Journal of Mathematics *Tạp chí Toán học 20(2)(1992), 60-64*

DIFFERENTIAL FORMS OF TYPE (0, q) ON COMPLETE CONVEX BORNOLOGICAL VECTOR SPACES

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Abstract. *In 1976, E. Ligocka proved that the $\overline{\partial}$ - problem has a solution for closed $(0,1)$ - forms with bounded support in a Banach space. The aim of this note is to extend this result for closed $(0,q)$ - forms with bounded support in complete convex bornological vector space E, i.e. E is the algebraic induction limit of a family of Banach spaces (E_i) $i \in I$ such that for every $i \leq j$ the canonical map $E_i \longrightarrow E_j$ is continuous. The main result is following*

Theorem: *Let ω be a differential form of type $(0,q)$ of class C^1 in complete convex bornological vector space E with a Schauder basis. Suppose that $\overline{\partial}\omega = 0$ and $\operatorname{supp}\omega \cap \mathrm{C}_x$ is bounded for every $x \in E$. Then there exists a differential form η of type $(0, q-1)$ and of class C^1 on E such that $\overline{\partial}\eta = \omega$.*

In 1976, E. Ligocka [2] proved that the $\overline{\partial}$ - problem has a solution for closed $(0,1)$ - forms with bounded support in a Banach space. The aim of this note is to extend this result for closed $(0,q)$ - forms with bounded support in complete convex bornological vector spaces [1].

1.

We first recall the definitions of (p,q) - forms and operators ∂ and $\overline{\partial}$ in open subsets of complete convex bornological vector spaces.

1.1. Suppose that E is a complete convex bornological vector space, i.e. E is the algebraic induction limit of a family of Banach spaces (E_i) $i \in I$ such that for every $i \leq j$ the canonical map $E_i \longrightarrow E_j$ is continuous.

Let U be an open subset of E, i.e. $U \cap E_i$ is an open subset of E_i for every $i \in I$. A function $f\colon U \longrightarrow \mathbf{C}$ is called to belong to class C^k if the restriction of f on $U \cap E_i$ is of class C^k for every $i \in I$.

Definition. *Let E be a complete convex bornological vector space and U be an open subset of E. Denote by $L_n^\alpha(E)$ the space of all mappings n - linear over $\mathbf{R}$ antisymetric and continuous of E into $\mathbf{C}$. The mapping $\omega: U \longrightarrow L_n^\alpha(E)$ is called a differential form of type (p,q) and of class C^k if the following conditions are satisfied:*

1) is a mapping of class C^k

2) $p+q=n$

3) for every $x \in U$: $x_1,\ldots,x_n \in E$ and for every $z \in \mathbf{C}$ we have

$$\omega(x)(zx_1,\ldots,zx_n) = z^p\overline{z}^q\omega(x)(x_1,\ldots,x_n).$$

The space of all such (p,q) - forms is denoted by $\Omega_{pq}^k(U)$.

1.2. Definition. *Suppose that ω is a differential form of type (p,q) and of class C^k, $k \geq 1$ on U. The C^{k-1} - mappings*

$$\overline{\partial}\omega\ : U \longrightarrow L_{p+q+1}^\alpha(E) \quad \text{and} \quad U \longrightarrow L_{p+q+1}^\alpha(E)$$

are defined as follows: for every $x \in U$ and $x_1,\ldots,x_{p+q+1} \in E$,

$$\overline{\partial}\omega(x)(x_1,\ldots,x_{p+q+1}) = \frac{1}{2}\sum_{j=1}^{p+q+1}(-1)^{j+1}\big[D\omega(x,x_j)+iD\omega(x,ix_j)\big](x_1,\ldots,\hat{x}_j,\ldots,x_{p+q+1}),$$

$$\partial\omega(x)(x_1,\ldots,x_{p+q+1}) = \frac{1}{2}\sum_{j=1}^{p+q+1}(-1)^{j+1}\big[D\omega(x,x_j)-iD\omega(x,ix_j)\big](x_1,\ldots,\hat{x}_j,\ldots,x_{p+q+1}).$$

