take $v_{n+1} = v_n$. Then $G_n(v_{n+1}) = 0$ and condition (15) is true for $\|\mu_n\| = r_n = 0$.

It can be verified that (9)-(14) are true for $n=0$, hence there exists $x_1=$ $u_1 + v_1$ by (3). Direct computation shows that (9)-(14) are true for $n = 1$, hence there exists x_2 such that the condition (15) is true, soits mediant vd works lliw eW

The conditions (9)-(15) are proved by mathematical induction for $n \geq 2$. Assume that there exists $x_n = u_n + v_n$, then $x_n \in \Omega$, $n \geq 0$,

$$
||u_{n+1}-u_n|| = ||\lambda_n|| \leq M [\alpha t \gamma \beta + \frac{L}{2}(1+t \beta \gamma)] ||\lambda_{n-1}|| = q ||\lambda_{n-1}||.
$$

Hence $\|\lambda_n\| \leq q^{n-1} \|\lambda_1\|$

$$
||v_{n+1}-v_n|| = ||\mu_n|| \le t \gamma \beta ||\lambda_n|| \le t \gamma q^{n-1} ||\lambda_1||.
$$

 (12)

Therefore
$$
u_n \to u^*
$$
, $v_n \to v^*$, $x_n = u_n + v_n \to x^* = u^* + v^*$.

It can easily be seen that x^* is a solution of the equation (1). The estimation (6) is true. Indeed, assume that for $n \geq 1$ such that $x_n \in \Omega$, then

Theorem 2.2. Assume that hypotheses of Theorem 2.1 hold. In addition assume that $[PF'(x)]_{X_2} = 0$ and the norm in X satisfies the inequality $C||x|| \ge ||u||$ for $x = u + v$, $u \in X_1$, $v \in X_2$ and $C > 0$ is a constant, then the sequence (x_n) , constructed by (3) , converges to x^* at quadratic rate:

 $||x_{n+1}-x_n|| \leq h ||x_n-x_{n-1}||^2$.
 $||x_{n+1}-x_n|| \leq h ||x_n-x_{n-1}||^2$.

On an approximation method...

Proof.

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For $n\geq 2$ we have $||x_{n+1} - x_n|| \leq q(1 + t\gamma\beta)||\lambda_n||$. Since $[PF'(x)]_{X_2} = 0$ it follows that $\alpha = 0$; hence

$$
(-\frac{1}{2}x_{n})\frac{\left\Vert \lambda_{n}\right\Vert }{2}\leq M\frac{L}{2}(1+t\gamma\beta)\left\Vert \lambda_{n-1}\right\Vert _{2\leq n(0,1)}=\frac{1}{2}x=\frac{1}{2}x
$$

Therefore $||x_{n+1} - x_n|| \leq h ||x_n - x_{n-1}||^2$ where $h = C^2 M \frac{L}{2} (1 + t \gamma \beta)^2$. If $(1 + t \gamma \beta)^2$

Remark: By our algorithm we can get a quadratic convergence rate different from [4] in which the rate is linear. an aniwollot eds missed ow 8.8 metod T gniaU

Theorem 2.3. Let F be continuously differentiable in an open neighbourhood of a solution x^* of (1). Assume that $QF'(x^*)_{X_2}$ has a bounded inverse and the restriction $[A + PF'x^*]_{X}$, of $[A + PF'x^*]$ to X_1 has a bounded inverse and

$$
\| [A + PF'(x^*)]_{X_1}^{-1} \| \| [PF'x^*]_{X_2} \| \| QF'x^* \| \| \| [QF'x^*]_{X_2}^{-1} \| < 1. \tag{17}
$$

If the initial approximation x_0 is sufficiently close to x^* , then the sequence $\{x_n\}$, constructed by (3), converges to a solution of (1).
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$\left(1 - \frac{3. \text{ AN EXAMPLE}}{3} - (1 - \frac{\pi}{3})\epsilon + \frac{\pi^3}{3} \text{ niz}\right)$

Consider the following periodic boundary-value problem:
 $(0)_{1+n}$ = $(0)_{1+n}$

$$
\ddot{x} + \sin \frac{1}{4}x + 5(e^{\ddot{x}} - 1) = 0, \quad \text{Hence } \text{This is}
$$
\n
$$
x(0) = x(1),
$$
\n
$$
\dot{x}(0) = \dot{x}(1).
$$
\n
$$
(18)
$$

converges to a solution of (18).

