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ON A FREE BOUNDARY PROBLEM ARISING IN THE NONCATALYTIC GAS-SOLID REACTION

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Dedicated to the memory of Professor Le Van Thiem

Abstract. *A free boundary problem arising in the noncatalytic gas-solid reaction is considered. The local existence of solutions is established by using the classical fixed point argument.*

1. INTRODUCTION

In [6] the following problem was investigated: Find a triple $(T, s(t), u(x,t))$ such that $T > 0, s(t) \in C^1[0,T]$, $u(x,t) \in C^{2,1}(D_T) \cap C(\overline{D}_T)$, where $D_T = \{(x,t): 0 < x < s(t), 0 < t < T\}$, u_x is continuous up to the boundary $x = s(t)$ and such that

$$u_{xx} - u_t = 0 \text{ in } D_T, \tag{1.1}$$

$$u(0,t) = v_0, \qquad 0 < t < T \tag{1.2}$$

$$s(0) = 0, \tag{1.3}$$

$$u_x(s(t),t) = g(u(s(t),t)), \qquad 0 < t < T, \tag{1.4}$$

$$\dot{s}(t) = f(u(s(t),t)), \qquad 0 < t < T, \tag{1.5}$$

where v_0 is a given positive constant, f and g are given functions.

This problem is a mathematical model of an isothermal diffusion-reaction process of a gas with a solid. We assume that the solid has a very low permeability and is chemically attacked from the surface with a quick and irreversible reaction. As a result of the chemical reaction an inert layer is formed, which is permeable to the gas and the process will exhibit a free boundary $x = s(t)$ (the reaction front), $u(x,t)$ represents the gas concentration. The condition (1.4) describes an empirical law, which relates the rate of mass consumption of the gas and the incoming flux of the gas. The condition (1.5) states the same balance in terms of free boundary velocity. In Wen's model the empirical law is described by

$$g(x) = -x^n = -f(x),$$

(see [7]). In Langmuir model

$$g(x) = -\frac{ax^n}{b + cx^n} = -f(x),$$

where a, b, c are positive constants, $n > 0$ (see [2]).

In this paper we consider the following Wen-Langmuir-like model:

Problem I. Find a triple $(T, s(t), u(x,t))$ such that $T > 0, s(t) \in C^1[0,T]$, $u(x,t) \in C^{2,1}(D_T) \cap C(\overline{D}_T)$, where $D_T = \{(x,t : 0 < x < s(t), 0 < t < T\}$, u_x is continuous up to the boundary $x = s(t)$, and such that

$$u_{xx} - u_t = 0 \qquad \text{in } D_T, \tag{1.6}$$

$$u(0,t) = v_0, \qquad 0 < t < T, \tag{1.7}$$

$$s(0) = 0, \tag{1.8}$$

$$u_x(s(t),t) = g(u(s(t),t),t), \qquad 0 < t < T, \tag{1.9}$$

$$\dot{s}(t) = f(u(s(t),t), s(t)), \qquad 0 < t < T, \tag{1.10}$$

where f and g are given functions.

The following assumptions are made:

$$f \in C^1((R^+)^2), f_v(v,r) > 0 \text{ and } F \geq f_r(v,r) > 0 \text{ for } v \geq 0, r \geq 0,$$

$$P \geq f(v_0, r) \text{ for } r \geq 0, f(0,0) = 0, \ F \text{ and } P \text{ being positive constants,} \tag{1.11}$$

$$g \in C^1((R^+)^2), \ g_v(v,t) < 0 \text{ and } g_t(v,t) < 0 \text{ for } v \geq 0, t \geq 0,$$

$$g(v_0,t) \geq -G \text{ and } g(0,t) = 0 \text{ for } t \geq 0, G \text{ being a positive constant.} \tag{1.12}$$

Moreover, there exist positive constants f_0, g_0 such that

$$|f(v_2,r) - f(v_1,r)| \leq f_0|v_2 - v_1|, \tag{1.13}$$

for $v_1, v_2 \in \left[\frac{v_0}{2}, v_0\right]$, $r \in R^+$,

$$|g(v_2,t) - g(v_1,t)| \leq g_0|v_2 - v_1| \text{ for } v_1, v_2 \in \left[\frac{v_0}{2}, v_0\right], \ t \in R^+. \tag{1.14}$$

