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UNIQUENESS OF GLOBAL QUASI-CLASSICAL SOLUTIONS OF THE CAUCHY PROBLEM FOR THE EQUATION $\partial u/\partial t+(\partial u/\partial x)^2=0$ (*)

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Dedicated to the memory of Professor Le Van Thiem

Abstract. *A notion of quasi-classical solutions for the Cauchy problem* $\partial u/\partial t+$ $+(\partial u/\partial x)^2=0$, $u(0,x)=u_0(x)$ *in n-dimensional space* $(n\geq 1)$ *is presented and a uniqueness theorem is established by the method based on the theory of differential inclusions.*

Key words. *Cauchy problems, quasi-classical solutions, multivalued functions, differential inclusions.*

In this paper we study the Cauchy problem for the Hamilton-Jacobi equation $\partial u/\partial t+(\partial u/\partial x)^2=0$ in n-dimensional space $(n\geq 1)$ and present a notion of global quasi-classical solutions for this problem. We establish a uniqueness theorem for global quasi-classical solutions by the method based on the theory of multivalued mappings and differential inclusions. In particular, we give an answer to a problem posed by S.N. Kružkov in [1].

Let T be a positive number, $\Omega_T=(0,T)\times\mathbf{R}^n=\{(t,x)\in\mathbf{R}^{n+1}|\,0<t<T\}$, $\|\cdot\|$ and $<.,.>$ be the norm and the scalar product in $\mathbf{R}^n$ respectively. We consider the Cauchy problem

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$$\frac{\partial u(t,x)}{\partial t} + \left(\frac{\partial u(t,x)}{\partial x}\right)^2 = 0, (t,x) \in \Omega_T, \tag{1}$$

$$u(0,x) = u_0(x), x \in \mathbf{R}^n, \tag{2}$$

where $u_0(.)$ is a known function, $\partial u/\partial x = (\partial u/\partial x_1, ..., \partial u/\partial x_n)$, $(\partial u/\partial x)^2 = (\partial u/\partial x_1)^2 + ... + (\partial u/\partial x_n)^2$.

Definition 1. *A function u in $C^1(\Omega_T) \cap C([0,T) \times \mathbf{R}^n)$ is called a global classical solution of the problem (1), (2) if and only if $u(t,x)$ satisfies (1) everywhere in Ω_T and (2) on $\{t = 0, x \in \mathbf{R}^n\}$.*

In [3] we obtained some new uniqueness results for global classical solutions of Cauchy problems for general Hamilton-Jacobi equations.

Denote by $\mathcal{A}$ the set of all closed sets G in $\mathbf{R}$ with *mes* $(G) = 0$, where *mes* is the Lebesgue measure. The Cantor-set belongs to $\mathcal{A}$ [4]. We remind that the Cantor-set is the set of all numbers of the from

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{3^i},$$

where ε_i is either 0 or 2. It is bounded, complete, nowhere dense on $\mathbf{R}$ and possesses a continuum capacity.

We denote by $Lip(\Omega_T)$ the set of all locally Lipschitz continuous functions u defined in Ω_T, i.e. for any compact set $K \subset \Omega_T$ there exists a number $L \geq 0$ such that:

$$|u(t_1,x_1) - u(t_2,x_2)| \leq L(|t_1 - t_2| + \|x_1 - x_2\|),$$
$$\forall\ (t_1,x_1) \in K, (t_2,x_2) \in K.$$

Further, we set $Lip([0,T) \times \mathbf{R}^n) = Lip(\Omega_T) \cap C([0,T) \times \mathbf{R}^n)$.

Definition 2. *A function u in $Lip([0,T) \times \mathbf{R}^n)$ is called a global quasi-classical solution of (1), (2) if and only if there exists a set $G \in \mathcal{A}$ such that $u \in C^1(((0,T) \setminus G) \times \mathbf{R}^n)$ and $u(t,x)$ satisfies (1) everywhere in $((0,T) \setminus G) \times \mathbf{R}^n$ and (2) on $\{t = 0, x \in \mathbf{R}^n\}$.*

We are now able to formulate the main result in this paper.

Theorem 1. *If u_1 and u_2 are global quasi-classical solutions of the Cauchy problem (1), (2) with*

$$\underset{(t,x)\in\Omega_T}{ess.sup} \|\partial u_i(t,x)/\partial x\| < \infty, i = 1,2,$$

then $u_1(t,x) = u_2(t,x)$ *in* Ω_T.