One can check by immediate computations that if ω is a (p,q) - form then $\overline{\partial}\omega$ is a $(p,q+1)$ - form and $\partial\omega$ is a $(p+1,q)$ - form. It is obvious that $\overline{\partial}+\partial = d$. This implies that $\overline{\partial}\,\overline{\partial} = 0$; $\partial\,\partial = 0$ and $\overline{\partial}\,\partial + \partial\,\overline{\partial} = 0$ for the forms of class C^k, $k \geq 2$, because $dd = 0$

2. THE MAIN RESULT

2.1. The following result has been proved by E. Ligocka [1].

Let ω be a differential form of type $(0,1)$ and of class C^1 on a Banach space E. Suppose that ω has the bounded support and $\overline{\partial}\omega = 0$. Then there exists a function u of class C^1 on E such that $\overline{\partial}u = \omega$.

2.2. Now we prove the main result of this note.

Theorem 2.2.1. *Let ω be a differential form of type $(0,q)$ and of class C^1 in complete convex bornological vector space E with a Schauder basis. Suppose that $\overline{\partial}\omega = 0$ and supp $\omega \cap Cx$ is bounded for every $x \in E$. Then there exists a differential form η of type $(0, q-1)$ and of class C^1 on E such that $\overline{\partial}\eta = \omega$.*

Let E be a complete convex bornological vector space with a Schauder basis $\{e_j\}$. For each open subset U of E, by $\Omega_q(U)$ we denote the space consisting of C^1 - forms of type $(0,q)$ on U. Then every $\omega \in \Omega_q(U)$ can be written uniquely in the form

$$\omega(x) = \sum_{n=n(q)}^{\infty} \sum_{\substack{1\le j_1<\cdots<j_q \\ j_1+\cdots+j_q=n}} h_{j_1\dots j_q}(x) d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}, \tag{1}$$

where $n(q) = 1+2+\cdots+q$; $d\overline{z}_j = \overline{e}_j^* \;\; \forall j \ge 1$; $\{e_j^*\}$ is the dual basis of $\{e_j\}$.
We need the following

Lemma 2.2.2. *Let ω be a differential form of type $(0,q)$ and of class C^1 on U. If $\overline{\partial}\omega = 0$ then we have*

$$\sum_{k=1}^{q}(-1)^{k-1} \frac{\partial h_{j_1\dots \hat{j}_k \dots j_{q+1}}}{\partial \overline{z}_{j_k}} = 0 \quad \text{for} \quad 1 \le j_1 < \cdots < j_{q+1}.$$

P r o o f. $\omega \in \Omega_q(U)$ can be written uniquely in the form (1) hence we have

$$\overline{\partial}\omega(x) = \sum_{n=n(q+1)}^{\infty} \sum_{\substack{1\le j_1<\cdots<j_{q+1} \\ j_1+\cdots+j_{q+1}=n}} g_{j_1\dots j_{q+1}}(x) d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_{q+1}}.$$

For each $n \ge n(q+1)$ consider

$$S_n(x) = \sum_{\substack{1\le j_1<\cdots<j_{q+1} \\ j_1+\cdots+j_{q+1}=n}} \Big(\frac{\partial h_{\hat{j}_1\dots j_{q+1}}(x)}{\partial \overline{z}_{j_1}} + \cdots + (-1)^q \frac{\partial h_{j_1\dots \hat{j}_{q+1}}(x)}{\partial \overline{z}_{j_{q+1}}} \Big) d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_{q+1}}.$$

Formaly

$$\sum_{n=n(q+1)}^{\infty} S_n(x) = \overline{\partial}\omega(x).$$

Moreover

$$S_n(x)\Big(\sum_{j=1}^{q+1} e_j^*(u)e_j\Big) = \overline{\partial}\omega(x)\Big(\sum_{j=1}^{n} e_j^*(u)e_j\Big) \quad \forall n \ge n(q+1).$$