2. AUXILIARY PROBLEMS

1. Case 1: $s(0) = b > 0$.

For each Lipschitz continuous function $s(t)$, satisfying $s(0) = b > 0$, we consider the following problem:

Problem II. Find the function $u(x,t)$ such that

$$u_{xx} - u_t = 0 \quad \text{in } D_T, \tag{2.1}$$

$$u(0,t) = v_0(t), \qquad 0 < t < T, \tag{2.2}$$

$$u(x,0) = \psi(x), \qquad 0 \le x \le b, \tag{2.3}$$

$$u_x(s(t),t) = g(u(s(t),t),t), \qquad 0 < t < T. \tag{2.4}$$

(For the sake of completeness we consider $v_0(t)$ instead of v_0 in the right hand side of (2.2).) We shall prove the following:

Theorem 2.1. *Under the hypotheses*

i) $\exists L > 0 \ \ |s(t_1) - s(t_2)| \le L|t_1 - t_2|, \forall t_1, t_2 \in [0,T]$, $0 < a_0 \le s(t) \le A_0$, $\forall t \in [0,T], a_0, A_0$ *are constants,*

ii) $\psi \in C[a,b], \psi(0) = v_0(0)$, $\psi(x) > 0$ *in* $[a,b]$, $\psi' \in C[b-\varepsilon, b]$ *for* $\varepsilon > 0$, $\psi'(b) \le 0$,

iii) $g(v,t)$ *is a strictly decreasing function with respect to* v *for* $v \ge 0, t \ge 0$, *satisfying (1.14) and* $g(0,t) = 0$ *for* $t \ge 0$,

iv) $v_0 \in C[0,T], v_0(t) > 0$ *in* $[0,T]$, $\max\limits_{0 \le t \le T} v_0(t) \ge \max\limits_{0 \le x \le b} \psi(x)$,

there exists a unique solution of Problem II.

P r o o f. a) First, we prove the following a priori estimate for the solution u of Problem II:

$$0 < u(x,t) \le \max_{0 \le t \le T} v_0(t) \text{ in } \overline{D}_T. \tag{2.5}$$

The right hand side inequality of (2.5) follows from the maximum principle and the fact that, $g < 0$. Let $T_0 > 0$ be the first time such that $u(s(T_0),T_0) = 0$, $0 < T_0 \le T$. By the strong maximum principle (see [5]), we get $u_x(s(T_0),T_0) < 0$, which contradicts $u_x(s(T_0),T_0) = g(u(s(T_0),T_0),T_0) = g(0,T_0) = 0$. It follows that $u(s(t),t) > 0, 0 < t \le T$. Using the maximum principle we get $u(x,t) > 0$ in $\overline{D}_T$. This proves (2.5).

b) Uniqueness. It follows from (iii) and the maximum principle.

c) Existence. For each given function $h(t) \in C[0,T]$ with $h \ge 0$ and $g(h(0),0) = \psi'(b)$, there exists a unique solution of the following problem:

$$\begin{aligned} v_{xx} - v_t &= 0 \quad \text{in } D_T, \\ v(0,t) &= v_0(t), \qquad 0 < t < T, \\ v(x,0) &= \psi(x), \qquad 0 \le x \le b, \\ v_x(s(t),t) &= g(h(t),t) = H(t), \quad 0 < t < T. \end{aligned}$$

(see [6]). Moreover, the solution v is continuous in $\overline{D}_T$. Hence, for every $h \in C[0,T]$ we can define $\tilde{h}(t) \equiv v(s(t),t) \in C[0,T]$ and therefore we have the application

$$F_1 : h(t) \in C[0,T] \longmapsto \tilde{h}(t) \in C[0,T]. \tag{2.6}$$

By an argument used in [4], we can prove that F_1 is a contractive mapping of $C[0,T]$ into itselt. Indeed, there exists an increasing continuous function $Q(T)$, vanishing for $T = 0$ and depending continuously on the parameters a_0, A_0, L, g_0 such that

$$\|\tilde{h}_2 - \tilde{h}_1\|_t \le Q(T)\|h_2 - h_1\|_t, \text{ for } t \in [0,T]$$

where $\|\varphi\|_t = \max\limits_{0 \le \tau \le T} |\phi(\tau)|$.