The problem (18) may be reduced to the form (1) by introducing the following spaces and operators:

$$
X = \left\{ x \in C^2[0,1]: \ x(0) = x(1), \ \dot{x}(0) = \dot{x}(1) \right\}
$$

$$
Y = C[0,1], \ ||x||_X = \max_{0 \le t \le 1} |x(t)| + \max_{0 \le t \le 1} |\dot{x}(t)| + \max_{0 \le t \le 1} |\ddot{x}(t)|, \ \text{as given}
$$

 $||y||_Y = max ||y||$

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$$
||y||_Y = \max_{0 \le t \le 1} |y(t)|
$$

\n
$$
X_1 = \left\{ x \in X : \int_0^t x(t)dt = 0 \right\}, \quad Y_1 = \left\{ y \in Y : \int_0^t y(t)dt = 0 \right\}.
$$

\n
$$
X_2 = Y_2 = \left\{ \text{const} \right\}, \quad Ax = \ddot{x}, \quad Fx = \left(\sin \frac{1}{4}x + 5(e^{\ddot{x}} - 1) \right).
$$

It can be verified that problem (18) has the solution $x^* \equiv x^*(t) \equiv 0$, $\forall t \in [0,1]$, $[A + PF'(0)]_{X_1}$ has a bounded inverse, $[QF'(0)]_{X_2}$ has a bounded inverse and $||[A + PF'(0)]_{X_1}^{-1}|| \leq \frac{50}{11}$, $||[QF'(0)]_{X_2}^{-1}|| \leq 4$, $||[PF'(0)]_{X_2}|| = 0$ is two \forall is the set

Using Theorem 2.3 we obtain the following result.^{Il at} size add dointw ni [4]

Theorem 3.1. If the initial approximation x_0 is sufficiently close to $x^* \equiv x^*(t) \equiv$ $0, \forall t \in [0,1],$ then the sequence $\{x_n\}$, constructed by the formulae:

$$
\ddot{u}_{n+1} + \frac{u_{n+1}}{4} \cos \frac{x_n}{4} + \ddot{u}_{n+1.5e} \ddot{x}_n - \int \left(\frac{u_{n+1}}{4} \cos \frac{x_n}{4} + \ddot{u}_{n+1.5e} \ddot{x}_n \right) dt
$$
\n
$$
= \frac{u_n}{4} \cos \frac{x_n}{4} + \ddot{u}_n.5e^{\ddot{x}_n} - \int \left(\frac{u_n}{4} \cos \frac{x_n}{4} + \ddot{u}_n.5e^{\ddot{x}_n} \right) dt
$$
\n
$$
- \left[\sin \frac{x_n}{4} + 5(e^{\ddot{x}_n} - 1) - \int \left(\sin \frac{x_n}{4} + 5(e^{\ddot{x}_n} - 1) \right) dt \right],
$$
\n
$$
u_{n+1}(0) = u_{n+1}(1), \quad \dot{u}_{n+1}(0) = \dot{u}_{n+1}(1),
$$
\n
$$
\int_0^1 \sin \frac{u_{n+1} + v_{n+1}}{4} + 5(e^{\ddot{u}_{n+1}} - 1) dt = 0,
$$
\n
$$
x_{n+1}(t) = u_{n+1}(t) + v_{n+1}
$$
\n
$$
(1) = (0) \ddot{x}
$$

converges to a solution of (18) .

It is to be noticed that the approximation method developed by Pham Ky Anh in [4] for problem (18) at the solution $x^* = x^*(t) = 0$ is not applicable.

Aknowledgement. I express my deepest gratitude to Prof. Nguyen Minh Chuong and Prof. Le Dinh Thinh for their help and encouragement.

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On an approximation method...

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Abstract. Let (Φ, Ψ) be a pair of Young's functions and (φ, ψ) their density functions

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Let $\{X_n\}$ be an integrable martingale. Then, the following maximal inequality holds:

$$
E\left[\xi\left(\sup_{1\leq k\leq n}|X_k|/\rho\|X_n\|_\Phi\right)\right]\leq \frac{1}{\rho-1}
$$

where $\rho > 1$ is a constant and $\|\cdot\|_{\Phi}$ denotes the Luzemburg's norm.

1. INTRODUCTION

Throughout this paper we shall work with a fixed probability space (0, A, P) and a sequence of σ -fields $B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots \subset A$, such that $\bigvee_{n=1}^{\infty} B_n =$ $B_{\rm ion} \subset A$.

The martingales or submartingales are always supposed to be adapted to the sequence of σ -fields $\{\beta_n\}_{n\in\mathbb{N}}$.

For a pair of Young's conjugate functions (Φ, Ψ) , we shall denote by φ (resp. ψ) the density function of Φ (resp. Ψ), and we shall set by ξ the increasing function from R + to R + defined by

$$
\xi(u) = u\varphi(u) - \Phi(u) = \Psi[\varphi(u)] \tag{1}
$$

For a sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ we shall set