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THE AR-PROBLEM IN LINEAR METRIC SPACES

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1. INTRODUCTION.

We have two aims in this paper: Our first aim is to survey some partial answers to the AR-problem in linear metric spaces obtained mainly by the author during the last few years. The second aim is to supply the reader with some open problemss stemming from the AR-problem. The orginal problems were posed by the founders of functional analysis: Banach and Schauder which are stated as follows:

(i) (Banach). Is every complete linear metric space homeomorphic to a Hilbert space ?

(ii) (Schauder). Has every compact convex set the fixed point property?

We call problems (i) and (ii) Banach-Schauder problems. Banach-Schauder problems were posed in early 1930's but they are still open untill now. It turns out that Banach-Schauder problems in fact come from the AR-problem. We shall discuss this problem in detail in Section 2. It is of interest to know that for more than half a century of being strongly attacked Banach-Schauder problems are still firmly standing, no one has been able to knock them down. Although the final solutions of these problems have not yet been found, the searching for their answers have received marvellous successes: a lot of important results were discovered. For instance, a new branch of mathematics, called infinite dimensional topology, see [BP] [M], was born on the way of searching for a solution to Banach problem. We hope that our list of open questions will provide young researchers a good source of open problems for their research study. These problems are still very active nowadays and play an important role in the development of modern functional analysis and topology.

We have been working on Banach-Schauder problems for several years and have found some partial answers. Our work has also discovered a lot of new problems for further investigation of Banach-Schauder problems. Most of problems discussed in this papers were posed by the author. However we also recall some old problems from other sources in this area in a hope that the reader will have a complete list of open problems which all come from Banach-Schauder problems. We believe that they are very good problems and hope that the answers to many of problems in our list will be soon found by the reader.

For convenience for the reader we state some criteria for recorgnizing ANR-spaces.

Notation and Conventions: In this paper all maps are assumed to be continuous. By a linear metric space we mean a topological linear space X which is metrizable. We write $||x-y|| = \rho(x,y)$, where ρ is an invariant metric, see [Re]. We may assume that $||\cdot||$ is monotonous, that is

$$||\lambda x|| \le ||x|| \text{ for every } x \in X \text{ and } \lambda \in \mathbf{R} \text{ with } |\lambda| \le 1.$$

We call $||\cdot||$ an F-norm.

The zero element of X is denoted by θ. A locally convex space is a linear metric space which possesses a basis of neighbourhoods of θ consisting of convex sets.

Let E be a subset of a linear space X. By conv E we denote the convex hull of E and span E denotes the linear subspace of X spanned by E.

Let E be a subset of a metric space X and $x \in X$. We denote

$$||x - E|| = \inf\{||x-y|| : y \in E\}.$$

We recall that for $p \in [0,1)$ the linear metric space L_p is defined by

$$L_p = \{f : [0,1] \to \mathbf{R}, \int_0^1 |f(t)|^p dt < \infty\} \text{ for } 0 < p < 1 \text{ and}$$

$$L_0 = \{f : [0,1] \to \mathbf{R}, \int_0^1 \frac{|f(t)|}{1+|f(t)|} dt < \infty\}.$$

For other notation, see [Bo][BP].

2. THE AR-PROBLEM.

We say that a metric space X is an ANR if and only if for any metric space Y which contains X topologically as a closed subset there exist a neighbourhood U of X in Y and a map (called a retraction) $r : Y \to X$ such that $r(x) = x$ for every $x \in X$.

We say that X is an AR if in the above definition we can take $U = Y$. The AR-problem in linear metric spaces is stated as follows:

2.1. Problem. (The AR-problem). Is every convex set in a linear metric space an AR? See [G] [W], Problems LS1, LS6.

For locally convex spaces Problem 2.1 was settled affirmatively by Dugundji [D] in 1951. However this problem remains open for non-locally convex linear metric spaces and is one of the most resistant difficult open problems in infinite dimensional topology.

Problem 2.1 is extremely important because of the following two reasons:

The first reason comes from Schauder's conjecture. In 1935 Schauder proved that every compact convex set in a locally convex space has the fixed point property. Schauder conjectured that his theorem holds true without the local convexity.

2.2. Schauder's conjecture. Every compact convex set in a linear metric space has the fixed point property ?

At a first glance one may think that Schauder's conecture is not very hard, however it is one of the most difficult problems in fixed point theory. In fact Schauder posed Problem 2-2 in the Scottish book in 1935 and despite great efforts by topologists for over fifty years his conjecture is still unproved. Schauder's conjecture is still open even in some very special cases: For instance, it not known whether compact convex subsets of the spaces $L_p, 0 \leq p < 1$, have the fixed point property.