Hence, there exists $T_0 = T_0(a_0, A_0, L, g_0) > 0$ such that $Q(T) \le Q(T_0) < 1$ for all $T \le T_0$ and then F_1 is a contractive mapping and its fixed point is a solution of Problem II. Moreover, $Q(t)$ and T_0 do not depend on the data $\psi(x)$ and $v_0(t)$, so that the same method can be repeated in $[T_0, 2T_0]$. Thus, there exists a unique solution of Problem II for any $T > 0$.

2. Case 2: $s(0) = b = 0$.

For each given function $s(t) \in C^1[0,T]$ with $s(0) = 0$ we consider the following problem:

Problem III. Find a function $u(x,t) \in C^{2,1}(D_T) \cap C(\overline{D}_T)$ such that:

$$u_{xx} - u_t = 0 \qquad \text{in } D_T, \tag{2.7}$$

$$u(0,t) = v_0, \qquad 0 < t < T, \tag{2.8}$$

$$u_x(s(t),t) = g(u(s(t),t),t), \quad 0 < t < T, \tag{2.9}$$

where v_0 is a positive constant and g satisfies (1.12).

We have the following a priori estimates:

Lemma 2.1 *i)*

$$0 \le u(x,t) \le v_0 \quad in\ \overline{D}_T, \tag{2.10}$$

$$-G \le u_x(x,t) \le 0 \quad in\ \overline{D}_T. \tag{2.11}$$

ii) If s also satisfies the condition

$$\exists K_2 > 0, s(t) \le K_2 t, \forall t \in (0,t_0), t_0 = \frac{v_0}{2K_2G} > 0,$$

then u verifies

$$0 < \frac{v_0}{2} \le u(x,t) \le v_0 \ in\ \overline{D}_{t_0}, \tag{2.12}$$

$$-G \le u_x(x,t) \le g(\frac{v_0}{2},0) < 0\ in\ \overline{D}_{t_0}. \tag{2.13}$$

P r o o f. i) The proof of (2.10) is similar to the one of (2.5).

We get (2.11) by using the maximum principle and the fact that $g < 0$, $g_v < 0, u_{xx}(0,t) = 0$ in $(0,t_0]$.

ii) For $(x,t) \in D_{t_0}$, we get

$$u(x,t) = v_0 + \int_0^x u_y(y,t)dy \ge v_0 + s(t). \min_{0\le t\le t_0} g(v_0,t) \ge$$
$$\ge v_0 - GK_2t \ge v_0 - GK_2t_0 = \frac{v_0}{2}.$$

By using the maximum principle and the fact that $g_v < 0, u_{xx}(0,t) = 0$ in $(0,t_0]$ we obtain the right hand side of (2.13).

As in [4] we can establish the following:

Lemma 2.2. *If $v_0(t) \in C[0,T], v_0(t) > 0$ in $[0,T], g(v,t)$ is a continuous function with respect to v, for $v \ge 0, t \in [0,T]$ and $s \in C[0,T]$ with $s(0) = 0$, then there exists $t' \in (0,T)$ such that the equation*

$$f(y,t) \equiv y - v_0(t) - g(y,t)s(t) = 0$$

has at least one solution y for each $t \in (0,t')$.

