Journal of Mathematics *Tạp chí Toán học* **21**(1 & 2)(1993), 55–61

ON AN APPROXIMATION METHOD FOR SOLVING QUASILINEAR OPERATOR EQUATIONS

NGUYEN VAN KHAI

Abstract. *In this paper the author presents an approximation method for solving quasilinear operator equation $Ax + Fx = 0$ where A is a bounded linear Fredholm operator and F is a nonlinear operator. Convergence theorems and theorem of rate of convergence of an approximation method are proved. Some numerical examples are given.*

1. INTRODUCTION

In this paper we consider the following operator equation:

$$Ax + Fx = 0, \tag{1}$$

where A is a bounded linear Fredholm operator (index zero), F is a nonlinear operator and $A, F : X \to Y$; X, Y are Banach spaces. It is well known that by the assumption of A, we have: $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$, $X_2 = \mathrm{Ker}A$, $Y_1 = \mathrm{Im}A$, $\dim X_2 = \mathrm{codim} Y_1 = m < +\infty$, Y_1 is closed in Y and the restriction $\hat{A}$ of A to X_1 has a bounded inverse.

Let us denote by P a bounded linear projection from Y on Y_1, $PY = Y_1$, $Q = (I - P)$, where I is a unit operator in Y. Then the equation (1) is equivalent to system:

$$\begin{aligned} \hat{A}u + PF(u+v) &= 0, \\ QF(u+v) &= 0. \end{aligned} \tag{2}$$

where $u \in X_1$, $v \in X_2$.

Note that the operator equation (1) has been investigated by many authors (see [1, 2, 3, 4] for instance). By using the degree theory, we can obtain existence theorems for the equation (1) (see [1, 2]). This equation may be solved by projection methods or by a special iterative method (see [3, 4]).

We shall solve (1) by the following approximation method: knowing the $n-$th approximation x_n (the 0-th approximation x_0 is supposed to be given), we construct the $(n+1)-$th approximation by the formulae:

$$\hat{A}u_{n+1} + PF'(x_n)u_{n+1} = PF'(x_n)u_n - PFx_n, \tag{3a}$$

$$QF(u_{n+1} + v_{n+1}) = 0, \tag{3b}$$

$$x_{n+1} = u_{n+1} + v_{n+1}$$

where $u_{n+1} \in X_1$, $v_{n+1} \in X_2$.

2. CONVERGENCE THEOREMS AND THEOREM ON THE RATE OF CONVERGENCE

Let us denote by $[PF'(x)]_{X_i}$ a restriction of the derivative $PF'(x)$ to X_i $(i = 1, 2)$ and $[QF'(x)]_{X_2}$ a restriction of the derivative $QF'(x)$ to X_2.

Theorem 2.1. *Let F be continuously differentiable (in the Fréchet sense) in an open neighbourhood*

$$\Omega = \{x \in X \ : \ \|x - x_0\| < R\},$$

$$\text{and} \quad \|[PF'x]_{X_2}\| \le \alpha, \quad \|QF'x\| \le \beta \quad \text{for all } x \in \Omega.$$

Assume that $[QF'x]_{X_2}$ has a uniformly bounded inverse $\|[QF'x]^{-1}_{X_2}\| \le \gamma$ and the restriction $[A + PF'(x)]_{X_1}$ of $[A + PF'(x)]$ to X_1 has a uniformly bounded inverse $\|[A + PF'(x)]^{-1}_{X_1}\| \le M$ for all $x \in \Omega$.

Furthermore, assume that $\|PF'x - PF'y\| \le L\|x - y\|$ for all $x, y \in \Omega$.

$$\text{If } q_0 = M\big[\alpha\beta\gamma + \frac{L}{2}(1 + \gamma\beta)\delta_0\big] < 1, \tag{4}$$

$$\delta_0 = (1 + \gamma\beta)M\|Ax_0 + PFx_0\| + \gamma\|QFx_0\| < (1 - q_0)R, \tag{5}$$

then the sequence $\{x_n\}$, constructed by (3) converges to a solution x^* of equation (1), and

$$\|x_n - x^*\| \le Rq^n. \tag{6}$$

Proof.

We may choose $t > 1$ such that

$$q = M\big[\alpha t\gamma\beta + \frac{L}{2}(1 + t\gamma\beta)\delta\big] < 1, \tag{7}$$

$$\delta = M(1 + t\gamma\beta)\|Ax_0 + PFx_0\| + t\gamma\|QFx_0\| < (1 - q)R. \tag{8}$$

For $n \geq 0$ let us define $x_n = u_n + v_n$, $\overline{x}_n = u_{n+1} + v_n$,

$$\Omega_n = \{x \in X \ : \ \|x - \overline{x}_n\| \leq r_n\}, \quad S_n = \{v \in X_2 \ : \ \|v - v_n\| \leq r_n\},$$
$$\lambda_n = u_{n+1} - u_n; \quad \mu_n = v_{n+1} - v_n; \quad r_n = t\gamma\|QF\overline{x}_n\|;$$
$$G(v) = QF(u_{n+1} + v).$$

