OPTIMAL RECOVERY OF FUNCTIONS OF A CERTAIN MIXED SMOOTHNESS

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Abstract. *Multivariate functions with one or several bounded partial derivatives have approximation properties completely different from ones of univariate smooth functions. We investigate the optimal recovery for functions of Hölder classes of a certain smoothness by means of some characteristics of optimal recovery of these classes. Some asymptotic estimates of these quantities are obtained by use of linear methods of recovery by trigonometric polynomials of hyperbolic crosses. These estimates in some cases coinside with asymptotic degrees of charateristics of optimal recovery of Hölder classes.*

1. INTRODUCTION

Multidimensional classes of smooth functions with one or several bounded partial derivatives have approximation properties completely different from ones of unidimensional classes of smooth functions. First of all, for functions of a given mixed smoothness one must understand, which polynomials or splines are reasonable to select for best approximations. Questions also arise concerning the comparison of approximation methods with widths, entropy and other approximation characteristics.

In this paper for multivariate periodic functions of Hölder classes of a certain mixed smoothness, we shall investigate the optimal recovery by means of some characteristics of optimal recovery. This problem is closely related to the problems of the n-width and of the best approximation by trigonometric polynomials of so-called hyperbolic crosses for these classes. We refer to [D1], [T1] for surveys on the latter problems. Special lattices and methods were constructed for recovering functions of a given mixed smoothness from their values at these lattices and some estimates of the recovery error were obtained in [S], [HW], [T2].

Let us introduce some characteristics of optimal recovery.

Let X be a normed linear space of functions defined on the torus $\mathbf{T}^d := [-\pi,\pi]^d$ and $W \subset X$. For $\ell_1,\ldots,\ell_s$, any s functionals in W and $P_s(t_1,\ldots,t_s)$, any mapping from $\mathbf{R}^s$ into a linear manifold in X of dimensions at most s, one can naturally consider recovering $f \in W$ from $\ell_1(f),\ldots,\ell_s(f)$ by the element $P_s(\ell_1(f),\ldots,\ell_s(f))$. We define the following characteristics of optimal recovery:

$$R_n(W,X) := \inf_{\substack{x^1,\ldots,x^s \in T^d;\\ P_s,\ s\le n}} \sup_{f\in W} \|f - P_s(f(x_1),\ldots,f(x_s))\|_X,$$

$$R_n^*(W,X) := \inf_{\substack{\ell_1,\ldots,\ell_s;\\ P_s,\ s\le n}} \sup_{f\in W} \|f - P_s(\ell_1(f),\ldots,\ell_s(f))\|_X.$$

These quantities express, in some sense, the optimal recovery of functions in the class W from their values or functionals with a preassigned information quantity. A characteristic of optimal recovery, similar to $R_n^*(W,X)$ was introduced in [K]. Note that the above definitions imply the following important inequalities:

$$R_n^*(W,X) \ge d_n(W,X); \tag{1.1}$$

$$R_n(W,X) \ge R_n^*(W,X), \tag{1.2}$$

where $d_n(W,X)$ denotes the n-width of W in X (see the definition in §3).

In this paper we shall discuss the asymptotic degree of $R(SH_p^A, L_q(\mathbf{T}^d))$ and $R^*(SH_p^A, L_q(\mathbf{T}^d))$ for given $A \subset \mathbf{R}^d$ and various pairs $1 \le p,\ q \le \infty$, where SH_p^A denotes the intersection of the unit balls SH_p^α in the spaces H_p^α, $\alpha \in A$; of functions on $\mathbf{T}^d$, satisfying the mixed Hölder condition α. The results of this paper have been proved [D3] for the class SH_p^α.

In §2, as preliminaries, we establish some properties of trigonometric polynomials and of de la Vallée Pussin's integral and sum convolution operators $I_{m,r}$ and $S_{m,r}$. In particular, we prove a modification of a theorem of Marcinkiewicz and an estimate of the L_p-norm of $S_{m,r}f$ for trigonometric polynomials f.

In §3 we introduce Hölder spaces H_p^α and H_p^A and associated classes SH_p^α and SH_p^A of mixed smoothness $\alpha \in \mathbf{R}^d$ or $A \subset \mathbf{R}^d$. Some dual descriptions of these spaces by harmonic diadic decompositions in terms of theoremes of equivalence of seminorms are given. These decompositions are constructed by means of $I_{m,r}$ and $S_{m,r}$. As auxiliary results, we obtain some estimates of the n-width $d_n(SH_p^A, L_q)$ and $G_n(SH_p^A, L_q)$, the best approximation by Fourier sums in the space L_q of the classes SH_p^A for various pairs p,q.