Remark 1. By virtue of Definition 2, if u is a global quasi-classical solution, there exists at least one interval (α,β) such that $u \in C^1((\alpha,\beta) \times \mathbf{R}^n)$. It is well known that in the case $n = 1$ we have the continuum set of global generalized Lipschitz continuous solution for (1), (2) with $u_0(x) \equiv 0$:

$$v_\lambda(t,x) = \min\{0, \lambda|x| - \lambda^2 t\}, \ \lambda = \text{const} \geq 0.$$

For $\lambda > 0$ we can not find any interval $(\alpha,\beta) \subset (0,T)$, such that $v_\lambda(t,x)$ is differentiable in $(\alpha,\beta) \times \mathbf{R}$, because $v_\lambda(t,x)$ is not differentiable on $|x| = \lambda t$ or on $x = 0, t \in (0,T)$ (see Fig. 1)

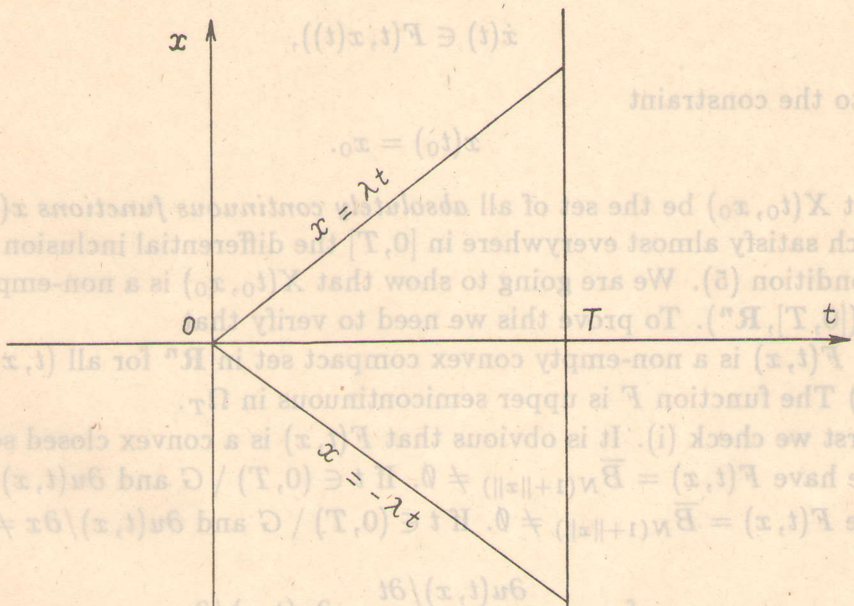

Fig.1

Thus, only the function $v_0(t,x) \equiv 0$ (i.e. $\lambda = 0$) is the unique global quasi-classical solution for (1), (2) with $u_0(x) \equiv 0$.

The proof of Theorem 1 will be based on the following result which is of independent interest, in our opinion.

Theorem 2. *Let* u *be a function in* $C^1(((0,T)\setminus G)\times\mathbf{R}^n) \cap Lip([0,T)\times\mathbf{R}^n)$, *where* $G \in \mathcal{A}$, $u(0,x) \equiv 0$ *on* $\{t = 0, x \in \mathbf{R}^n\}$. *Suppose that there exists a number* $N \geq 0$ *such that for any* $(t,x) \in ((0,T) \setminus G) \times \mathbf{R}^n$:

$$\left|\frac{\partial u(t,x)}{\partial t}\right| \leq N(1 + \|x\|) \left\|\frac{\partial u(t,x)}{\partial x}\right\|. \qquad (3)$$

Then $u(t,x) \equiv 0$ *in* Ω_T.

P r o o f. Let $(t_0, x_0) \in \Omega_T$ be an arbitrary point in $\Omega_T, \overline{B}_r$ be the ball $\overline{B}^n_r = \{f | \, \|f\| \leq r\} \subset \mathbf{R}^n$. We have to show that $u(t_0, x_0) = 0$. For this we define in Ω_T a multivalued function $F : \Omega_T \to \mathbf{R}^n$ in the following way

$$F(t,x) = \begin{cases} \overline{B}_{N(1+\|x\|)}, & t \in G, \\ \{f \in \overline{B}_{N(1+\|x\|)} : \partial u(t,x)/\partial t & + < f, \partial u(t,x)/\partial x >= 0\}, \\ & t \in (0,T) \setminus G. \end{cases}$$

