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COFAITHFUL MODULES AND GENERATORS

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Dedicated to the memory of Professor Le Van Thiem

Abstract. *In this article we survey right co-FPF rings, defined to be those rings for which every finitely generated cofaithful right module is a generator. This interesting family of rings was recently introduced by Le van Thuyet as a generalization of the class of right self-injective rings and the class of right FPF rings (those for which every finitely generated faithful module is a generator).*

1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unitary. For a module M over a ring R we write M_R $({}_RM)$ to indicate that M is a right (left) R-module. We denote the category of all right R-modules by Mod-R. The texts by Anderson and Fuller [1], Faith [13], Goodearl and Warfield [18], Rowen [23], Stenström [24] and Wisbauer [29] are general references for ring-theoretic notions not defined in the paper.

If I is a set of cardinality α and M is a module, then we will denote the direct sum of α copies of M by $M^{(I)}$ or $M^{(\alpha)}$. (If I is the empty set then we take $M^{(I)}$ to be the zero module.) The direct product of α copies of M will be denoted by M^I or M^α.

Given two R-modules A and M we say that M is *generated by A* (or A *generates M*) if there is an R-epimorphism from the direct sum $A^{(I)}$ to M for some suitable index set I. The module A_R is called a *generator* of Mod-R (or is said to *generate* Mod-R) if it generates every M_R in Mod-R. A well-known result in ring theory says that every R-module is an epimorphic image of a free R-module and so it follows from this that R_R is a generator of Mod-R. In this survey paper we will examine the structure of rings R for which certain types of R-modules are generators.

The definition of generator can be easily dualised in category theory terms. Thus, given two R-modules B and M we say that M is *cogenerated by* B (or B *cogenerates* M) if there is an R-monomorphism from M to the direct product B^I for some suitable index set I. Similarly, B_R is called a *cogenerator* of Mod-R (or is said to *cogenerate* Mod-R) if it cogenerates every M_R in Mod-R.

For any nonempty subset X of the R-module M_R,the *(right) annihilator* of X is denoted by $r(X)$ and defined to be the right ideal $\{r \in R : Xr = 0\}$. The *left annihilator* of a subset Y of a left R-module is defined similarly and denoted by $l(Y)$. If $r(M)=0$ then M_R is called *faithful*. The right and left annihilators of an element x of a module are denoted by $r(x)$ and $l(x)$ respectively.

If A is a generator of Mod-R then, since R_R is faithful and an epimorphic image of $A^{(I)}$ for some index set I, A is faithful. However a faithful module need not be a generator. Indeed a ring R for which every faithful right R-modunle is a generator is called *right pseudo-Frobenius* (briefly *right PF*) and the following theorem, due to Azumaya [2], Osofsky [20] and Utumi [28] characterizes right PF rings:

Theorem 1.1. *A ring is right PF if and only if it is a right self-injective semiperfect ring with essential right socle.*

A ring is called *quasi-Frobenius* (briefly QF) if it is right self-injective and right Artinian.

Every QF ring is PF. However if R is the trivial extension of the ring $\mathbf{Z}_{(p)}$ by the Prufer group $\mathbf{Z}_{(p^\infty)}$, where p is a fixed prime and $\mathbf{Z}_{(p)}$ is the ring of p-adic integers, then, by Osofsky [20], R is PF but not QF. The long-standing question of whether every right PF ring is also left PF was finally settled in the negative by an example of Dischinger and Müller [11]. In [10] we provided a necessary and sufficient condition for à semiperfect right self-injective ring to be QF.

Faithful modules in Mod-R can be characterized as those modules cogenerate every projective module in Mod-R. (See [1, page 217].) This description allows us to define the following dual notion:

A module M_R is called *cofaithful* if it generates every injective right R-module.

The next result gives an alternative and useful description of cofaithful modules. For its proof see [3, Proposition 4.5.4] or [29, 15.3]. It shows in particular that any cofaithful module is faithful.

Lemma 1.2. *For a right R-module M the following statements are equivalent:*

(a) M is cofaithful.

(b) There is a finite subset F of M such that $r(F) = 0$.

(c) For some positive integer n there is an R-monomorphism from R to the direct sum $M^{(n)}$.