In §4 we establish some asymptotic estimates of $R_n^*(SH_p^A, L_q)$ and $R_n(SH_p^A, L_q)$ for various pairs p, q. The asymptotic degree of the first quantity in some cases is obtained from the estimates of $d_n(SH_p^A, L_q)$ and $G_n(SH_p^A, L_q)$ in §3. To obtain the asymptotic estimate of the second quantity, we construct a linear method for recovering functions in SH_p^A from their values at a subset of the diadic lattice by trigonometric polynomials of certain hyperbolic crosses depending on p, q and A using the decomposition generated by the operator $S_{m,m}$. We prove that in the case $1 < p < q \leq 2$ this estimate coinsides with the asymptotic degree of $R_n(SH_p^A, L_q)$.

2. PRELIMINARIES

Let

$$\Phi_{m,r}(x) := \prod_{j=1}^{d} \varphi_{m_j,r_j}(x_j), \qquad m, r \in \mathbf{N}^d,$$

be de la Vallée Pussin's kernel of d variables where

$$\varphi_{m,r}(t) := 1 + 2\sum_{k=1}^{m} \cos kt + 2 \sum_{k=m+1}^{m+r} \frac{m+r-k}{r} \cos kt$$

for natural numbers m, r and x_j denotes the j-th coordinate of $x \in \mathbf{R}^d$. For functions f on $\mathbf{T}^d$ consider the integral convolution operator

$$I_{m,r}f := f * \Phi_{m,r},$$

and the sum convolution operator

$$S_{m,r}f := \prod_{j=1}^{d} (2m_j + r_j)^{-1} \sum_{k} f(hk)\Phi_{m,r}(\cdot - hk),$$

where the sum is taken over all $k \in \mathbf{Z}^d$ such that $0 \leq k < 2m+r$; $h = 2\pi/(2m+r)$; $\pi/x = (\pi/x_1, \dots, \pi/x_d)$, $xy = (x_1y_1, \dots, x_dy_d)$ and the inequality $x \leq y$ $(x < y)$ is understood as $x_j \leq y_j$ $(x_j < y_j)$, $j = 1, \dots, d$. For abbreviation we write ϕ_m, S_m and I_m instead of $\phi_{m,m}$, $S_{m,m}$ and $I_{m,m}$.

Denote by T_m, $m \in \mathbf{Z}_+^d$, the set of trigonometric polynomials of order at most m_j at the variable x_j, $j = 1, \dots, d$, where $\mathbf{Z}_+^d := \{k \in \mathbf{Z}^d : k \geq 0\}$. For a fixed number a, $0 < a \leq 1$, let M_a be the set of all pairs (m, r), $m, r \in \mathbf{N}^d$, satisfying the condition $am \leq r \leq m$. Below C, C' denote various absolute positive constants, while $C_{a,b,\dots}$, $C'_{a,b,\dots}$ denote various positive constants depending on $a, b, \dots$

We shall need the following properties:

$$\|\phi_{m,r}\|_1 \le C_a \quad \text{for any} \quad (m,r) \in M_a \tag{2.1}$$

$$\|I_{m,r}f\|_p \le C_a \|f\|_p \quad \text{for any} \quad (m,r) \in M_a,\ (1 \le p \le \infty) \tag{2.2}$$

$$I_{m,r}f = f \quad \text{for any} \quad f \in T_m \tag{2.3}$$

$$(S_{m,r}f)(hk) = f(hk), \quad k \in \mathbf{Z}^d, \quad 0 \le k < 2m + r \tag{2.4}$$

$$S_{m,r}f = f \quad \text{for any} \quad f \in T_m \tag{2.5}$$

The properties (2.1-4) can be easily verified. The latter one follows from a more generalized assertion. Namely, if $m, n, s \in \mathbf{Z}^d_+$ and $m + n < s$, then the following equality holds for any $f \in T_m$ and $g \in T_n$

$$f * g = \frac{1}{s} \sum_k f(hk) g(\cdot - hk), \quad h = 2\pi/s, \tag{2.6}$$

where the sum is taken over all $k \in \mathbf{Z}^d$ such that $0 \le k < s$. Indeed, in view of the formula

$$f * g = \sum_k f_k g_k e^{i\langle k, \cdot\rangle},$$

where f_k and g_k are the k-th Fourier coefficients of f and g, it sufficies to prove that for any $g \in T_n$ and $f = e^{i\langle k,\cdot\rangle}$, $k \le m$, the right side of (2.6) is equal to $g_k e^{i\langle k,\cdot\rangle}$. This fact can be checked directly by replacing $g(\cdot - hk)$ in the right side of (2.6) by its Fourier series. Because of (2.3) the above assertion implies (2.5).