Moreover, the function $y_0(t) > 0$ can be defined in $(0,t')$ such that

$$f(y_0(t),t) = 0 \ and \ \lim_{t\to 0^+} y_0(t) = v_0(0).$$

Theorem 2.2. *If g verifies (1.12), $s \in C^1[0,T], s(0) = 0$ and $s(t) \geq K_1 t, (K_1 > 0)$ in $[0,T]$, then there exists a unique solution of Problem III for a suitable small $T > 0$.*

P r o o f. a) The uniqueness is deduced as in the proof of Theorem 2.1.
b) Existence. We introduce a decreasing sequence $\{t_n\}$ such that

$$T > t' > t_1 > t_2 > \ldots > t_n > \ldots, \lim_{n\to\infty} t_n = 0,$$

where t' is defined in Lemma 2.2. We define the sequence $\{u_n\}$ such that $u_n = u_n(x,t)$ is the unique solution of the following problem:

$$u_{n_{xx}} - u_{n_t} = 0 \text{ in } D_{n,T}, \tag{2.14}$$

$$u_n(0,t) = v_0, \qquad t_n < t < T, \tag{2.15}$$

$$u_{n_x}(s(t),t) = g(u_n(s(t),t),t), \quad t_n < t < T, \tag{2.16}$$

$$u_n(x,t_n) = \psi_n(x), \quad 0 \leq x \leq s(t_n), \tag{2.17}$$

where $D_{n,T} = \{(x,t) : 0 < x < s(t), t_n < t < T\}$, and

$$\psi_n(x) = v_0 + g(\psi_n(s(t_n)), t_n)x. \tag{2.18}$$

which is justified by Lemma 2.2 and choosing $\psi_n(s(t_n)) = y_0(t_n) > 0$ for each n that satisfies

$$\lim_{n\to\infty} \psi_n(s(t_n)) = v_0.$$

Let z_n be a solution of the following problem:

$$z_{n_{xx}} - z_{n_t} = 0 \qquad \text{in } D_{n_t}, \tag{2.19}$$

$$z_n(0,t) = 0, \qquad t_n < t < T, \tag{2.20}$$

$$z_n(x,t_n) = \psi_n''(x) = 0, \quad 0 \leq x \leq s(t_n), \tag{2.21}$$

$$\begin{aligned} z_{n_x}(s(t),t) + \dot{s}(t) z_n(s(t),t) = g_v(\gamma(t),t)[\dot{s}(t)g(\gamma(t),t) + \\ + z_n(s(t),t)] + g_t(\gamma(t),t), \quad t_n < t < T, \end{aligned} \tag{2.22}$$

$$\begin{aligned} \gamma(t) = \int_{t_n}^{t} [\dot{s}(\tau)g(\gamma(\tau),\tau) + z_n(s(\tau),\tau)]d\tau + \\ + \psi_n(s(t_n)), \qquad t_n < t < T. \end{aligned} \tag{2.23}$$

As in [1] we can see that there exists a small enough $T_1 > 0$ such that

$$\|z_n\|_{D_{n,T_1}} \le \sup_{t_n \le t \le T_1} \dot{s}(t). \sup_{\substack{\frac{V_0}{2} < V < V_0 \\ t_n \le \tau < T-1}} |g(v,\tau)| \le \text{const.} \tag{2.24}$$

Define $\tilde{u}_n(x,t)$ in $D_{n,T}(T \le T_1)$ by putting

$$\tilde{u}_n(x,t) = v_0 + x[g(\psi_n(s(t_n)),t_n) + \int_{t_n}^{t} z_{n_x}(0,\tau)d\tau] +$$

$$+ \int_0^x d\xi \int_0^\xi z_n(y,t)dy. \tag{2.25}$$

We have the following properties:

i) $\tilde{u}_{n_{xx}}(x,t) = \tilde{u}_{n_t}(x,t) = z_n(x,t)$ in $D_{n,T}$.

ii) $\tilde{u}_n(0,t) = v_0,\ 0 < t < T$.

iii) $\tilde{u}_n(x,t_n) = v_0 + xg(\psi_n(s(t_n)),t_n) = \psi_n(x),\ 0 \le x \le s(t_n)$.

