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PROJECTION–REGULARIZATION METHOD AND ILL–POSEDNESS FOR EQUATIONS INVOLVING ACCRETIVE OPERATORS

NGUYEN BUONG

Abstract. *The purpose of this paper is to approximate the ill-posed problems involving accretive operators in Banach spaces by a sequence of well-posed finite-dimensional problems depending on small parameter and to give a necessary condition for ill-posedness, when the operator is Fréchet differentiable. Some examples from theory of integral equations of second kind are also considered.*

1. INTRODUCTION

Let X be a real reflexive Banach space with the norm $\|\cdot\|$ and X^* be its adjoint space with the norm $\|\cdot\|_*$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let A be a m–accretive operator in X, i.e. [10]

i) $$\langle A(x+h) - A(x),\ J(h)\rangle \geq 0, \quad \forall x, h \in X$$

where J is a dual mapping of X, i.e. a mapping from X onto X^*, satisfying the condition

$$\langle J(x), x\rangle = \|x\|^2 = \|J(x)\|_*^2, \quad \forall x \in X$$

and

ii) $R(A + \lambda I) = X$ for each $\lambda > 0$ where $R(A)$ denotes the range of A and I is an identical operator in X.

We are interested in solving the equation

$$A(x) = f. \tag{1.1}$$

Without additional conditions on the structure of A, such as strongly or uniformly accretive property, the problem (1.1) is, in general, an ill-posed one [1]. By this we mean that solutions of (1.1) do not depend continuously on the data f. To solve it we have to use stable methods. An widely used and effective method is the variational Tikhonov regularization [9] that consists of minimizing the functional

$$F_{\alpha\delta}(x) = \|A(x) - f_\delta\|^2 + \alpha\|x\|^2, \tag{1.2}$$

where $\alpha > 0$ is a parameter of regularization and f_δ are approximations for f:

$$\|f_\delta - f\| < \delta.$$

The aspects of existence, convergence and stability for the solutions of (1.2) have been established in [3, 9].

For problems involving accretive operators there is another version of Tikhonov regularization that consists of solving the equation

$$A(x) + \alpha(x - x_*) = f_\delta, \tag{1.3}$$

where x_* is some fixed element of X.

This equation has been investigated in [1, 7] and [8]. For finding the solution $x_{\alpha\delta}$ of (1.3) one can use iterative methods in infinite–dimensional spaces X in [2, 5, 6]. But in order to realize them first we must approximate (1.3) by a sequence of finite-dimensional problems as follows.

Let X_n be a sequence of finite–dimensional subspaces of X such that

$$X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

and P_n be a sequence of projections from X onto X_n and $P_n x \to x, \forall x \in X$. Here and below, the symbols $\to$ and $\rightharpoonup$ denote the strong convergence and the weak convergence, respectively.

Consider the problem

$$A_n(x) + \alpha(x - x_{*n}) = f_{\delta n}, \quad x \in X, \tag{1.4}$$

where $f_{\delta n} = P_n f_\delta$, $x_{*n} = P_n x_*$ and $A_n = P_n A P_n$. It's easy to see that if A is accretive, A_n are accretive, too. The existence and the convergence of the solutions $x^\delta_{\alpha n}$ of (1.4) to the solution $x_{\alpha\delta}$ of (1.3) for each $\alpha > 0$ has been studied in [9].

It is still open the questiosn under which conditions the sequence $x^\delta_{\alpha n}$ converges to a solution of (1.1), as α, $\delta \to 0$ and $n \to \infty$. We note that for the variational Tikhonov regularization this question has been studied in [3]. In [4] it was shown that the solutions of the variational Tikhonov regularization converge to a solution x_0 of (1.1) if $\|(I - P_n)x_0\| = o(\alpha(n))$ for the case when A is linear and bounded.

In Section 2 of this paper we prove that this result remains valid for the operator method of regularization (1.4) and we also give a necessary condition for ill-posedness for (1.1). In Section 3 we consider an example.

2. MAIN RESULTS

We suppose that A is Fréchet differentiable in some neighbourhood of x_0, a unique solution of (1.1), and there exists a constant $L > 0$ such that

$$\|A'(x_0) - A'(x)\| \leq L\|x_0 - x\|, \quad \forall x \in S(x_0, r), \tag{2.1}$$

where $S(x_0, r)$ is a ball with center x_0 and radius $r > 0$. For X we require that J is weakly continuous.

Theorem 1. *If $\delta/\alpha \to 0$, $\alpha(n) \to 0$ as $n \to \infty$ and $\|(I - P_n)x_0\| = o(\alpha(n))$ then the solution $x^\delta_{\alpha n}$ for (1.4) converges to the solution x_0 of (1.1).*

Proof. From (1.4) it follows

$$\alpha\|x^\delta_{\alpha n} - x_{0n}\|^2 \leq \delta\|x^\delta_{\alpha n} - x_{0n}\| + \langle f_n - P_n A P_n x_0, J_n(x^\delta_{\alpha n} - x_{0n})\rangle$$

$$+ \alpha\langle x_{*n} - x_{0n}, J_n(x^\delta_{\alpha n} - x_{0n})\rangle, \tag{2.2}$$

where $x_{0n} = P_n x_0$ and J_n is a dual mapping of X_n.