For a sequence $\{a_k : k \in \mathbf{Z}^d, 0 \le k < s\}$, $s \in \mathbf{N}^d$, of real or complex numbers, we define the norm

$$\|\{a_k\}\|^p_{p,s} := (2\pi)^d \prod_{j=1}^d \frac{1}{s_j} \sum_k |a_k|^p, \quad 1 \le p < \infty,$$

the sum is taken over all $k \in \mathbf{Z}^d$ such that $0 \le k < s$. The sum norm is changed to the max norm when $p = \infty$.

For any $f \in T_m$ we have

$$\sup_x \|\{f(x - hk)\}\|_{p,s} \le \prod_{j=1}^d (1 + h_j m_j)^{1/p} \|f\|_p, \tag{2.7}$$

where $h = 2\pi/s$.

For the case $p = \infty$, this inequality is obvious. The case $p = 1$ can be proved in a way similar to a proof of an analogous inequality for functions of exponential type [N]. The case $1 < p < \infty$ follows easily from the cases $p = 1, \infty$ by interpolation properties of the spaces L_p.

Lemma 2.1. *If $1 \le p \le \infty$ and $(m,r) \in M_a$, then for any $f \in C(\mathbf{T}^d)$ we have*

$$\|S_{m,r}f\|_p \le C_a \|\{f(hk)\}\|_{p,2m+r}, \quad h = 2\pi/(2m+r).$$

P r o o f. We prove the lemma for $1 \le p < \infty$, the case $p = \infty$ can be proved in a similar way. By use of Hölder's inequality we have

$$\prod_{j=1}^{d} (2m_j + r_j)^p |(S_{m,r}f)(x)|^p$$

$$\le \Big(\sum_k |\Phi_{m,r}(x - hk)|\Big)^{p/p'} \sum_k |f(hk)|^p |\Phi_{m,r}(x - hk)|,$$

where the sums are taken over all $k \in \mathbf{Z}^d$ such that $0 \le k < 2m + r$ and $1/p + 1/p' = 1$. Therefore, by (2.1) , (2.7) we obtain

$$\|S_{m,r}f\|_p$$

$$\le (\sup_x \|\{\Phi_{m,r}(x - hk)\}\|_{1,2m+r})^{1/p'} \|\Phi_{m,r}\|_1^{1/p} \|\{f(hk)\}\|_{p,2m+r}$$

$$\le C'_a \|\Phi_{m,r}\|_1 \|\{f(hk)\}\|_{p,2m+r} \le C_a \|\{f(hk)\}\|_{p,2m+r}. \quad \square$$

From Lemma 2.1 it follows that

$$\|S_{m,r}\|_{C(\mathbf{T}^d) \to C(\mathbf{T}^d)} \le C_a \quad \text{for any} \quad (m,r) \in M_a, \tag{2.9}$$

and, moreover, for any $f \in C(\mathbf{T}^d)$

$$\lim_{\substack{m_j \to \infty \\ j=1,\dots,d}} \|f - S_m f\|_{C(\mathbf{T}^d)} = 0. \tag{2.10}$$

As a modification of a theorem of Marcinkiewicz (cf., e.g. [Z]), from (2.5), (2.7) and Lemma 2.1 we obtain the following

Lemma 2.2. *If $1 \le p \le \infty$ and $(m,r) \in M_a$, then for any $f \in T_m$ we have*

$$C_a\|f\|_p \le \|\{f(hk)\}\|_{p,2m+r} \le C'_a\|f\|_p, \quad h = 2\pi/(2m+r).$$