We now consider the differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \tag{4}$$

subject to the constraint

$$x(t_0) = x_0. \tag{5}$$

Let $X(t_0, x_0)$ be the set of all *absolutely continuous functions* $x(\cdot) : [0,T] \to \mathbf{R}^n$, which satisfy almost everywhere in $[0,T]$ the differential inclusion (4) and the initial condition (5). We are going to show that $X(t_0, x_0)$ is a non-empty compact set in $C([0,T], \mathbf{R}^n)$. To prove this we need to verify that

(i) $F(t,x)$ is a non-empty convex compact set in $\mathbf{R}^n$ for all $(t,x) \in \Omega_T$.

(ii) The function F is upper semicontinuous in Ω_T.

First we check (i). It is obvious that $F(t,x)$ is a convex closed set in $\mathbf{R}^n$. If $t \in G$ we have $F(t,x) = \overline{B}_{N(1+\|x\|)} \neq \emptyset$. If $t \in (0,T) \setminus G$ and $\partial u(t,x)/\partial x = 0$ we also have $F(t,x) = \overline{B}_{N(1+\|x\|)} \neq \emptyset$. If $t \in (0,T) \setminus G$ and $\partial u(t,x)/\partial x \neq 0$. We put

$$f = -\frac{\partial u(t,x)/\partial t}{\|\partial u(t,x)/\partial x\|^2} \, \partial u(t,x)/\partial x.$$

By virtue of (3) we obtain

$$\|f\| = \frac{|\partial u(t,x)/\partial t|}{\|\partial u(t,x)/\partial x\|} \leq N(1 + \|x\|).$$

On the other hand,

$$\frac{\partial u(t,x)}{\partial t} + < f, \frac{\partial u(t,x)}{\partial x} >=$$

$$= \frac{\partial u(t,x)}{\partial t} - \frac{\partial u(t,x)/\partial t}{\|\partial u(t,x)/\partial x\|^2} < \partial u/\partial x, \partial u/\partial x >= 0,$$

so $f \in F(t,x)$. Thus we have shown for any $(t,x) \in \Omega_T$ the set $F(t,x)$ is non-empty. Besides that $F(t,x)$ is a convex, closed and bounded subset in $\mathbf{R}^n$. Hence $F(t,x)$ is a compact set in $\mathbf{R}^n$.

To verify (ii) we observe that the function F is bounded in a neighborhood of any $(t,x) \in \Omega_T$, i.e. there exist numbers $l > 0, r > 0$ such that

$$\sup\{\|f\| \mid f \in F(\tau,y), (\tau,y) \in B_l^1(t) \times B_r^n(x) \subset \Omega_T\} < +\infty,$$

where $B_l^1(t)$(resp. $B_r^n(x)$) is an open ball in $\mathbf{R}^1$(resp. $\mathbf{R}^n$) centered in t (resp. x) with radius l (resp. r). In addition, it is easy to see that the function F is closed because for any sequence $(t_k,x_k) \in \Omega_T(k=1,2,...), (t_k,x_k) \to (t,x) \in \Omega_T$, and for any sequence $f_k \in F(t_k,x_k)(k=1,2,...), f_k \to f$, we have $f \in F(t,x)$. Then the function F is upper semicontinuous in Ω_T

Thus, we have shown that the multivalued function F satisfies (i),(ii) and from the definition of $F(t,x)$ we have

$$\sup\{\|f\| \mid f \in F(t,x)\} \leq N(1+\|x\|).$$

By virtue of Theorem 3, p. 206 in [2] the set $X(t_0,x_0)$ of absolutely continuous solutions of (4),(5) is non-empty and compact in $C([0,T],\mathbf{R}^n)$.

Now let $x(.) \in X(t_0,x_0)$. We consider the function $\varphi(t) \equiv u(t,x(t))$. Since $u \in Lip([0,T) \times \mathbf{R}^n)$ and $x(t)$ is absolutely continuous on $[0,T]$, we conclude that $\varphi(.)$ is absolutely continuous on $[\varepsilon, T-\varepsilon]$ for any $\varepsilon \in (0,T/2)$. On the other hand,

$$\dot{\varphi}(t) = \frac{\partial u(t,x)}{\partial t} + < \dot{x}(t), \frac{\partial u(t,x)}{\partial x} >= 0$$

almost everywhere on $[\varepsilon, T-\varepsilon]$. Then $\varphi(t)$ is constant on $[\varepsilon, T-\varepsilon]$. Since ε is an arbitrary positive number and $\varphi(t)$ is continuous at $t=0$, we obtain that $\varphi(t) = \varphi(0) = u(0,x(0)) = 0$ for $t \in [0,T)$. In particular, $\varphi(t_0) = u(t_0,x(t_0)) = u(t_0,x_0) = 0$. The proof of Theorem 2 is complete.