Since

$$A(P_n x_0) = A(x_0) + A'(x_0)(P_n - I)x_0 + r_n$$

with

$$\|r_n\| \leq \frac{L}{2}\|(I - P_n)x_0\|^2$$

the inequality (2.2) implies

$$\alpha\|x^\delta_{\alpha n} - x_{0n}\|^2 \leq (\delta + c_1\|(I - P_n)x_0\| + c_2\|(I - P_n)x_0\|^2)$$

$$\times \|x^\delta_{\alpha n} - x_{0n}\| + \alpha\langle x_{*n} - x_{0n}, J_n(x^\delta_{\alpha n} - x_{0n})\rangle, \quad c_i > 0, \ i = 1, 2. \tag{2.3}$$

Due to this inequality we obtain the boundedness of the sequence $x^\delta_{\alpha n}$. Now, let

$$x^\delta_{\alpha n} \rightharpoonup x_1, \quad \text{as } n \to \infty.$$

Then, the inequality (1.4) gives

$$A_n(x^\delta_{\alpha n}) \to f, \quad \text{as } n \to \infty.$$

Because A_n is accretive

$$\langle A_n(x_n) - A_n(x^\delta_{\alpha n}), J_n(x_n - x_{\alpha n})\rangle \geq 0, \quad \forall x \in X,\ x_n = P_n x.$$

Since $J_n(x) = J(x)$ for $x_n \in X_n$ [8] the last inequality is equivalent to

$$\langle A(x_n) - A_n(x^\delta_{\alpha n}), J_n(x_n - x_{\alpha n})\rangle \geq 0.$$

By letting $n \to \infty$ we get

$$\langle A(x) - f,\ J(x - x_1)\rangle \geq 0, \quad \forall x \in X.$$

Therefore $A(x_1) = f$ [10]. By our assumption that (1.1) has a unique solution, the sequence $x^\delta_{\alpha n}$ converges weakly to the solution. Finally, the strong convergence of $x^\delta_{\alpha n}$ follows from (2.3) and the conditions of the theorem.

Remark. We can consider the case, when instead of A we know only its approximations A_h which also are accretive.

Now consider the relation between nonlinear ill-posed problems and their linearizations. This relation is not strong as one might think for linear ill-posed problems. A nonlinear ill-posed problem may have a well-posed linearization at certain points, even at a point where the nonlinear problem is ill-posed. On the other hand, a well-posed nonlinear problem may have ill-posed linearizations. In [3] H. W. Engl gave a sufficient condition for ill-posedness for nonlinear problem in the case where A is compact. Here, we present a necessary condition for ill-posedness in the case when A is Fréchet differentiable.

Definition. *An operator A in X is called strongly accretive if there exists a constant $m > 0$ such that*

$$\langle A(x+h) - A(x), J(h)\rangle \geq m\|h\|^2, \quad \forall x, h \in X.$$

We prove the following result.

Theorem 2. *If in some neighbourhood of a point x_0 at which problem (1.1) is ill-posed and the operator A has a Fréchet derivative $A'(x_0)$ at x_0, then $A'(x_0)$ can not be a strongly accretive operator*

P r o o f. Indeed, if $A'(x_0)$ is strongly accretive, from

$$A(x) = A(x_0) + A'(x_0)(x - x_0) + r_0,$$

with

$$\|r_0\| \leq \frac{L}{2}\|x - x_0\|^2$$

we have

$$\langle A(x) - A(x_0), J(x - x_0)\rangle = \langle A'(x_0)(x - x_0), J(x - x_0)\rangle + \langle r_0, J(x - x_0)\rangle.$$

Therefore,

$$\|A(x) - A(x_0)\| \geq m\|x - x_0\| - \frac{L}{2}\|x - x_0\|^2.$$

For x in a ball with the center x_0 and the radius $r \leq m/L$

$$\|A(x) - A(x_0)\| \geq m\|x - x_0\|/2,$$

i.e. Problem (1.1) is well-posed at the point x_0 contradicting the hypothesis.

3. EXAMPLE

We consider the linear integral equation of the second kind

$$x(s) - \int_{\Omega} k(s,t)x(t)dt = f(s),$$

where $f(s) \in L_p[\Omega]$, the space of p - summable functions in σ - finite measure set $\Omega \subseteq R^n$. Let the kernel function $k(s,t)$ be such that the operator K in $L_p[\Omega]$ defined by

$$(Kx)(s) = \int_{\Omega} k(s,t)x(t)dt$$

has an eigeinvalue $\lambda = 1$ and $\|K\| = 1$. Then the operator $I - K$ is accretive. Indeed,

$$\langle (I - K)(x + h) - (I - K)x, J(h)\rangle = \langle (I - K)h, J(h)\rangle =$$

$$= \|h\|^2 - \langle Kh, J(h)\rangle \geq \|h\|^2(I - \|K\|) \geq 0,$$

where J is a dual mapping of $L_p[\Omega]$.

We consider the subsets Ω_j, $j = 1, 2, \ldots, m$, such that $\bigcup_{j=1}^{m} \Omega_j = \Omega$ and denote by $f_j(t)$ the characteristic function of Ω_j. Then we can choose

$$p_m f = \sum_{j=1}^{m} \frac{1}{(m\mu(\Omega_j))^{1/q}} \int_{\Omega_j} f(t)dt,$$

where $p^{-1} + q^{-1} = 1$ (see, e.g. [10]).

We can also apply the results obtained in Section 2 to solve the nonlinear integral equations of Hammerstein's type

$$x(s) - \int_{\Omega} k(s,t)F(x(t))dt = f(s),$$

under the condition

$$|F(t)| \leq a + b|t|, \quad a, b > 0.$$

In this case the operator G defined by

$$G(x)(s) = \int_{\Omega} k(s,t)F(x(t))dt$$

maps $L_p[\Omega]$ into $L_p[\Omega]$ [11]. If we suppose, in addition, that F is Lipschitz continuous with Lipschitz constant $\|K\|^{-1}$, then it is easy to verify that $I - G$ is accretive.

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Institute of Informatics
Nghia Do, Tu Liem
Hanoi, Vietnam