Since the class of faithful modules contains all cofaithful modules the following simple analogue of Theorem 1.1, due to Beachy [3], is of particular interest:

Proposition 1.3. *The ring R is right self-injective if every cofaithful right R-module is a generator.*

We now introduce a generalization of PF rings first considered by Endo [12]. A ring R is called *right finitely pseudo-Frobenius* (briefly *right FPF*) if every finitely generated faithful right R-module is a generator. FPF rings have been the subject of much research and two recent monographs, Faith and Page [15] and Faith and Pillay [16], have been devoted to them. While the general theory still has several major questions unanswered, the class of commutative FPF rings has been completely described. In particular, the next theorem, due to Faith, gives a nice "internal" characterization. The proof of this result takes up much of Faith and Pillay's work [16] and involves an intricate examination of rings lying between the ring R and its maximal quotient ring $Q_{max}(R)$ and the behaviour of submodules of the injective hull $E(R)$ of R.

Theorem 1.4. *A commutative ring R is FPF if and only if it satisfies the following two conditions:*

(i) Every finitely generated faithful ideal of R is a generator.

(ii) The classical ring of questions $Q_{cl}(R)$ of R is self-injective.

In the sections ahead we will describe the structure of special classes of co-FPF rings, in relation to corresponding results for FPF rings.

A ring R is defined to be *right co-FPF* if every finitely generated cofaithful right R-module generates Mod-R.

The class of right co-FPF rings was introduced by Le Van Thuyet in Part III of his Ph.D. thesis [26], written under the direction of Dinh van Huynh, and we will summarize here his main results.

We note that, by Theorem 1.3, every right self-injective ring is right co-FPF and, since every cofaithful module is faithful, every right FPF ring is also right co-FPF. If we let V be an infinite-dimensional vector space over a division ring D and let $S = \text{End}_D(V)$ be the ring of D-linear transformations from V to V then S is a right self-injective ring and so right co-FPF. However one can show that S is not right FPF.

As a simple consequence of Theorem 1.4, we see that **Z**, the ring of integers, is an example of a FPF ring. On the other hand it is easily seen that the direct sum of two co-FPF rings is again co-FPF and so the ring $R = \mathbf{Z} \oplus S$, where S is as above, is a co-FPF ring which is neither self-injective nor FPF. This shows that the class of co-FPF rings properly contains both the class of FPF rings and the class of self-injective rings. This makes the study of co-FPF rings useful both as a class in its own right and as an alternative approach to the study of self-injective rings.

The following result states precisely when co-FPF rings are self-injective.

Proposition 1.5. *A ring is right self-injective if and only if it is right co-FPF and its right injective hull is finitely generated.*

2. Co-FPF RINGS WHICH ARE FPF

In this section we indicate that in a number of circumstances the two classes of co-FPF and FPF rings coincide. The first of these circumstances involve conditions close to commutativity.

A (von Neumann) regular ring is called *strongly* regular if all its idempotent elements are central.

Theorem 2.1. *Let R be a strongly regular ring. Then the following conditions are equipvalent:*

(a) R is right FPF.

(b) R is right co-FPF.

(c) R is right self-injective.

We note that Theorem 2.1 generalizes [15, 3.9B]. Of course, an infinite direct product of skew fields is an example of a non-semisimple ring satisfying the conditions of Theorem 2.1.

A ring R is said to be *strongly right bounded* if every nonzero right ideal of R contains a nonzero two-sided ideal. Now let M be a finitely generated faithful right R-module, say $M = x_1R + \cdots + x_nR$, where R is strongly right bounded. Set $A = r(\{x_1, ..., x_n\})$. If $A \neq 0$, there is a nonzero ideal B of R such that $B \subseteq A$. Then $MB = (x_1R + \cdots + x_nR)B = x_1B + \cdots + x_nB = 0$, a contradiction. Hence $A = 0$ and so M is cofaithful. From this we have the following result which considerably improves [15,5.11D] (by replacing self-injectivity with the co-FPF condition):

Proposition 2.2. *If R is a strongly right bounded right co-FPF ring then R is FPF. In particular any commutative co-FPF ring is FPF.*

If for each prime ideal P of the ring R every essential right ideal of the factor ring R/P contains a nonzero ideal then R is called *right fully bounded.* The abbreviation *right FBN* is commonly used for a right Noetherian right fully bounded ring.