iv) $\tilde{u}_{n_x}(s(t),t) = g(\psi_n(s(t_n)),t_n) + \int_{t_n}^t z_{n_x}(0,\tau)d\tau + \int_0^{s(t)} z_n(x,t)dx = g(\psi_n(s(t_n)),t_n) + \int_0^t \frac{d}{d\tau}g(\gamma(\tau),\tau)d\tau = g(\gamma(t),t),\quad 0 < t < T$. (It follows by using Stoke's theorem : for $t \in (t_n,T)$ we get $0 = \iint\limits_{D_{n,t}} (z_{n_{xx}} - z_{n_\tau})dxd\tau = \int\limits_{D_{n,t}} z_n dx + z_{n_x} d\tau$.)

v) $\frac{d}{dt}\tilde{u}_n(s(t),t) = \dot{s}(t)g(\gamma(t),t) + z_n(s(t),t) = \dot{\gamma}(t),\ 0 < t < T$, and by integration, we find $\tilde{u}_n(s(t),t) = \gamma(t),\ 0 < t < T$. Hence

$$\tilde{u}_{n_x}(s(t),t) = g(\tilde{u}_n(s(t),t),t). \tag{2.26}$$

From i) - ii), (2.26) and the uniqueness of solution of (2.14)-(2.17), we deduce $\tilde{u}_n \equiv u_n$ in $\overline{D}_{n,T}$. By (2.24) it follows that

$$\|u_{n_{xx}}\|_{D_{n,T}} \le \text{const},\quad \|u_{n_x}\|_{D_{n,T}} \le \text{const},\quad \forall n. \tag{2.27}$$

Denote by $u(x,t)$ the limit function of u_n for $n \to \infty$. Then u satisfies (2.7), (2.8). We have only to verify (2.9). Let $t \in (0,T)$, $x \in (0,s(t))$ be fixed, then we have

$$\begin{aligned} u(s(t),t) - u(x,t) &= [u(s(t),t) - u_n(s(t),t)] + \\ &+ [u_n(s(t),t) - u_n(x,t)] + [u_n(x,t) - u(x,t)] = \\ &= [u(s(t),t) - u_n(s(t),t)] + [u_n(x,t) - u(x,t)] + \\ &+ g(u_n(s(t),t),t)(s(t) - x) + \frac{1}{2}u_{n_{xx}}(\tilde{x},t)(s(t)-x)^2 \end{aligned}$$

for some $\tilde{x} \in (x, s(t))$. By (2.27), we obtain

$$|u(s(t), t) - u(x,t) - g(u_n(s(t), t), t), t).(s(t) - x)| \leq \\ \leq 2\|u - u_n\| + \text{const}.(s(t) - x)^2.$$

Letting n tend to ∞ and then x to $s(t)$, we get (2.9).

3. LOCAL EXISTENCE AND UNIQUENESS

We return to Problem I with f and g satisfying (1.11)-(1.14).

Since $f_v > 0$, $f_r > 0$ and (2.12), it follows that

$$\dot{s}(t) > f(\frac{v_0}{2}, 0) > 0, \ \forall t \in (0, t_0), \tag{3.1}$$

and $s(t) = s(0) + \int_0^t \dot{s}(\tau) d\tau > 0, \ \forall t \in (0, t_0)$.

We choose T such that

$$T \leq \min(t_0, t', T_1), \tag{3.2}$$

where t_0, t', T_1 are defined by Lemma 2.1, Lemma 2.2 and (2.24) respectively.

We consider the following auxiliary problem: For each given function $r(t) \in C^1[0,T]$ such that $r(0) = 0$ and $0 < K_1 \leq \dot{r}(t) \leq K_2$ in $(0,T)$, let $v(x,t)$ be the unique solution of the problem:

$$v_{xx} - v_t = 0 \qquad \text{in } D_{r,T}, \tag{3.3}$$

$$v(0,t) = v_0, \qquad 0 < t < T, \tag{3.4}$$

$$v_x(r(t), t) = g(v(r(t), t), t), \qquad 0 < t < T, \tag{3.5}$$

where $D_{r,T} = \{(x,t) : 0 < x < r(t), \ 0 < t < T\}$. Then $v(x,t)$ satisfies (2.12), (2.13) in $\overline{D}_{r,T}$, i.e.