Let us observe that the following theorem of Borsuk [Bo] has reduced Schauder's conjecture to the AR-problem:

2.3. Theorem. (Borsuk, 1937) *Every compact AR-space has the fixed point property.*

The second reason comes from the problem of topological classification of convex sets in linear metric spaces which is, in our opinion, even more important than Schauder's conjecture. It asks.

2.4. Problem. (i) Is every infinite dimensional compact convex set in a linear metric space homeomorphic to the Hilbert cube $Q = [0,1]^\infty$?.

(ii) Is every complete separable infinite dimensional linear metric space homeomorphic to a Hilbert space ?

Problem 2.4 was posed by Banach in early 1930's and is the most fundamental question in infinite dimensional topology. In fact, infinite dimensional topology was born on the way of searching for a solution of this problem.

In the late seventies Torunczyk established very powerful characterizations of Hilbert cube manifolds and Hilbert space manifolds which reduce Problem 2.4 to the AR-problem.

2.5. Theorem. [DT1]. *(i) An infinite dimentsional compact convex set X in a linear metric space is homeomorphic to the Hilbert cube if and only if X is an AR.*

(ii) A complete separable linear metric space X is homepmorphic to a Hilbert space if and only if X is an AR.

From Theorems 2.3 and 2.4 it follows that a solution to the AR-problem would solve both Banach problem and Schauder's conjecture which have been open for six decades.

3. A CHARACTERIZATION OF ANR-SPACES.

The problem of detecting the ANR-property of a metric space is, in general, very difficult. There are a lot of criteria for recorgnizing ANR-spaces, see [Bo] [BP] [Hu] [vM]. In this section we present a criterion for ANR-spaces extablished by the author in [N1], see also [N3] [AN]. This charaterization of ANR-space is very convenient in applications and has been used frequently by the author and others to obain new ANR-spaces, see [N1] [N2] [N4] [N5] [N6] [NT3] [NS].

Let X be a metric space. For an open cover $\mathcal{U}$ of X let $\mathcal{N}(\mathcal{U})$ denote the nerve of $\mathcal{U}$ equipped with the *Whitehead topology.* Let $\{\mathcal{U}_n\}$ be a sequence of open covers of X. We say that $\{\mathcal{U}_n\}$ is a *zero sequence* if and only if

$$\sup\{\operatorname{diam} U : U \in \mathcal{U}_n\} \to 0 \text{ as } n \to \infty.$$

We denote

$$\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n \text{ and } \mathcal{K}(\mathcal{U}) = \bigcup_{n=1}^{\infty} \mathcal{N}(\mathcal{U}_n \bigcup \mathcal{U}_{n+1}).$$

For each $\sigma \in \mathcal{K}(\mathcal{U})$ we write

$$n(\sigma) = \sup\{n \in \mathbf{N} : \sigma \in \mathcal{N}(\mathcal{U}_n \bigcup \mathcal{U}_{n+1})\}.$$

We say that a map $f : \mathcal{U} \to X$ is a *selection* if and only $f(U) \in U$ for every $U \in \mathcal{U}$.

The characterization of ANR's established by the author in [N1] is simplified to the following due to observations of J.Luukkainen and K. Sakai:

3.1. Theorem. [N1] [N3] [AN] *A metric space X is an ANR if and only if there exist a zero sequence of open covers $\{\mathcal{U}_n\}$ of X together with a map $f : |\mathcal{K}(\mathcal{U})| \to X$ and a selection $g : \mathcal{U} \to X$ such that for any sequency $\{\sigma_k\}$ with $n(\sigma_k) \to \infty$ we have $diam\{g(\sigma_k^0) \cup f(\sigma_k)\} \to 0$.*

As a consequence of Theorem 3.1 we obtain Dugundji theorem.

3.2. Corrolary. (Dungundji theorem [D]) *Every convex subset of a locally convex space is an AR.*

P r o o f. Let X be a convex set in a locally convex space. Since X is contractible, it suffices to show that X is an ANR, see [Bo]. We shall verify the condition of Theorem 3.1.