A ring R is said to satisfy the *(Gabriel) H-condition* if for any finitely generated M_R there is a finite subset S of M such that $r(M) = r(S)$. An important result due to Cauchon [9] says that a right Noetherian ring R is right FBN if and only if R satisfies the H-condition. From this we get the following result:

Proposition 2.3. *A right FBN ring is right FPF if and only if it is right co-FPF.*

The next result describes the relationship between right FPF rings and right co-FPF rings under the additional assumption that the ring is prime right Goldie.

Proposition 2.4. *A prime right Goldie ring is right FPF if and only if it is right bounded and right co-FPF.*

We note that, while every right FPF ring is right bounded, this is not the case for right co-FPF rings in general as an example in [26] shows.

Finally in this section we consider some conditions which imply that a right co-FPF ring is QF. The first of these involves the descending chain condition (briefly DCC) on right annihilators.

Theorem 2.5. *(i) The ring R is QF if and only if R is a right co-FPF right QF-3 ring satisfying DCC on right annihilators.*

(ii) The ring R is QF if and only if R is right co-FPF, it satisfies DCC on right annihilators and its injective hull is finitely generated.

The second characterization of QF rings uses chain conditions on essential one-sided ideals. We also note that a right (left) R-module M is said to be *right (left) CS* if every submodule of M is contained as an essential submodule in a direct summand of M. The ring R is called *right (left) CS* if R_R (respectively $_RR$) is a CS module.

Theorem 2.6. *The following conditions are equivalent for a ring R:*

(a) R is QF.

(b) R is right co-FPF and there is a finitely generated quasi-injective cofaithful right R-module with ACC or DCC on essential submodules.

(c) R is a left CS right co-FPF ring and there is a finitely generated continuous cofaithful right R-module with ACC on essential submodules.

(d) R is right co-FPF, it satisfies ACC on essential left or right ideals and its infective hull is finitely generated.

3. Co-FPF RINGS AND THE ILAS CONDITIONS

A ring R is said to satisfy the *intersection left annihilator sum property* (briefly the *ILAS property*) if, whenever X_1 and X_2 are two *right* ideals of R such that $X_1 \cap X_2 = 0$, then $l(X_1)R + l(X_2)R = R$.

We say that R satisfies the *ideal intersection left annihilator sum property* (briefly the *IILAS property*) if in the above definition X_1 and X_2 are restricted to *two-sided* ideals.

Moreover, we say that R satisfies the *finite intersection left annihilator sum property* (briefly the *FILAS property*) if, given any finite set $\{X_i : i = 1, ..., n\}$ of right ideals of R such that $\cap_{i-1}^{n} X_i = 0$, then $R = \sum_{i=1}^{n} l(X_i)R$.

It is shown in [26] that every right co-FPF ring satisfies the ILAS property and more generally the FILAS property. From this and the work of Birkenmeier [5] we get the following result for semiprime rings:

Proposition 3.1. *Let R be a semiprime right co-FPF ring. Then*

(i) R is quasi-Baer,i.e.,the right annihilator of every ideal of R is generated by an idempotent,

(ii) every ideal of R is essential as a right ideal in a ring direct summand of R,

(iii) every ideal of R which is closed as a right ideal in R is a ring direct summand of R,

(iv) R satisfies the IILAS property and

(v) for every ideal X of R the right annihilator $r(X)$ is essential in a direct summand of R.

Moreover rings with the FILAS property are also characterized as those for which the finitely generated cofaithful right R-modules can be used to construct generators, as detailed in the next result:

Theorem 3.2. *The ring R satisfies the FILAS property if and only if, given any finitely generated cofaithful right R-module M with generating set $\{m_1, ..., m_n\}$, then, for any $m_{n+1}, ..., m_t \in M$ such that $\cap_{i=n+1}^{t} r(m_i) = 0$, the external direct sum $\oplus_{i=1}^{t} m_i R$ is a generator of Mod-R.*

While every co-FPF satisfies the FILAS property the converse is not true. Indeed, let F be a field with a proper subfield K, set $F_n = F$ and $K_n = K$ for each $n \in \mathbf{N}$, set $Q = \prod_{n=1}^{\infty} F_n$ and define the ring R by

$$R = \{(x_n) \in Q : x_n \in K_n \text{ for all but a finite number of } n\}.$$

Then (see Goodear [17, Example 13.8]) R is a strongly regular continuous ring which is not self-injective. It follows from the work of Birkenmeier [6, Lemma 2.2] that R satisfies the FILAS condition. However, since R is not-injective, Theorem 2.1 above shows that R is not co-FPF.