$$\frac{v_0}{2} \leq v(x,t) \leq v_0, \tag{3.6}$$

$$|v_x(x,t)| \leq G. \tag{3.7}$$

By an argument used in the proof of Theorem 2.2 and taking [1] into account, we find that v_{xx} is bounded in $D_{r,T}$ by a constant Z_0 depending on K_2, G for $T > 0$ small enough. Define

$$\begin{aligned} B = \{s \in C^1[0,T] : s(0) = 0,\ \ 0 < K_1 \le \dot{s}(t) \le K_2, \\ |\dot{s}(t_2) - \dot{s}(t_1)| \le K_3|t_2 - t_1| \ \text{ for } t_1, t_2 \in (0,T]\}, \\ \Omega = \{(y,p) : \frac{v_0}{2} \le y \le v_0,\ 0 \le p \le \infty\}, \end{aligned} \tag{3.8}$$

where the coefficients K_1, K_2, K_3 satisfy the conditions

$$\begin{aligned} 0 < K_1 \le \inf_{(y,p)\in\Omega} f(y,p) = f(\frac{v_0}{2}, 0), \\ 0 < \sup_{(y,p)\in\Omega} f(y,p) \le K_2,\ f_0(GK_2 + Z_0) + FK_2 \le K_3. \end{aligned} \tag{3.9}$$

Since $0 < K_1 \le \dot{s}(t) \le K_2$ and $s(t) = s(0) + \int_0^t \dot{s}(\tau)d\tau = \int_0^t \dot{s}(\tau)d\tau$, we have $K_1 t \le s(t) \le K_2 t,\ \ 0 \le t \le T$.

It is clear that B is a closed subset of $C[0,T]$. In particular, we can choose

$$K_1 = f(\frac{v_0}{2}, 0),\ K_2 = P,\ K_3 = f_0(GK_2 + Z_0) + FK_2. \tag{3.10}$$

Define F_2 to be the application

$$F_2 : r \in B \mapsto \tilde{r},$$

where

$$\tilde{r}(t) = \int_0^t f(v(r(\tau),\tau), s(\tau))d\tau,\ t \in [0,T], \tag{3.11}$$

and $v(x,t)$ is the unique solution of (3.3)-(3.5), which satisfies the estimates

$$\frac{v_0}{2} \le v \le v_0,\ \ |v_x| \le G,\ |v_{xx}| \le Z_0 \text{ in } \overline{D}_{r,T}. \tag{3.12}$$

Then we have $\tilde{r} \in B$ because

$$\begin{aligned} |\dot{\tilde{r}}(t_2) - \dot{\tilde{r}}(t_1)| \le |f(v(s(t_2),t_2), s(t_2)) - f(v(s(t_1),t_1), s(t_2))| + \\ + |f(v(s(t_1),t_1), s(t_2)) - f(v(s(t_1),t_1), s(t_1))| \le \\ \le f_0|v(s(t_2),t_2) - (v(s(t_1),t_1)| + F|s(t_2) - s(t_1)| \le \\ \le f_0\{GK_2 + Z_0\}|t_2 - t_1| + FK_2|t_2 - t_1| = K_3|t_2 - t_1|, \end{aligned} \tag{3.13}$$

for $t_1, t_2 \in (0,T)$. We have the following theorem:

Theorem 3.1. *The application F_2 is a contractive mapping of B into itself in the metric of $C[0,T]$ for a small enough $T > 0$.*

P r o o f. Let u and v be the corresponding solutions of the problem (3.3)-(3.5) for $s, r \in B$, respectively.