Remark 2. We show by the following example that the Lipschitz continuity of $u(t,x)$ is essential in Theorems 1 and 2

Let $G \subset [0,1]$ be the Cantor set, i.e the set of all numbers in the form

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{3^i},$$

where ϵ_i is either 0 or 2. We define the function $v(.)$ which is called the Cantor ladder in the following way [4]. For $t \in G$ and

$$t = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{3^i}, \quad \varepsilon_i \in \{0,1\}, \quad i=1,2,\ldots$$

we put

$$v(t) = \sum_{i=1}^{\infty} \frac{b_i}{2^i}, \quad b_i = \frac{\varepsilon_i}{2}.$$

If (α, β) is an open maximum interval in $(0,1)\setminus G$ (i.e. $\alpha, \beta \in G$), then $v(\beta) = v(\alpha)$. We set for $t \in (\alpha, \beta) : v(t) = const = v(\alpha) = v(\beta)$. It follows that $v(.) \in C[0,1]$, and $\dot{v}(t) = 0$ almost everywhere in (0,1). In fact, $\dot{v}(t) = 0$, $\forall t \in (0,1) \setminus G$.

Putting $u(t,x) = v(t)$, $(t,x) \in \Omega_1$, we see that $u \in C^1(((0,1) \setminus G) \times \mathbf{R}^n)$, but u does not belong to $Lip([0,1) \times \mathbf{R}^n)$. The function u satisfies the condition (3) in Theorem 2, and $u(0,x) = 0$, for all $x \in \mathbf{R}^n$ but $u(t,x) = v(t) \not\equiv 0$ in Ω_1.

P r o o f of Theorem 1. Consider the function $u(t,x) = u_1(t,x) - u_2(t,x)$, $u(0,x) \equiv 0, x \in \mathbf{R}^n$. Let

$$k = \max_{i=1,2} \left\{ \operatorname*{ess.sup}_{(t,x)\in\Omega_T} \|\partial u_i(t,x)/\partial x\| \right\}.$$

From definition 2, there exist $G_1, G_2 \in \mathcal{A}$ such that

$$u_i \in Lip([0,T) \times \mathbf{R}^n) \cap C^1(((0,T) \setminus G_i) \times \mathbf{R}^n)$$

and u_i satisfies (1) everywhere in $((0,T) \setminus G_i) \times \mathbf{R}^n$. So, $\partial u_i/\partial x_j \in C(((0,T) \setminus G_i) \times \mathbf{R}^n)$. Hence,

$$\operatorname*{ess.sup}_{(t,x)\in\Omega_T} \|\partial u_i(t,x)/\partial x\| = \sup_{(t,x)\in((0,T)\setminus G_i)\times\mathbf{R}^n} \|\partial u_i(t,x)/\partial x\|, i = 1,2.$$

It is easily seen that there exists a constant L such that for all $p_1, p_2 \in \overline{B}_k$

$$\big| \|p_1\|^2 - \|p_2\|^2 \big| \le L\|p_1 - p_2\|.$$

By virtue of the last inequality we have

$$\left|\frac{\partial u(t,x)}{\partial t}\right| \le \left| \left\|\frac{\partial u_1(t,x)}{\partial x}\right\|^2 - \left\|\frac{\partial u_2(t,x)}{\partial x}\right\|^2 \right| \le$$

$$\le L.\left\|\frac{\partial u(t,x)}{\partial x}\right\|,$$

for any $(t,x) \in ((0,T) \setminus G) \times \mathbf{R}^n$, $G = G_1 \cup G_2 \in \mathcal{A}$. Applying Theorem 2 to the function u we obtain that $u(t,x) \equiv 0$ in Ω_T, which proves Theorem 1.

The uniqueness of global quasi-classical solutions of Cauchy problems for general nonlinear partial differential equations of first order will be considered in a forthcoming paper by the method used here.

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