Let $\{\mathcal{U}_n\}$ be a sequence of open covers of X consisting of convex subsets of X with the following properties:

(1) diam conv $U < 2^{-n}$ for every $U \in st\mathcal{U}_n$;

(2) $\mathcal{U}_{n+1} < \mathcal{U}_n$ for every $n \in \mathbf{N}$.

Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ and let $g : \mathcal{U} \to X$ be any selection. Using the convexity of X we extend g to a map $f : |\mathcal{K}(\mathcal{U})| \to X$. It is easy to see that the map satisfies the condition of Theorem 3.1, therefore X is an ANR and the corrolary is proved.

4. THE LOCALLY CONVEX APPROXIMATION PROPERTY.

In this section we shall provide some our partial answers to Problem 2-1. Our idea of attacking Problem 2.1 is to approximate convex sets in linear metric spaces by convex sets in locally convex spaces. We introduce the notion of *the locally convex approximation property* (the LCAP) for convex sets in linear metric spaces and prove that the LCAP implies the AR-property. Roughly speaking, our theorem states that if a convex set X can be "approximated", in some sence, by convex subsets in locally convex spaces then X is an AR. In the compact case the LCAP is equivalent to the notion of admissibility introduced by Klee [K1] [K2].

4.1. Definition. [N3] Let us say that a convex set X in a linear metric space is LC-convex if and only if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, X)$ such that for every finite set $A \subset X$ with diam $A < \delta$ we have diam conv $A < \varepsilon$.

Obviously, any convex set in a locally convex space is LC-convex.

4.2. Definition. [N3] We say that a convex set X in a linear metric space Y has the locally convex approximation property (the LCAP) if and only if there exist an F-norm $||.||$ on Y, a sequence $\{X_n\}$ of LC-convex subsets of X and a sequence of continuous maps $r_n : X \to X_n$ such that for some summable sequence $\{a_n\}$ of positive numbers we have

$$(LC) \qquad \lim_{n\to\infty} \inf\ (a_n)^{-1}||x - r_n(x)|| = 0 \text{ for every } x \in X.$$

We have proved the following theorem which indicated the importance of the LCAP for investigating the AR-problem.

4.3. Theorem. [N3] *Any convex set with the LCAP is an AR.*

Our Theorem 4.3 reduces Problem 2.1 to

4.4. Problem. Has every convex set the LCAP ?

Theorem 4.3 also suggests the following problem:

4.5. Problem. Assume that X is a convex set with the LCAP. Is every convex subset of X an AR ?

We shall discuss some applications of Theorem 4.3.

Application 1. The following result is an obvious application of our Theorem 4.3.

4.6. Corrolary. *Any convex set which is a countable union of LC-convex subsets is an AR.*

From Corrolary 4.6 it follows that a positive answer to the following problem would imply the AR-property of all separable convex sets.

4.7. Question. Let X be a convex set in a linear metric space E. Assume that X has the LCAP. Has the closure $\overline{X}$ of X in E the LCAP ?

Application 2. In 1940 Krein and Milman proved the following theorem:

4.8. Theorem. [KM] *Any compact convex set in a locally convex space is the closure convex hull of its extreme points.*

The following question was open for a long time.

4.9. Question. Does Krein-Milman theorem holds true for non-locally convex linear metric spaces ?

In 1976 Roberts constructed a striking example of a linear metric space which contains a compact convex set without any extreme points. Thus the Krein-Milman theorem does not hold true for non-locally convex linear metric spaces.

One may ask whether Krein-Milman theorem holds true for compact convex AR-sets. In other words, is it true that any compact convex AR-set in a linear metric space is the closed convex hull of its extreme points ?. The answer to this question is "no". In fact as an application of our Theorem 4.3 we get the following result which was proved first in [NT].

4.10. Corrolary. [N3] *Some of compact convex sets with no extreme points constructed by Roberts has the LCAP and hence is an AR.*

4.11. Question. Has every compact convex set the LCAP ?

Let us note that by Theorems 4-3 and 2-3 a positive answer to Problem 4-11 would also provide an affirmative solution to Schauder's conjecture.

4.12. Question. Let X denote the linear metric space constructed by Roberts [R1]. We ask:

(i) Has every convex subset of X the LCAP ?
(ii) Has *every compact convex* of X the LCAP ?
(iii) Has *every linear subspace* of X the LCAP ?
(iv) Has *the whole space* X the LCAP ?
(v) Has every compact convex set of X the fixed point property ?