A ring R is defined by Birkenmeier [7] to be *right quasi*-FPF if whenever a faithful module M_R is a direct sum of finitely many cyclic modules then M is a generator. Birkenmeier proves the following internal characterization of right quasi-FPF rings.

Theorem 3.3. *A ring R is right quasi-FPF if and only if, given any finite set $\{X_1, ..., X_n\}$ of right ideals of R for which $\cap_{i=1}^n X_i$ contains no nonzero ideal of R, then*

$$R = l\,(\bigcap_{i=1}^{n} X_i)R = \sum_{i=1}^{n} l\,(X_i)R.$$

It follows from this result that every right quasi-FPF ring satisfies the FILAS condition. However the converse to this is not true. In fact if R is an integral domain which is not right Ore domain then its maximal right quotient ring $Q_{max}(R)$ is a right self-injective ring since R is right non-singular. Thus $Q_{max}(R)$ is right co-FPF and so satisfies the FILAS property. Moreover, by [15, Corollary 3.8B], $Q_{max}(R)$ has nonzero nilpotent elements and there is no bound on the nilpotency index of these elements. Using this in conjunction with results of Birkenmeier, it can be shown that $Q_{max}(R)$ is not right quasi-FPF.

The next result provides conditions under which a right co-FPF ring is right quasi-FPF.

Proposition 3.4. *Let R be a ring which has the FILAS property. Then R is right quasi-FPF if it satisfies the following two conditions:*

(i) If X is a right ideal of R containing a nonzero ideal then the right complement of X in R is left faithful.

(ii) $l\,(X_1 \cap X_2)R = l\,(X_1)R + l\,(X_2)R$ for any two right ideals X_1 and X_2 of R.

4. MODULES FLAT OVER THEIR ENDOMORPHISM RINGS

The flatness of certain modules over endomorphism rings can sometimes be used to characterize certain ring properties. For example, let $E(R_R)$ denote the right injective hull of the ring R. Then a result of Camillo and Fuller [8] shows that R is a right self-injective ring if and only if either $E(R_R) \oplus (E(R_R)/R)$ or

$E(R_R) \oplus (E(E(R_R)/R)$ is flat as a left module over its endomorphism ring. (See also [15, Corollary 1.17].)

Moreover, Camillo and Fuller [8] also characterize right PF rings as those for which every quasi-injective faithful right module is finitely generated flat over its endomorphism ring. Similarly there is the following nice description of right FPF rings, given in [15, 1.19]:

Theorem 4.1. *A ring R is right FPF if and only if every finitely generated faithful right R-module is finitely generated flat as a left module over its endomorphism ring.*

There is an analogous result for right co-FPF rings which shows in particular that right self-injective rings possess the above property. We provide the proof of this as a general illustration of the techniques involved in these characterizations.

Theorem 4.2. *A ring R is right co-FPF if and only if every finitely generated cofaithful right R-module is finitely generated flat as a left module over its endomorphism ring. In particular every finity generated cofaithful right module over a right self-injective ring is finitely generated flat over its endomorphism ring.*

P r o o f. The necessity is clear since every generator is a finitely generated projective module over its endomophism ring.

Conversely, let M be a finitely generated cofaithful right R-module. Then, by Lemma 1.2, for some $n > 0$ there is an exact sequence

$$0 \longrightarrow R \xrightarrow{f} M^{(n)}.$$

Set $U = M^{(n)} \oplus (M^{(n)}/f(R))$. Then, since R can be embedded in U, U is also a finitely generated cofaithful right R-module. By our hypothesis, U is then a flat left module over its endomorphism ring. Now define $g \in End_R(U)$ by

$$g(x, y + f(R)) = (0, x + f(R)) \text{ where } x, y \in M^{(n)}.$$

Then we have the exact sequence

$$0 \longrightarrow f(R) \oplus (M^{(n)}/f(R)) \overset{i}{\hookrightarrow} U \xrightarrow{g} U.$$

Now by [8] (or see [15, Proposition 1.15]), U generates the kernel of g. However, $\ker(g) = f(R) \oplus (M^{(n)}/f(R))$ and so, since R is a homomorphic image of $f(R) \oplus (M^{(n)}/f(R))$, it follows that U generates R_R and so is a generator. Finally, from the exact sequence

$$M^{(2n)} = M^{(n)} \oplus M^{(n)} \longrightarrow M^{(n)} \oplus (M^{(n)}/f(R)) = U \longrightarrow 0$$

we see that M is also a generator, completing the proof.