Lemma 2.3. *If $1 \le p \le \infty$ and $(m,r) \in M_a$, then for any $f \in T_n$, $n \ge m$, we have*

$$\|S_{m,r}f\|_p \le C_a \prod_{j=1}^{d} (n_j/m_j)^{1/p}\|f\|_p.$$

3. HÖLDER SPACES AND HARMONIC DECOMPOSITIONS

First we introduce H_p^α, $1 \le p \le \infty$, the Hölder space of smoothness $\alpha \in \mathbf{R}^d$. This space consists of all those distributions $f \in \varphi'(\mathbf{T}^d)$ with zero mean in each variable such that the seminorm

$$\|f\|_{H_p^\alpha} := \sup_{h \in \mathbf{T}^d} \left\|\Delta_h^2 f^{(r)}\right\|_p \prod_{j=1}^{d} |h_j|^{-\beta_j}$$

is finite, where $\Delta_h^2 := \Delta_h \circ \Delta_h$, $\Delta_h := \prod_{j=1}^{d} \Delta_{h_j}$, $(\Delta_{h_j} f)(x) := f(x_1, \dots, x_j + h_j, \dots, x_d) - f(x)$ and by definition $f^{(r)}$ is the Weyl partial derivative of order r of f; the vectors $r \in \mathbf{Z}^d$ and $\beta \in (0,1]^d$ are defined by the equality $\alpha = r + \beta$. There is another equivalent definition of H_p^α for $\alpha > 0$, $\alpha \in \mathbf{R}^d$, using a mixed higher–order finite difference operator (cf. [T1]).

In recovering we shall take functions of a given mixed smoothness $A \subset \mathbf{R}^d$, belonging the intersection $H_p^A := \bigcap_{\alpha \in A} H_p^\alpha$. Below we can see that H_p^A coinsides with the space of all those distributions f for which the seminorm

$$\|f\|_{H_p^A} := \sup_{\alpha \in A} \|f\|_{H_p^\alpha} \tag{3.1}$$

is finite.

To formulate and prove our results, we describe two harmonic diadic decompositions of the space H_p^A. Let U_k and V_k be the unidimensional operators defined by

$$U_1 := S_1, \quad U_k := S_{2^{k-1}} - S_{2^{k-2}}, \quad k = 2, 3, \dots;$$

$$V_1 := I_1, \quad V_k := I_{2^{k-1}} - I_{2^{k-2}}, \quad k = 2, 3, \dots$$

The multidimensional mixed operators U_k and V_k, $k \in \mathbf{N}^d$ are defined by

$$U_k := \prod_{j=1}^{d} U_{k_j}; \qquad V_k := \prod_{j=1}^{d} V_{k_j},$$

where U_{k_j} and V_{k_j} are the unidimensional operators at the variable x_j.

Note that for any $m \in \mathbf{N}^d$

$$S_{2^m} = \sum_{k \le m} U_k; \qquad I_{2^m} = \sum_{k \le m} V_k,$$

where $2^x := (2^{x_1}, \dots, 2^{x_d})$ for $x \in \mathbf{R}^d$. Hence, using the properties (2.2-3) and (2.10), one can verify that any $f \in L_p$, $1 \le p \le \infty$, can be represented by the series

$$f = \sum_k V_k f, \tag{3.2}$$

converging in the L_p - norm, and any $f \in C(\mathbf{T}^d)$ can be represented by the series

$$f = \sum_k U_k f, \tag{3.3}$$

converging uniformly on $\mathbf{T}^d$.

We shall use the notations $\ll$ and $\asymp$ to denote the inequality and equivalence of asymptotic degrees (orders) (for details cf. [D1]). For $G \subset \mathbf{R}^d$ let $sG(x) := \sup\{\langle x, y\rangle : y \in G\}$ be the support function of G.

Lemma 3.1. *For arbitrary $A \subset \mathbf{R}^d$ and $1 \le p \le \infty$, a distribution f belongs to H_p^A iff the quasinorm (3.1) is finite. Moreover, the following equivalence of quasinorms holds*

$$\|f\|_{H_p^A} \asymp \sup_k 2^{sA(k)} \|V_k f\|_p, \quad f \in H_p^A.$$

P r o o f. Obviously, the lemma will be proved if we establish the inequalities

$$C_p \|f\|_{H_p^\alpha} \le \sup_k 2^{\langle \alpha, k\rangle} \|V_k f\|_p \le C'_p \|f\|_{H_p^\alpha}, \tag{3.4}$$

for each $f \in H_p^\alpha$. Inequalities analogous to (3.4) were proved for functions of a space similar to H_p^α and defined by use of a mixed higher–order finite difference operator [T1]. In particular, from those inequalities [T1] and their proof it follows that (3.4) holds for arbitrary $\alpha \in (0,1]^d$. We shall prove (3.4) for the unidimensional case ($d = 1$), the multidimensional case can be proved in a similar way without essential changes. Let $\alpha \in \mathbf{R}$ and $\alpha = r + \beta$, $r \in \mathbf{Z}$, $\beta \in (0,1]$. By the above mentioned remark and the equality $V_k f^{(r)} = (V_k f)^{(r)}$ we have