5. AMIDSSIBLE CONVEX SETS.

The notion of admissibility of convex sets in linear metric spaces was introduced by Klee in [K1] [K2] and plays an important role in discovering the AR-property in linear metric space. In this section we shall see that the LCAP is an extension of the notion of admissibility of Klee to the non-compact case.

5.1. Definition. ([K1] [K2]) We say that a convex set X is *admissible* if and only if for every compact subset A of X and for every $\varepsilon > 0$ there is a map f from A into a finite dimensional subset of X such that $||x - f(x)|| < \varepsilon$ for every $x \in A$.

The relation between the LCAP and the admissibility of Klee is established in the following theorem.

5.2. Theorem. [AN] *A compact convex set X is admissible if and only if X has the LCAP.*

Klee [K1] [K2] showed that any convex admissible convex set X has the *compact extension property*, that is, any map into X defined on a compact subset of metric space can be extended to the whole space. Observe that Theorem 4.3 can be thought of as an extension of Klee theorem. We are not able to prove Theorem 5.2 for non-compact convex sets.

5.3. Question. Has every admissible convex set the LCAP ?

The following problem is still open:

5.4. Question. Is every convex set in a linear metric space admidssible ?
Even the following special case of Question 5.4 has no answer.

5.5. Question. Assume that X is an admissble convex set in a linear metric space. Is every convex subset of X admissible ?

6. NEEDLE POINT SPACES.

The idea of Roberts of constructing a compact convex set with no extreme points is to introduce the notion of needle point spaces.

6.1. Definition. [R1] [R2] We say that a non-zero point a in a linear metric space X is a *needle point* if and only if for every $\varepsilon > 0$ there exists a finite set $A(a, \varepsilon) = \{a_1, ..., a_m\}$ satisfying the following conditions:

(i) $||a_i|| < \varepsilon$ for every $i = 1, ..., m$;

(ii) For every $x \in conv(A(a,\varepsilon) \cup \{0\})$ there exists an $\alpha \in [0,1]$ such that $||x - \alpha a|| < \varepsilon$;

(iii) $a = m^{-1}(a_1 + ... + a_m)$.

It is easy to see that condition (iii) can be replaced by the following one.

(iii)' $||a - m^{-1}(a_1 + ... + a_m)|| < \varepsilon$.

A linear metric space X is a *needle point space* if and only if X is a complete separable space in which every non-zero point is a needle point.

Roberts proved the following theorems.

6.2. Theorem. [R2] *Every needle point space contains a compact convex set without any extreme points.*

6.3. Theorem. [R2] *For every $p \in [0,1)$ the space L_p is a needle point space.*

We observe that the proof of Theorem 6.2 is quite simple. So if we have a needle point space at hand we can easily to construct a compact convex set with no extreme points. However it is not easy to give an example of a needle point space: The proof of Theorem 6-3 is far more complicated.

Because Theorem 6.2 is suprisingly interesting and its proof is simple we shall outline the proof. Let X be a needle point space. At first we take any point $a_0 \neq 0$ and let $A_0 = \{a_0\}$.

Assume that $A_n = \{a_1^n, ..., a_{m(n)}^n\}$ has been defined. For every $a \in A_n$ we use Definition 6-1 to take $A(a, \varepsilon_{n+1})$, where

$$\varepsilon_{n+1} = 2^{-n-1}(\text{card } A_n)^{-1}.$$

Put

$$A_{n+1} = \cup\{A(a, \varepsilon_{n+1}) : a \in A_n\};$$

$$A = \overline{\text{conv} \bigcup_{n=0}^{\infty} a_n} \subset X.$$

Observe that A is a compact convex set in X and the only possible extreme point of A is 0. Therefore the set $B = \text{conv } (A \cup (-A))$ is a compact convex set without any extreme points.

For sometime it was hoped that Roberts' example would provide a counter-example to Schauder's conjecture. However this is not the case: In fact in [NT2] it is shown that *all compact convex sets* constructed by Roberst have the fixed point property. Let us observe that in [KPR] it was claimed that all compact convex sets constructed by Roberts' method have the fixed point property but no detail proof was given.

It was proved earlier in [NT1] that every needle point space contains a compact convex AR-set with no extreme points. However let us observe that the proof given in [NT1] has not yet reached *all Roberts' compact convex sets.* Some of Roberts compact convex sets were still standing away from the arguments given in [NT1].