As a consequence of the theorem we have:

Corollary 4.3. *For a ring R the following statements are equivalent:*

(a) R is right co-FPF.

(b) Every finitely generated cofaithful right R-module is projective over its endomorphism ring.

(c) Every finitely generated cofaithful right R-module is (finitely generated) flat over its endomorphism ring.

(d) If M is a finitely generated cofaithful right R-module then the left $End(M)$-module $Hom_{End(M)}(End(M), M)$ is injective.

5. SEMIPERFECT co-FPF RINGS

To investigate the theory of co-FPF rings in more detail, in this section we restrict our attention to the class of semiperfect rings. Here the theory closely parallels that of FPF rings as detailed in the monograph by Faith and Page [15]. However, as we have emphasised above, the co-FPF development has the advantage that it automatically applies to self-injective rings. Moreover the co-FPF approach also leads to distinct improvements in several FPF results.

The first result we mention is fundamental to the investigation.

Theorem 5.1. *A ring R is right co-FPF if and only if R is Morita equivalent to a right co-FPF ring.*

As a consequence of this we have the following two corollaies:

Corollary 5.2. *A ring R is right co-FPF if and only if for each $n > 0$, the ring $M_n(R)$ of $n \times n$ matrices over R is right co-FPF.*

Corollary 5.3. *A semiperfect ring is right co-FPF if and only if its basic ring is right co-FPF.*

The next result gives specific information about the internal structure of semiperfect right co-FPF rings.

Theorem 5.4. *Let R be a semiperfect right co-FPF ring. Then eR is a uniform right ideal for each primitive idempotent e of R. Consequently, R is a direct sum of uniform right ideals.*

Corollary 5.5. *Let I be an ideal of the semiperfect right co-FPF ring R. If I is closed as a right ideal then it is a direct summand of R_R.*

The following improves [15, Proposition 2.2A and 2.2B].

Theorem 5.6. *Let R be a semiperfect right co-FPF ring.*

(i) If the basic ring of R is strongly right bounded then R is a right FPF ring.

(ii) If the Jacobson radical of R is nil then R is right self-injective.

Corollary 5.7. *Let R be a local ring. Then the following conditions are equivalent:*

(a) R is a right co-FPF ring and every element in the Jacobson radical of R is a zero divisor.

(b) R is a right self-injective ring.

The question now arises as to whether every semiperfect right co-FPF ring is in fact right FPF. As a consequence of part (ii) of the above result we have the following partial answer to this. It is a generalisation of a well-known theorem of Tachikawa [25]. (See also [15, Theorem 2.2E].)

Theorem 5.8. *Let R be a left perfect ring. Then the following conditions are equivalent:*

(a) R is a right PF ring.

(b) R is a right FPF ring.

(c) R is a right co-FPF ring.

Corollary 5.9. *A left co-FPF right PF ring with nil Jacobson radical is left PF.*

The next corollary extends a well-known result on self-injective rings of Osofsky[20] and Kato [19] to co-FPF rings.

Corollary 5.10. *A left perfect right and left co-FPF ring is QF.*

It is well-known that a left Noetherian right or left self-injective ring is QF (see[13, Theorem 24.5]). Replacing the self-injectivity by co-FPF we are able to get the following theorem. It generalizes an interesting result due to S.Page [22] from FPF rings to co-FPF rings.

Theorem 5.11. *If R is a left Noetherian right co-FPF semiperfect ring then the classical left quotient ring of R is a QF ring.*

The proof of this theorem requires several lemmas of independent interest and for this reason we shall give them below. First we need some notation.