$$C_p \|f^{(r)}\|_{H_p^\alpha} \le \sup_k 2^{\beta k} \|(V_k f)^{(r)}\|_p \le C'_p \|f^{(r)}\|_p, \tag{3.5}$$

for each $f \in H_p^\alpha$. Since $V_k f \in T_{2^{k+1}}$ and $V_k f$ is orthogonal to the trigonometric polynomials of order at most 2^{k-2}, by Bohr's and Bernstein's inequalities (cf., e. g., [N]) we obtain

$$C_p \|(V_k f)^{(r)}\|_p \le 2^{rk} \|V_k f\|_p \le C'_p \|(V_k f)^{(r)}\|_p.$$

This and (3.5) imply (3.4) for arbitrary $\alpha \in \mathbf{R}$. □

Set $\mathbf{1} := (1, \ldots, 1) \in \mathbf{R}^d$ and $\mathcal{N}(G) := \text{co}\, G - \mathbf{R}_+^d$ for $G \subset \mathbf{R}^d$, where co G denotes the convex hull of G and $\mathbf{R}_+^d := \{x \in \mathbf{R}^d : x \ge 0\}$.

Lemma 3.2. *If* $1 \le p \le \infty$ *and* $\mathbf{1}/p \in \text{int}\, \mathcal{N}(A)$, *then the following equivalence of quasinorms holds*

$$\|f\|_{H_p^A} \asymp \sup_k 2^{sA(k)} \|U_k f\|_p, \quad f \in H_p^A.$$

P r o o f. From the condition $\mathbf{1}/p \in \text{int}\, \mathcal{N}(A)$ by use of Lemma 3.1 one can prove that H_p^A is compactly embedded into $C(\mathbf{T}^d)$. Let $f \in H_p^A$. By (3.2) we have for any $k \in \mathbf{N}^d$

$$U_k f = \sum_s U_k V_s f.$$

By (2.5) it is not hard to verify that $U_k V_s f \equiv 0$ whenever the inequality $s > k - \mathbf{1}$ does not hold. Therefore,

$$\|U_k f\|_p \le \sum_{s > k - \mathbf{1}} \|U_k V_s f\|_p. \tag{3.6}$$

Lemma 2.3 gives

$$\|U_k V_s f\|_p \ll 2^{\langle 1/p, s-k \rangle} \|V_s f\|_p.$$

Hence by (3.6) and Lemma 3.1 we have

$$\begin{aligned} 2^{sA(k)} \|U_k f\|_p &\ll 2^{sA(k)} \sum_{s > k-1} 2^{\langle 1/p, s-k \rangle} 2^{-sA(s)} \|f\|_{H_p^A} \\ &\ll \|f\|_{H_p^A} 2^{sB(k)} \sum_{s > k-1} 2^{sB(s)}, \end{aligned} \tag{3.7}$$

where $B = A - 1/p$. In virtue of the condition $1/p \in \operatorname{int} \mathcal{N}(A)$ the series in the last expression does not exceed a multiple of $2^{-sB(k)}$. Using this estimate, we obtain from (3.7) the asymptotic inequality

$$\sup_k 2^{sA(k)} \|U_k f\|_p \ll \|f\|_{H_p^A}. \tag{3.8}$$

The inverse asymptotic inequality can be proved in the same way by replacing roles of U_k and V_k by one other.

À inequality similar to (3.8) was proved [T2] for functions of a mixed smoothness $\alpha > 0$, $\alpha \in \mathbf{R}^d$, defined by means of a mixed higher-order finite difference operator.

Let f be a function represented by the series

$$f(x) = \sum_{k \in \mathbf{Z}_+^d} f_k(x),$$

with $f_k \in T_{2^k}$, $k \in \mathbf{Z}_+^d$, satisfying the condition

$$\sum_{k \in \mathbf{Z}_+^d} (2^{(1/p - 1/q)|k|} \|f_k\|_p)^q < \infty$$

for $1 \le p < q < \infty$. Then $f \in L_q$ and

$$\|f\|_q^q \le C_{p,q} \sum_{k \in \mathbf{Z}_+^d} (2^{1/p - 1/q)|k|} \|f_k\|_p)^q. \tag{3.9}$$

Here $|x| := \prod_{j=1}^d x_j$ for $x \in \mathbf{R}_+^d$.