Our result in [NT2] has settled completely the question about the fixed point property for all the compact convex sets constructed by Roberts. However the AR-property of Roberts compact convex sets has not yet been completely settled: The result of [NT2] does not say that all Roberts' compact convex sets are AR's. It seems to the author that the AR-property for all Roberts' compact convex sets can be established by using the arguments given in [NT2]. However this has not yet been done.

After Roberts constructed his example needle point spaces become the most important area for finding a solution of Problem 2.1. Because needle point spaces contain compact convex sets with no extreme points, it is hoped that needle point spaces (and in particular the spaces $L_p, 0 \leq p < 1$) would be a good place for constructing counter-examples to Problems 2.1. The following question arises naturally:

6.4. Problem. Is every convex set in a needle point space an AR ?

7. THE FINITE DIMENSIONAL APPROXIMATION PROPERTY.

Our aim is to search for a solution of Problem 6.4. Again we try to approximate convex sets in needle point spaces by convex sets in finite dimensional spaces. *The finite dimensional approximation property* (the FDAP) introduced in [N3] is the key to this problem. Our results produce linear metric spaces which contain compact convex sets with no extreme points such that all convex subsets of them are AR's. This result extends the earlier theorem established in [NT1].

As we have seen the LCAP is quite useful for detecting the AR-property in linear metric spaces. However it is not strong enough to attack Problem 6.4. For instance we are not able to show that convex subsets of a convex set with the LCAP are AR, see Question 4.5. We shall introduce the notion of the FDAP which is stronger than the LCAP and could be applied to the case of Problem 6.4. In fact applying the FDAP we obtain some partial answers to Problem 6.4 , see Theorem 7.4.

7.1. **Definition.** [N3] Let X be a convex set in a linear metric space Y. We say that X has the finite dimensional approximation property (the FDAP) if and only if there exist an F-norm $\|\ \|$ on Y and a sequence of continuous maps r_n from X into finite dimensional subsets X_n of X such that for some summable sequence $\{a_n\}$ of positive numbers we have.

$$(FD) \qquad \lim_{n\to\infty} \inf (a_n)^{-1} \dim X_n \, \|x - r_n(x)\| = 0 \text{ for every } x \in X.$$

Of course the FDAP is stronger than LCAP. So we also obtained the following stronger theorem.

7.2. **Theorem.** [N3] *Let X be a convex set in a linear metric space. If X has the FDAP then every convex subset $E \subset X$ is an AR. In particular any convex set with the FDAP is an AR.*

Theorem 7.2 reduces Problem 6.4 to

7.3. **Problem.** Has every needle point spaces the FDAP ?

As an application of Theorem 7.2 we obtain the following result which provides a partial answer to Problem 6.4.

7.4. **Theorem.** [N3] *Every needle point space X contains a dense linear subspace $E \subset X$ with the following properties:*

(i) E contains a compact convex set with no extreme points;

(ii) E has the FDAP, therefore every convex subset of E is an AR.

Unfortunately, we are not able to prove that $E = X$. Even the answer to the following question has not yet been found:

7.5. Question. Is there a *complete* linear metric space with properties (i) and (ii) of Theorem 7-4?

We are not able to prove Theorem 7-4 even for the spaces L_p, $0 \leq p < 1$. However for the spaces L_p, $0 \leq p < 1$, we get something more than Theorem 7-4.

7.6. Definition. We say that a subset $D \subset L_p$ is $\prod$-convex if and only if for any $f, g \in D$ and for every $\alpha \in [0,1]$ we have $\prod_\alpha(f,g) \in D$ where $\prod_\alpha(f,g)$ is defined by

$$\prod_\alpha(f,g) = \begin{cases} f(t) \; if \; t \in [0,\alpha); \\ g(t) \; if \; t \in [\alpha, 1]. \end{cases}$$

7.7. Theorem. [N1] [N2] *Every $\prod$-convex subset of L_p, $0 \leq p < 1$, is an AR.*

In particular we have:

7.8. Corrolary. *The whole spaces L_p, $0 \leq p < 1$ are AR.*

The following questions are still unanswered.

7.9. Question. (i) Is every linear subspace of L_p, $0 \leq p < 1$, an AR ?

(ii) Is every *compact convex set* in L_p, $0 \leq p < 1$ an AR ?

(iii) Has every compact convex set in the spaces L_p, $0 \leq p < 1$, the fixed point property ?

Our results provide new examples of convex sets with the AR-property and raise a lot of new problems for further investigation of Problem 2.1, one of the most difficult problems in infinite dimensional topology.

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