We first observe that $u_{n+1} + S_n \subset \Omega_n$ and

$$G_n'(v) = \big[QF'(u_{n+1} + v)\big]_{X_2}.$$

We will show by mathematical induction for n the following relations:

$$x_n \in \Omega, \quad n \geq 0, \tag{9}$$
$$\|\lambda_0\| \leq M\|Ax_0 + PFx_0\|, \tag{10}$$
$$\|\lambda_n\| \leq M\big[\alpha\|\mu_{n-1}\| + \frac{L}{2}\big(\|\lambda_{n-1}\| + \|\mu_{n-1}\|\big)\|\lambda_{n-1}\|\big], \ n \geq 1,$$
$$\|\lambda_n\| \leq q\|\lambda_{n-1}\|, \quad n \geq 2, \tag{11}$$
$$\overline{x}_n \in \Omega, \quad n \geq 0, \tag{12}$$
$$\|QF\overline{x}_0\| \leq \beta\|\lambda_0\| + \|QFx_0\|, \tag{13}$$
$$\|QF\overline{x}_n\| \leq \beta\|\lambda_n\|, \quad n \geq 1, \tag{14}$$
$$\Omega_n \subset \Omega, \quad n \geq 0.$$

Indeed, assume that for $n \geq 1$ such that $x_n \in \Omega$, then

$$\|\lambda_n\| = \|u_{n+1} - u_n\| \leq M\|(A + PF)x_n\|.$$

Therefore

$$\|\lambda_n\| \leq M\Big\|\int_0^1 PF'\big[x_{n-1} + t(x_n - x_{n-1})\big](x_n - x_{n-1})dt - PF'x_{n-1}(u_n - u_{n-1})\Big\|,$$

hence

$$\|\lambda_n\| \leq M\big[\alpha\|\mu_{n-1}\| + \frac{L}{2}(\|\lambda_{n-1}\| + \|\mu_{n-1}\|)\|\lambda_{n-1}\|\big].$$

Assume that for a $n \geq 0$, (9)-(14) are true. Then $(u_{n+1} + v) \in \Omega_n \subset \Omega$ and

$$\|G_n'(v)^{-1}\| \leq \gamma \quad \text{for all } v \in S_n.$$

If $\|QF(\overline{x}_n)\| > 0$ then $r_n = t\gamma\|QF\overline{x}_n\| > \gamma\|G_n(v_n)\|$ for $t > 1$.

It follows readily from the theorem of Hadamard of local topological isomorphism (see [5], p.139) that there exists $v_{n+1} \in S_n$ so that $G_n(v_{n+1}) = 0$, and

$$\|\mu_n\| = \|v_{n+1} - v_n\| \leq r_n = t\gamma\|QF(\overline{x}_n)\|. \tag{15}$$

If $QF(\overline{x}_n) = 0$, we take $v_{n+1} = v_n$. Then $G_n(v_{n+1}) = 0$ and condition (15) is true for $\|\mu_n\| = r_n = 0$.

It can be verified that (9)-(14) are true for $n = 0$, hence there exists $x_1 = u_1 + v_1$ by (3). Direct computation shows that (9)-(14) are true for $n = 1$, hence there exists x_2 such that the condition (15) is true.

The conditions (9)-(15) are proved by mathematical induction for $n \geq 2$. Assume that there exists $x_n = u_n + v_n$, then

$$\|u_{n+1} - u_n\| = \|\lambda_n\| \leq M\big[\alpha t\gamma\beta + \frac{L}{2}(1 + t\beta\gamma)\big]\|\lambda_{n-1}\| = q\|\lambda_{n-1}\|.$$

Hence $\|\lambda_n\| \leq q^{n-1}\|\lambda_1\|$

$$\|v_{n+1} - v_n\| = \|\mu_n\| \leq t\gamma\beta\|\lambda_n\| \leq t\gamma q^{n-1}\|\lambda_1\|.$$

Therefore $u_n \to u^*$, $v_n \to v^*$, $x_n = u_n + v_n \to x^* = u^* + v^*$.

It can easily be seen that x^* is a solution of the equation (1). The estimation (6) is true.

Theorem 2.2. *Assume that hypotheses of Theorem 2.1 hold. In addition assume that* $[PF'(x)]_{X_2} = 0$ *and the norm in* X *satisfies the inequality* $C\|x\| \geq \|u\|$ *for* $x = u + v$, $u \in X_1$, $v \in X_2$ *and* $C > 0$ *is a constant, then the sequence* (x_n), *constructed by (3), converges to* x^* *at quadratic rate:*

$$\|x_{n+1} - x_n\| \leq h\|x_n - x_{n-1}\|^2. \tag{16}$$

Proof.

For $n \geq 2$ we have $\|x_{n+1} - x_n\| \leq q(1 + t\gamma\beta)\|\lambda_n\|$. Since $\left[PF'(x)\right]_{X_2} = 0$, it follows that $\alpha = 0$; hence

$$\|\lambda_n\| \leq M\frac{L}{2}(1 + t\gamma\beta)\|\lambda_{n-1}\|.$$

Therefore $\|x_{n+1} - x_n\| \leq h\|x_n - x_{n-1}\|^2$ where $h = C^2 M\frac{L}{2}(1 + t\gamma\beta)^2$.