This inequality was proved in [T1] for a diadic decomposition and in [D2] in the general case.

In studying n-widths and other approximation characteristics of classes of functions of a mixed smoothness, the estimates of the approximations often reduce to the problem of determining the asymptotic degrees of the sums

$$Z(tG) := \sum_{k \in \mathbf{N}^d \cap tG} 2^{|k|}, \tag{3.10}$$

depending on a positive parameter t, where $G \subset \mathbf{R}^d_+$ is a convex compact set with non-empty interior, and of the sums of some exponents over integral points in the complements of tG (see (3.16)). We need some results of these estimates in [D2] for estimating the characterictics of optimal recovery of classes of functions of a mixed smoothness. To formulate these results we define some values and functions associated with the following problem of convex programing in $\mathbf{R}^d$:

$$\text{maximize } |x|, \text{ subject to } x \in G.$$

Let $\theta = \theta(G) := \sup\{|x| : x \in G\}$ be the value of the problem $s = s(G) := d-1-r$ and $r = r(G)$ be the linear dimension of the set of solutions; $\varphi(h) = \varphi(G;h)$, $h \geq 0$, be the d-1-dimensional volume of the set $\{x \in G : |x| = \theta - h\}$. It was proved that one can construct a function $\omega(h) = \omega(G;h)$, $h \geq 0$, such that $\omega(h)$ is a concave modulus of continuity if $s > 0$ and $\omega(h) \equiv 1$ if $s = 0$, and the relation

$$\varphi(h) \asymp \omega^s(h), \quad h \to 0, \tag{3.11}$$

holds. Moreover, if G is a polyhedral set, i. e. the intersection of a finite number of half-spaces, then

$$\varphi(h) \asymp h^r, \quad h \to 0. \tag{3.12}$$

Set

$$R(t) = R(G;t) := t^{d-1}\omega^s(1/t). \tag{3.13}$$

We have

$$Z(tG) \asymp 2^{\theta t} R(t), \quad t \to \infty. \tag{3.14}$$

For a given G one can construct a convex compact set $H \subset \mathbf{R}^d_+$ such that $H \supset G$,

$$Z(tH) \asymp Z(tG), \quad t \to \infty, \tag{3.15}$$

and

$$\sum_{k \in \mathbf{N}^d \setminus tH} 2^{-amG(k)} \asymp 2^{-at} R(t), \quad t \to \infty, \tag{3.16}$$

where $mG(x) := \inf\{t > 0 : x/t \in G\}$ is the Minkowski function of G, and $a > 0$ is a fixed number.

We shall take functions from

$$SH_p^A := \{f \in H_p^A \ : \|f\|_{H_p^A} \le 1\},$$

the unit ball of the space H_p^A, for recovering. As auxiliary results, we first establish some asymptotic estimates of n–widths and the best approximation by Fourier sums in L_q of the class SH_p^A for various pairs $1 \le p, q \le \infty$.

Let X be a normed linear space of functions defined in $\mathbf{T}^d$ and $W \subset X$. We recall that the n–width of W in X is

$$d_n(W, X) := \inf_{\dim\ L \le n} E(W, L, X),$$

where L is a linear manifold in X and

$$E(W, L, X) := \sup_{f \in W} \inf_{g \in L} \|f - g\|_X.$$

A characteristic of best approximation by Fourier sums was introduced in [D1]. This quantity is defined by

$$G_n(W, X) := \inf_{\mathrm{card}\ G \le n} \sup_{f \in W} \|f - F_G f\|_X,$$

where G is a finite subset in $\mathbf{Z}^d$, card G is the cardinality of G and

$$F_G f := \sum_{k \in G} f_k e^{i\langle k, . \rangle}.$$

From the definitions it follows that the following the inequality holds

$$d_n(W, X) \le R_n^*(W, X) \le G_n(W, X). \tag{3.17}$$

Set $G_+^0 := \{x \in \mathbf{R}_+^d : \langle x, y \rangle \le 1,\ y \in G\}$ and $C(G) := \sum_{k \in G \cap \mathbf{N}^d} \square_k$ for $G \subset \mathbf{R}^d$, $a_+ := \max(a, 0)$ for $a \in \mathbf{R}$, where

$$\square_k := \{s \in \mathbf{Z}^d :\ 2^{k_j - 1} \le |s_j| \le 2^{k_j},\ \ j = 1, \dots, d\}.$$

Theorem 3.1. *For $A \in \mathbf{R}^d$ and $1 \le p, q \le \infty$ let $B := A - (1/p - 1/q)_+ \mathbf{1}$. Suppose that $0 \in int\ \mathcal{N}(B)$ and one of the following conditions holds:*

(i) $\quad 1 \le q \le p \quad$ and $\quad p \ge 2$

(ii) $1 < p < q < \infty$ or $p = q \le 2$.