For the sake of simplicity we put

$$\delta_{F_2}(t) = |\tilde{s}(t) - \tilde{r}(t)|, \|\delta_{F_2}\|_t = \sup_{0\leq\tau\leq t} \delta_{F_2}(\tau),$$
$$\delta(t) = |s(t) - r(t)|, \ \|\delta\|_t = \sup_{0\leq\tau\leq t} \delta(\tau),$$
$$\sigma_1(t) = \inf(s(t), r(t)), \ \sigma_2 = \sup(s(t), r(t)).$$

Without loss of generality we may assume that $\sigma_1 = r$, $\sigma_2 = s$. Then

$$\sigma_{F_2}(t) = |\int_0^t [f(v(s(\tau),\tau), s(\tau)) - f(u(r(\tau),\tau), r(\tau))]d\tau| \leq$$
$$\leq f_0 t\|v_{|s} - u_{|r}\|_t + Ft \max_{0\leq\tau\leq t} |s(\tau) - r(\tau)|, \tag{3.14}$$

(we denote by $v_{|s}$ and $v_{|r}$ the restrictions of v on $x = s(t)$ and $x = r(t)$, respectively). We get

$$\|v_{|s} - u_{|r}\|_t = \max_{0\leq\tau\leq t} |v(s(\tau),\tau) - u(r(\tau),\tau)| \leq$$
$$\leq \max_{0\leq\tau\leq t} |v(s(\tau),\tau) - v(r(\tau),\tau)| +$$
$$+ \max_{0\leq\tau\leq t} |v(r(\tau),\tau) - u(r(\tau),\tau)| \equiv b(t) + a(t). \tag{3.15}$$

Applying the average value theorem we get further

$$b(t) \leq G.\|\delta\|_t. \tag{3.16}$$

Notice that $v(r(t),t) = v_0 + \int_0^{r(t)} v_x(x,t)dx$, $u(r(t),t) = v_0 + \int_0^{r(t)} u_x(x,t)dx$. This yields

$$|v(r(t),t) - u(r(t),t)| \leq K_2 t\|v_x - u_x\|_{D_{r,t}} = K_2 t \sup_{0\leq\tau\leq t} C(\tau),$$

where

$$C(\tau) = |v_x(r(\tau),\tau) - u_x(r(\tau),\tau)| \leq$$
$$\leq |v_x(r(\tau),\tau) - v_x(s(\tau),\tau)| + |v_x(s(\tau),\tau) - u_x(r(\tau),\tau)| \leq$$
$$\leq z_0\|\delta\|_t + g_0\|v_{|s} - u_{|r}\|_t.$$

Hence, we find

$$a(t) \leq K_2 t[z_0\|\delta\|_t + g_0\|v_{|s} - u_{|r}\|_t]. \tag{3.17}$$

From (3.15)-(3.17) it follows

$$\|v_{|s} - u_{|r}\|_t \leq G\|\delta\|_t + K_2 t[z_0\|\delta\|_t + g_0\|v_{|s} - u_{|r}\|_t.$$

Then

$$\|v_{|s} - u_{|r}\|_t \leq \frac{G + K_2 Z_0 t}{1 - K_2 g_0} \leq \alpha_0\|\delta\|_t \text{ for } 0 < t < t^*, \tag{3.18}$$

where $t^* = \frac{1}{2K_2 g_0} > 0,\ \alpha_0 = 2G + \frac{z_0}{g_0} > 0.$
By using (3.14) and (3.18) it follows that

$$\delta_{F_2}(t) \leq f_0\alpha_0 t\|\delta\|_t + Ft\|\delta\|_t = (f_0\alpha_0 + F)t\|\delta\|_t, 0 < t < t^*.$$

If we choose T such that $0 < T \leq T_0 = \min(t^*, \frac{1}{f_0\alpha_0 + F})$, then F_2 is a contractive mapping.

Theorem 3.2. *Problem I admits a unique solution for $T \leq T_0$.*

P r o o f. This is a straightforward consequence of Theorem 3.1 and of the Banach's fixed point theorem.

REFERENCES

1. Comparini E., Ricci R., On the swelling of a glassy polymer in contact with a well-stirred solvent, Math. Mecth. Appl. Sci., 7, (1985), 238-250.
2. Do D. D., On the validity of the shringking core model in noncatalytic gas solid reaction, Chem. Eng. Sci. 37, (1982), 1477-1481.
3. Fasano A., Primicerio M., Esistenza e unicita della soluzione per uno classe di problemi di diffusione con condizioni al contorno non lineari, