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EXTENSION OF HOLOMORPHIC AND MEROMORPHIC MAPS

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In the theory of holomorphic functions of one and several complex variables the extension of holomorphic and meromorphic maps attracted a great attention of many mathematicians. In this note we announce some our results about the extension of holomorphic and meromorphic, also about the extension of separately holomorphic and meromophic maps. Their detailed proofs will be given in [17], [18].

1. EXTENSION OF HOLOMORPHIC MAPS

The problem of extension of holomorphic maps is considered in two ways:

a) Extending holomorphic maps defined on a complement of an analytic subset A in a complex manifold X to X (a Riemann holomorphic extension).

b) Extending holomorphic maps from a Riemann domain D over a Stein manifold to its envelope of holomorphy $^\wedge D$ (a Hartogs holomorphic extension).

During the 50-th the problem of Riemann holomorphic extension was solved for holomorphic functions defined on complex manifolds. The main result had been obtained as follows:

Let X be a complex manifold and A be an analytic subset in X. Assume that $f : X \setminus A \longrightarrow \mathbf{C}$ is holomorphic. If f is locally bounded on X then f can be holomorphically extended to X [5].

If codim $A \geq 2$ then the local boundedness of f is not necessary. It should be remarked that the result as above is still true if $\mathbf{C}$ is replaced by a Stein complex space Y. Hence, in order to investigate the problem of extension of holomorphic maps more fully and deeply we must consider the problem of extension for holomorphic maps with values in complex manifolds, or more general, in complex spaces. The Hartogs holomorphic extension for holomorphic maps with values in complex manifolds having a Stein covering has been investigated by A. Andreotti and

W. Stoll [1]. After that, results of the Hartogs holomorphic extension for holomorphic maps with values in the following complex manifolds have been obtained: *Lie* groups [2], *Hermitian manifolds with a negative holomorphic* curvature [6], [24], locally pseudoconvex domains over homogeneous projective spaces [7], hyperbolic manifolds [20], holomorphically convex Kahler manifolds [12].

Concerning the Riemann holomorphic extension, we note the result obtained by P. Kiernan for holomorphic maps with values in complex manifolds imbeded hyperbolically [19]. Here we give some results about the extension of holomorphic maps [10]. Let X be a complex space. X is called a holomorphic extension space if the following two conditions are satisfied:

(i) every holomorphic map $f : D \longrightarrow X$, can be holomorphically extended to $\hat{}D$, where D is a Riemann domain over a Stein manifold and $\hat{}D$ denotes an envelope on holomorphy of D.

(ii) every holomorphic map $f : Z \setminus S \longrightarrow X$, where Z is a normal complex space and S is an analytic subset of codimension ≥ 2 in Z, can be holomorphically extended to Z.

If X satisfies (i), X is called a Hartogs holomorphic extension space and if X satisfies (ii), X is called a Riemann holomorphic extension space.

In [10] we give examples about holomorphic extension spaces. Also, we proved the following interesting result about an invariance of holomorphic extension spaces under finite proper surjective holomorphic maps.

1.1. Theorem. *Let X and Y be complex spaces and $\theta : X \longrightarrow Y$ be a finite proper surjective holomorphic map. Then X is a holomorphic extension space if and only if Y has the same property.*

There is an example showing that Hartogs holomorphic extension spaces are not invariant under finite proper surjective holomorphic maps.

2. EXTENSION OF MEROMORPHIC MAPS

Let X and Y be complex spaces. A meromorphic map $f : X \longrightarrow Y$ is an analytic set $\Gamma(f)$ in $X \times Y$ which is called the graph of f, such that the canonical $\sigma : \Gamma(f) \longrightarrow X$ is proper and there exists an analytic $I(f)$ in X of codimension ≥ 2 together with a holomorphic map f_0 from $X \setminus I(f) \longrightarrow Y$ for which $\Gamma(f)$ Cl $\Gamma(f_0)$.

The extension of meromorphic maps is more difficult than the extension of holomorphic maps because extending meromorphic maps means that we must extend analytic sets. Hence, results about extension of meromorphic maps with values in various complex spaces are not so many. The extension of meromorphic functions from a Riemann domain over a Stein manifold to its envelope of holomorphy has been established by E. E. Levi [21] and J. Kajiwara and E. Sakai [13]. In 1969 H. Rossi has proved that every meromorphic function defined on a domain of projective space CP^{m-1} containing an analytic subset of positive dimension, can be meromorphically extended to CP^{m-1} [23]. The result of Rossi was proved at the same time, by W. Barth [3] and after that, in a more general case, has been proved by W. L Chow [4] for meromorphic maps with values in algebraic varieties. The another result about an extension of meromorphic maps with values in compact Kahler manifolds has been investigated by P. Griffiths [6] and later in a more general case has been established by Y. T. Siu [27]. In [25] B. Shiffman has proved the extension of holomorphic and meromorphic maps with values in compact manifolds satisfying the curvature condition. We have obtained the results about extension of meromorphic maps.

2.1. Definition. *Let X be a complex space X is called a meromorphic extension space if two the following conditions are satisfied:*

(i) every meromorphic map $f : D \longrightarrow X$, can be meromorphically extended to $^\wedge D$, where D is a Riemann domain over a Stein manifold and $^\wedge D$ denotes an envelope of holomorphy of D.

(ii) every meromophic map $f : Z \setminus S \longrightarrow X$, where Z is a normal complex space and S is an analytic subset of codimension ≥ 2 in Z, can be meromorphically extended to Z.

We proved that the following spaces are meromorphic extension spaces: complex Lie groups, compact homogeneous Kahler manifolds, elliptic non-singular Kahler surfaces.

The main result which will be proved in [17] is following

2.2. Theorem. *Let X and Y be complex spaces and $\theta : X \longrightarrow Y$ be a finite proper surjective holomorphic maps. Then X is a meromorphic extension space if and only if Y has the same property.*

In [8] we have extended the result of Barth-Rossi for meromorphic maps with values in compact manifolds. Recently, we proved that Barth-Rossi theorem is still valid for the case when D is a connected open subset in infinite dimensional projective spaces and a map f has the following forms: f is a meromorphic function with values in a sequentially complete locally convex space, a meromorphic map with values in a meromorphic extension space and, finally, a locally biholomorphic map with values in an infinite dimensional projective space [11].

3. EXTENSION OF SEPARATELY HOLOMORPHIC AND MEROMORPHIC MAPS

Together with the problem of extension of holomorphic and meromorphic maps, the extension of separately holomorphic and meromorphic maps has been investigated by many mathematicians. The results of J. Siciak [28] and V. P. Zakharyuta [29] say that every separately holomorphic function defined on special subsets of $\mathbf{C}^n$ can be extended to a holomorphic function on its envelope of holomorphy. In 1984 M. V. Kazarian has proved the following result about the extension of separately meromorphic functions.

3.1. Theorem *[14]. Let $f = f(z, w)$ be a separetely meromorphic function on a set $X = (D \times F) \cup (E \times G) \subset \mathbf{C}_z^n \times \mathbf{C}_w^m$, where $D \subset \mathbf{C}_z^n$ and $G \subset \mathbf{C}_w^m$ are domains, $E \subset D$ and $F \subset G$ are pluriregular compact subsets. Then there exists a meromorphic function $\tilde{f}$ defined on the domain.*

$$\Omega = \{(z, w) \in D \times G \ : \ \omega^*(z, E, D) + \omega^*(w, F, G) + 1 < 0\},$$

containing X such that $\tilde{f} = f$ on X.

Recently, in 1990 B. Shiffman has proved that the Hartogs theorem is still valid for separately holomorphic maps with values in Hartogs holomorphic extension spaces [26].

We have considered the theorem of Siciak-Zakharyuta for separately holomorphic functions with values in Banach-Lie groups. The following theorem has been proved in [16].

3.2. Theorem. *Let X and Y be Stein manifolds and let $E \subset X, F \subset Y$ be compact sets such that $\hat{E}, \hat{F}$ the holomorphically convex hull of E, F respectively, are regular in X, Y, respectively. Let Γ be a Banach-Lie group and $f \ : \ Z \ : X \times F \cup E \times Y \longrightarrow \Gamma$ be a continous and separately holomorphic function. Then f can be uniquely extended to a holomorphic function f in the set*

$$Z = \{(z, w) \in X \times Y \ : \omega^*(z, E, X) + \omega^*(w, F, Y) + 1 < 0\}$$

containing Z.

Another result about the extension for weak separately meromorphic functions with values in Banach spaces has been established.

3.3. Theorem. *Let G be an open set in $\mathbf{C}^n$ and F be a Banach space. Assume that f is a F-valued meromorphic function on an open subset X of G such that $x^* f$ can be extended to a meromorphic functions $\widehat{x^* f}$ on G for all $x^* \in F^*$, the dual space of F. Then f is meromorphically extended to G.*

This result will be proved in [18].

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