Then one can construct a convex compact set $G \subset \mathbf{R}^d_+$ such that card $C(m^\circ G) \le n$ *with some $m^\circ = m^\circ(n)$, and the following relations hold*

$$G_n(SH^A_p, L_q) \asymp \sup_{f \in SH^A_p} \|f - F_{C(mG)} f\|_q$$

$$\asymp n^{-1/\theta} R(\log n)^{1/\theta + 1/q^\circ}, \quad n \to \infty,$$

where $q^\circ = 2$ in the case (i) and $q^\circ = q$ in the case (ii) and $\theta = \theta(B^\circ_+)$, $R(.) = R(B^\circ_+; .)$.

P r o o f. This theorem can be proved in a way completely similar to a proof of an analogous theorem [D1] for the class $SW^r H^\Omega_p$, defined by the mixed modulus of continuity $\Omega(x) = \inf_{\alpha \in A'} x^\alpha$, $A' = A - r$. □

Theorem 3.2. *Under the hypotheses and notations of Theorem 3.2, let the condition (ii) be restricted by (ii'): $1 < p \le q \le 2$. Then*

$$d_n(SH^A_p, L_q) \asymp n^{-1/\theta} \{R(\log n)\}^{1/\theta + 1/q^\circ}, \quad n \to \infty.$$

P r o o f. The upper bound follows from (3.17) and Theorem 3.1. The lower bound in the case (i) can be proved in a way similar to establishing the lower bound in Theorem 1.2 [D1]. The lower bound in the case (ii') was proved in [G] for finite sets A. For arbitrary set A it can be proved analogously. We draw a sketch of proof and refer the reader to [G] for a more detailed proof. Let $T :=$ span $\{e^{i\langle k,.\rangle} : k \in C(S)\}$, $S := \{k \in \mathbf{N}^d : |k| = m, \langle \alpha, k\rangle \le m/\theta + C, \alpha \in B\}$. For a given n, one can choose a value of C such that $n \asymp \dim T = Z(S) \le n$, $n \to \infty$. On the other hand $Z(S) \asymp 2^m R(\log m)$ and $N :=$ card $S \asymp R(\log m)$, $m \to \infty$. Hence, we have

$$d_n(SH^A_p, L_q) \asymp d_n(SH^A_p \cap T, L_q \cap T)$$

$$\asymp 2^{-m/\theta + m/p - m/q} d_n(B^{2^m, N}_p, \ell^{2^m, N}_2) \asymp 2^{-m/\theta} N^{1/q}$$

$$\asymp n^{-1/\theta} R(\log n)^{1/\theta + 1/q}, \quad n \to \infty.$$

Here we use the Littlewood–Paley theorem (cf., e.g. [D1]), Lemma 2.2 and Theorem 2 [G]. For definitions of the set $B^{2^m, N}_p$ and the space $\ell^{2^m, N}_{2,q}$ see also in [G]. □

4. OPTIMAL RECOVERY

From Theorems 3.1-2 and (3.17) follows

Theorem 4.1. *Under the hypotheses and notations of Theorem 3.2 we have*

$$R_n^*(SH_p^A, L_q) \asymp n^{-1/\theta}\{R(\log n)\}^{1/\theta+1/q}, \quad n \to \infty.$$

Theorems 3.1 and 4.1 show that recovering functions in SH_p^A from the n Fourier coefficients associated with the hyperbolic cross $T(C(t^\circ G))$, G constructed in Theorem 3.1. by the corresponding Fourier sum, gives the asymptotic degree of $R_n^*(SH_p^A, L_q)$ in the cases of p and q considered in Theorems 3.1 and 4.1.