This yields

$$g_{j_1\dots j_{q+1}} = \sum_{k=1}^{q+1}(-1)^{k-1}\frac{\partial h_{j_1\dots\hat{j}_k\dots j_{q+1}}}{\partial \overline{z}_{j_k}}$$

$$\forall 1 \le j_1 < \dots < j_{q+1} \le n \quad \text{for} \quad j_1 + \dots + j_{q+1} = n$$

and hence

$$\overline{\partial}\omega(x) = \sum_{n=n(q+1)}^{\infty} \sum_{\substack{1\le j_1<\dots<j_{q+1}\\ j+1+\dots+j_{q+1}=n}} \Big(\sum_{k=1}^{q+1}(-1)^{k-1}\frac{\partial h_{j_1\dots\hat{j}_k\dots j_{q+1}}(x)}{\partial \overline{z}_{j_k}}\Big) d\overline{z}_{j_1}\wedge\dots\wedge d\overline{z}_{j_{q+1}}$$

and $\overline{\partial}\omega = 0$ if and only if

$$\sum_{k=1}^{q+1}(-1)^{k-1}\frac{\partial h_{j_1\dots\hat{j}_k\dots j_{q+1}}}{\partial \overline{z}_{j_k}} = 0 \quad \forall 1 \le j_1 < \dots < j_{q+1}.$$

The Lemma 2.2.2 is proved. □

P r o o f of Theorem 2.2.1

Let ω be a differential form of type $(0,q)$ and of class C^1. Suppose that supp $\omega \cap \mathbb{C}x$ is bounded for every $x \in E$ and $\overline{\partial}\omega = 0$. Put

$$g_{j_2\dots j_q}(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{t} h_{1j_2\dots j_q}(t+z_1, z')dt\wedge d\overline{t}$$

for $2 \le j_2 < \dots < j_q$, where $z = (z_1, z')$.

Consider C^1 - differential form η of type $(0, q-1)$:

$$\eta = \sum_{2\le j_2<\dots<j_q} g_{j_2\dots j_q} d\overline{z}_{j_2}\wedge\dots\wedge d\overline{z}_{j_q}.$$

From the definition of $g_{j_2\dots j_q}$ we have the relations [3]

$$\frac{\partial g_{j_2\dots j_q}}{\partial \overline{z}_1} = h_{1j_2\dots j_q} \qquad \forall 2 \le j_2 < \dots < j_q.$$

On the other hand

$$\overline{\partial}\eta = \sum_{2\le j_2<\dots<j_q} h_{1j_2\dots j_q} d\overline{z}_1\wedge\dots\wedge d\overline{z}_{j_q}$$

$$+ \sum_{n=n(q)+1}^{\infty} \sum_{\substack{2\le j_1<\dots<j_q\\ j_1+\dots+j_q=n}} \Big(\sum_{k=1}^{q}(-1)^{k-1}\frac{\partial g_{j_1\dots\hat{j}_k\dots j_q}}{\partial \overline{z}_k}\Big) d\overline{z}_{j_1}\wedge\dots\wedge d\overline{z}_{j_q}.$$

It remains to check that

$$\sum_{k=1}^{q}(-1)^k \frac{\partial g_{j_1\ldots\hat{j}_k\ldots j_q}}{\partial \overline{z}_k} = h_{j_1\ldots j_q}, \quad \forall 2 \le j_1 < \cdots < j_q.$$

Indeed, by the definition of $g_{j_1\ldots\hat{j}_k\ldots j_q}$ we have

$$\begin{aligned}
&\sum_{k=1}^{q}(-1)^k \frac{\partial g_{j_1\ldots\hat{j}_k\ldots j_q}}{\partial \overline{z}_k} \\
&= \sum_{k=1}^{q}(-1)^k \frac{1}{\pi}\int_{\mathbf{C}} \frac{1}{t}\frac{\partial h_{1j_1\ldots\hat{j}_k\ldots j_q}}{\partial \overline{z}_k}(t+z_1,z')dt\wedge d\overline{t} \\
&= \frac{1}{\pi}\int_{\mathbf{C}} \frac{1}{t}\frac{\partial h_{j_1\ldots j_q}}{\partial \overline{z}_1}(t+z_1,z')dt\wedge d\overline{t} \\
&= \frac{\partial}{\partial \overline{z}_1}\Big(\frac{1}{\pi}\int_{\mathbf{C}} \frac{1}{t} h_{j_1\ldots j_q}(t+z_1,z')dt\wedge d\overline{t}\Big) \\
&= h_{j_1\ldots j_q}(z), \quad \forall 2 \le j_1 < \cdots < j_q. \quad \square
\end{aligned}$$

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