For a ring R we denote the maximal right quotient ring of R by $Q^r_{max}(R)$. We let $Q^r_{cl}(R)$ and $Q^l_{cl}(R)$ denote its classical right and left quotient rings respectively. The right injective hull of R is denoted by $E(R_R)$. The singular submodule of an R-module M is denoted by $Z(M)$ while its Jacobson radical is denoted by $J(M)$.

Lemma 5.12. *Let R be a semiperfect right co-FPF ring and let M_R be a finitely generated submodule of $E(R_R)$ which contains R. Then*

(i) M is projective and

(ii) if R also satisfies ACC on left annihilators then M is isomorphic to R_R.

As a consequece of this lemma we have the following corollary:

Corollary 5.13. *Let R be a semiperfect right co-FPF ring satisfying ACC on left annihilators. Then*

(i) $qQ^r_{max}(R) + Q^r_{max}(R) \simeq Q^r_{max}(R)$ for each $q \in E(R_R)$,

(ii) $Z(E(R_R)) = Z(R_R)$ and

(iii) if $E(R_R) = Q^r_{max}(R)$ then $Z(R_R) = J(E(R_R)) \subseteq J(R)$.

Lemma 5.14. *Let R be a semiperfect right co-FPF ring satisfying ACC on left annihilators. Then*

(i) every right regular element of R is left regular and a unit in $Q^r_{max}(R)$ and

(ii) $Q^r_{max}(R) = Q^l_{cl}(R) = E(R_R)$.

Using the above results on semiperfect co-FPF rings one may now prove the following theorem which provides criteria for a semiperfect ring to be right FPF.

Theorem 5.15. *Let R be a semiperfect ring such that*

(i) $Q^r_{max}(R) = Q^r_{cl}(R) = E(R_R) = Q^l_{cl}(R)$,

(ii) the basic ring of R is strongly right bounded and

(iii) every finitely generated right ideal of R containing a regular element is a generator. Then R is right FPF.

We note that this theorem was proved by Page [21] under the additional assumption that $Q^r_{max}(R)$ is right FPF.

6. SEMIPRIME AND NONSINGULAR co-FPF RINGS

Recall that the right singular ideal of a ring R is denoted by $Z(R_R)$ and is defined to be the set of all elements x of R for which the right annihilator $r(x)$ is an essential right ideal. R is called *right nonsingular* if $Z(R_R) = 0$.

An important result in the theory of FPF rings, due to S. Page [21], says that a right FPF ring is semiprime if and only if it is right nonsingular. It is an open question as to whether the co-FPF analogue of this result is true. In this section we will present some results that relate to this question. The first of these is a generalization of a result of G.F.Birkenmeier [6] which describes right annihilators in semiprime rings in the presence of the IILAS condition.

Proposition 6.1. *A semiprime ring R satisfies the IILAS condition if and only if the right annihilator of every right ideal is generated by an idempotent.*

By Proposition 3.1 a semiprime right co-FPF ring satisfying the IILAS condition is quasi-Baer. This can be used to deduce the following result from Proposition 6.1.

Proposition 6.2. *Let R be a semiprime right co-FPF ring. If R satisfies DCC on right annihilators then R is right nonsingular.*

The next result gives a partial converse to the above.

Proposition 6.3. *Let R be a right nonsingular right co-FPF ring. If R satisfies DCC on right annihilators and* $r(l(X)) = X$ *for each right ideal X then R is semiprime.*

Now let R be a semiprime right FPF ring without an infinite set of orthogonal central idempotents. Then a decomposition theorem of [15, Theorem 3.4] says that R is a direct sum of finitely many prime (right FPF) rings. The following analogue of this shows in particular that the study of semiprime right self-infective rings without infinitely many orthogonal central idempotents reduces to prime right self-injective rings.

Theorem 6.4. *Let R be a semiprime right co-FPF ring without an infinite set of orthogonal central idempotents. Then R is a direct sum of finitely many prime right co-FPF rings.*

C. Faith and S. Page [15, Theorem 5.1] have also proved an important splitting theorem for FPF rings. The final result in this survey is the corresponding result for co-FPF rings. We note that this contains a splitting property for self-injective rings.

Theorem 6.5. *Let R be a right co-FPF ring. Then there is a two-sided ideal A of R such that*

(i) $R_R = A \oplus B$ *for some right ideal B,*

(ii) $Z(R_R)$ *is essential in* A_R *and*

(iii) the factor ring R/A is right nonsingular and right co-FPF.