To estimate the other charateristic $R_n(SH_p^A, L_q)$, we preliminarily construct a linear method of recovery on basis of the harmonic diadic decomposition (3.3). For $G \subset \mathbf{R}_+^d$ we define the operator S_G by

$$S_G f := \sum_{k \in \mathbf{N}^d \cap G} U_k f$$

for functions f on $\mathbf{T}^d$. The functions $S_G f$ are completely determined from the values of f at the lattice

$$L(G) := \bigcup_{k \in G} \{2^{-k}s/3 \,:\, 0 \le s < 3.2^k\}.$$

The error of recovering f from the values at $L(G)$ by $S_G f$ is estimated by the following

Lemma 4.1. *For any $G \subset \mathbf{R}_+^d$ and $f \in C(\mathbf{T}^d)$ we have*

$$\|f - S_G f\|_q \ll \Big(\sum_{k \in \mathbf{N}^d \setminus G} 2^{(1/q-1/p)_+|k|} \|U_k f\|_p^{q^\circ} \Big)^{1/q^\circ},$$

where $q^\circ = 1$ for $1 \le q \le p \le \infty$ or $q = \infty$ and $q^\circ = q$ for $1 \le p < q < \infty$.

P r o o f. The lemma in the case $q^\circ = 1$ is obvious and in the case $q^\circ = q$ obtained from (3.9). $\square$

Theorem 4.2. *Let $1 \le p, q \le \infty$, $A \subset \mathbf{R}^d$ be a bounded set and $1/p \in$ int $\mathcal{N}(A)$. Then one can construct a convex compact set $G \subset \mathbf{R}_+^d$ such that for a given n card $L(m^\circ G) \le n$ with some $m^\circ = m^\circ(n)$, and the following relations hold*

$$R_n(SH_p^A, L_q) \le \sup_{f\in SH_p^A} \|f - S_{m^\circ G} f\|_q$$

$$\asymp n^{-1/\theta}\{R(\log n)\}^{1/\theta+1/q^\circ}, \quad n \to \infty,$$

where $\theta = \theta(B_+^\circ)$, $R(.) = R(B_+^\circ;.)$ $B = A - (1/p - 1/q)_+\mathbf{1}$ *and* q° *is defined in Lemma 5.1.*

P r o o f. We prove this theorem in the case $1 \le p < q < \infty$, the theorem in the remaining cases can be proved in a similar way. Note that B_+° is a convex compact set because of the definition of B_+° and the boundedness of A. We construct by (3.14-16) a convex compact set $G \subset \mathbf{R}_+^d$ such that $G \supset B_+^\circ$ and

$$\operatorname{card} C(mG) \asymp Z(mG) \asymp 2^{-\theta m} R(m), \quad m \to \infty; \tag{4.1}$$

$$\sum_{k\in \mathbf{N}^d\setminus mG} 2^{-qmB_+^\circ(k)} \asymp 2^{-qm} R(m), \quad m \to \infty. \tag{4.2}$$

By Lemmas 3.2 and 4.1 and the equality $sB(k) = sA(k) - (1/p - 1/q)|k|$, we have for each $f \in SH_p^A$

$$\|f - S_{mG} f\|_q^q \ll \sum_{k\in \mathbf{N}^d\setminus mG} 2^{-qsB(k)}, \quad m \to \infty.$$

The equality $sB(x) = mB_+^\circ(x)$ for $x \in \mathbf{R}_+^d$ and (4.5) give

$$\|f - S_{mG} f\|_q \ll 2^{-m} R^{1/q}(m), \quad m \to \infty, \tag{4.3}$$

for each $f \in SH_p^A$. For a given natural number n, let $m^\circ := \sup\{m : \operatorname{card} C(mG) \le n\}$. Then, from (4.1), (4.3) and properties of the function $R(m)$ it follows that

$$\|f - S_{m^\circ G} f\|_q \ll n^{-1/\theta}\{R(\log n)\}^{1/\theta+1/q}, \quad n \to \infty.$$

Since the function $S_{m^\circ G} f$ belong to the space $T^\circ := T(C(m^\circ G))$ and $\dim T^\circ = \operatorname{card} C(m^\circ G) \le n$, the inequality (4.4) proves the theorem in the case $1 \le p < q < \infty$. □

A similar method for recovering in L_p functions of SH_p^α, $\alpha > 0$, $1 \le p \le \infty$, was constructed in [T2].

From Theorems 4.1 and 3.2 and (1.1) - (1.2) we obtain

Theorem 4.3. *Under the hypotheses and notations of Theorem 4.1 let* $1 < p < q \leq 2$. *Then*

$$R_n(SH_p^A, L_q) \asymp n^{-1/\theta}\{R(\log n)\}^{1/\theta+1/q}, \quad n \to \infty.$$

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