Remark: By our algorithm we can get a quadratic convergence rate different from [4] in which the rate is linear.

Theorem 2.3. *Let F be continuously differentiable in an open neighbourhood of a solution x^* of (1). Assume that $QF'(x^*)_{X_2}$ has a bounded inverse and the restriction $\left[A + PF'x^*\right]_{X_1}$ of $\left[A + PF'x^*\right]$ to X_1 has a bounded inverse and*

$$\|[A + PF'(x^*)]_{X_1}^{-1}\| \; \|[PF'x^*]_{X_2}\| \; \|QF'x^*\| \; \|[QF'x^*]_{X_2}^{-1}\| < 1. \tag{17}$$

If the initial approximation x_0 is sufficiently close to x^*, then the sequence $\{x_n\}$, constructed by (3), converges to a solution of (1).

3. AN EXAMPLE

Consider the following periodic boundary-value problem:

$$\begin{aligned} &\ddot{x} + \sin\frac{1}{4}x + 5(e^{\ddot{x}} - 1) = 0, \\ &x(0) = x(1), \\ &\dot{x}(0) = \dot{x}(1). \end{aligned} \tag{18}$$

The problem (18) may be reduced to the form (1) by introducing the following spaces and operators:

$$X = \left\{x \in C^2[0,1] : \; x(0) = x(1), \; \dot{x}(0) = \dot{x}(1)\right\}$$
$$Y = C[0,1], \; \|x\|_X = \max_{0\leq t\leq 1}|x(t)| + \max_{0\leq t\leq 1}|\dot{x}(t)| + \max_{0\leq t\leq 1}|\ddot{x}(t)|,$$

$$\|y\|_Y = \max_{0\le t\le 1} |y(t)|$$

$$X_1 = \Big\{x \in X \ : \int_0^1 x(t)dt = 0\Big\}, \quad Y_1 = \Big\{y \in Y \ : \int_0^1 y(t)dt = 0\Big\}$$

$$X_2 = Y_2 = \{\text{const}\}, \quad Ax = \ddot{x}, \quad Fx = \Big(\sin\frac{1}{4}x + 5(e^{\ddot{x}} - 1)\Big).$$

It can be verified that problem (18) has the solution $x^* \equiv x^*(t) \equiv 0$, $\forall t \in [0,1]$, $[A + PF'(0)]_{X_1}$ has a bounded inverse, $[QF'(0)]_{X_2}$ has a bounded inverse and $\|[A + PF'(0)]_{X_1}^{-1}\| \le \frac{50}{11}$, $\|[QF'(0)]_{X_2}^{-1}\| \le 4$, $\|[PF'(0)]_{X_2}\| = 0$.

Using Theorem 2.3 we obtain the following result.

Theorem 3.1. *If the initial approximation x_0 is sufficiently close to $x^* \equiv x^*(t) \equiv 0$, $\forall t \in [0,1]$, then the sequence $\{x_n\}$, constructed by the formulae:*

$$\ddot{u}_{n+1} + \frac{u_{n+1}}{4}\cos\frac{x_n}{4} + \ddot{u}_{n+1.5e}\ddot{x}_n - \int_0^1 \Big(\frac{u_{n+1}}{4}\cos\frac{x_n}{4} + \ddot{u}_{n+1.5e}\ddot{x}_n\Big).dt$$

$$= \frac{u_n}{4}\cos\frac{x_n}{4} + \ddot{u}_n.5e^{\ddot{x}_n} - \int_0^1 \Big(\frac{u_n}{4}\cos\frac{x_n}{4} + \ddot{u}_n.5e^{\ddot{x}_n}\Big)dt$$

$$- \Big[\sin\frac{x_n}{4} + 5(e^{\ddot{x}_n} - 1) - \int_0^1 \Big(\sin\frac{x_n}{4} + 5(e^{\ddot{x}_n} - 1)\Big)dt\Big], \tag{19}$$

$$u_{n+1}(0) = u_{n+1}(1), \quad \dot{u}_{n+1}(0) = \dot{u}_{n+1}(1),$$

$$\int_0^1 \sin\frac{u_{n+1} + v_{n+1}}{4} + 5(e^{\ddot{u}_{n+1}} - 1)dt = 0,$$

$$x_{n+1}(t) = u_{n+1(t)} + v_{n+1}$$

converges to a solution of (18).

It is to be noticed that the approximation method developed by Pham Ky Anh in [4] for problem (18) at the solution $x^* = x^*(t) = 0$ is not applicable.

Aknowledgement. I express my deepest gratitude to Prof. Nguyen Minh Chuong and Prof. Le Dinh Thinh for their help and encouragement.

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Institute of Mathematics
P.O. Box 631, Bo Ho
10000 Hanoi, Vietnam

Received June 13, 1991