If R is also left co-FPF then B is an ideal of R which is a right and left nonsingular *co-FPF ring.*

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REFERENCES

1. F.W. Anderson and K.R. Fuller: *Rings and Categories of Modules*, Springer-Verlag, Berlin, 1974.
2. G. Azumaya: Completely faithful modules and self-injective rings, *Nagoya J. Math.* **27** (1966), 697 - 708.
3. J.A. Beachy: Generating and cogenerating structures, *Trans. Amer. Math. Soc.* **158** (1971), 75-92.
4. J.A. Beachy and W.D.Blair: Rings whose faithful left ideals are cofaithful, *Pacific J. Math.* **58** (1975), 1-13.
5. G.F. Berkenmeier: Idempotents and completely semiprime ideals, *Comm. Algebra* **11** (1983), 567-580.
6. G.F. Birkenmeier: A generalization of FPF rings, *Comm. Algebra* **17** (1983), 855-884.
7. G.F. Birkenmeier: Rings generated by faithful direct sums, *Tamkang J. Math.* **20** (1989), 341-346.
8. F.W. Camillo and K.R.Fuller: Rings whose faithful modules are flat over their endomorphism rings, *Arch. Math.* **27** (1976), 522-525.
9. G. Cauchon: Les T-anneaux, la condition (H) de Gabriel et ses cos'equences, *Comm. Algebra* **4** (1976), 11-50.
10. J. Clark and D.V. Huynh: When is a self-injective semiperfect ring quasi-Frobenius ?, *Research paper no. 2, Dept. Math. and Stat., Univ. of Otago*, 1992.
11. F. Dischinger and W. Müller: Left PF is not right PF, *Comm. Algebra* **14** (1986), 1223-1227.
12. S.Endo: Completely faithful modules and quasi-Frobenius algebras, *J.Math. Soc. Japan* **19** (1967), 437-456.
13. C.Faith: *Algebra II: Ring Theory*, Springer - Verlag, Berlin, 1976.
14. C. Faith: Self-injective rings, *Proc. Amer. Math. Soc.* **77**(1979), 157-164.

15. C. Faith and S. Page: *FPF Ring Theory. Faithful modules and generators of mod-R*, London Math. Soc. Lecture Note Series **88**, Cambridge, 1984.
16. C. Faith and P.Pillay: *Classification of Commutative FPF Rings*, Notas de Matematica 4, Universidad de Murcia, 1990.
17. K.R. Goodearl: *Von Neumann Regular Rings*, Monographs Stud. Math. **4**, Pitman, London, 1979.
18. K.R. Goodearl and R.B. Warfield: *An Introduction to Noncommutative Noetherian Rings*, London Math. Soc. Student Texts **16**, Cambridge, 1989.
19. T. Kato: Self-injective rings, *Tohoku Math. J.* **19**(1967), 485-495.
20. B. Osofsky: A generalization of quasi-Frobenius rings, *J. Algebra* **4**(1966), 373-387.
21. S.S. Page: Semi-prime and non-singular FPF rings, *Comm. Algebra* **10** (1982), 2253-2259.
22. S.S. Page: Semiperfect FPF ring, *Proc. Amer. Math. Soc.* **89**(1983), 395-401.
23. L.H. Rowen: *Ring Theory. Volumes I, II*, Academic Press, London, 1988.
24. B. Stenström: *Ring of Quotients*, Springer-Verlag, Berlin, 1975.
25. H. Tachikawa: A generalization of quasi-Frobenius rings, *Proc. Amer. Math. Soc.***20**(1969), 471-476.
26. L.V. Thuyet: Quasi-Frobenius rings and generalizations, *Ph. D. thesis submitted to the Institute of Mathematics, Hanoi*, 1991.
27. L.V. Thuyet: On continuous rings with chain conditions, *(Vietnam) J.Math.* **19**(1991), 49-59.
28. Y. Utumi: Self-injective rings, *J. Algebra* **6**(1967), 56-64.
29. R. Wisbauer: *Foundations of Module and Ring Theory.* A Handbook for Study and Research, Gordon and Breach, Reading, 1991.

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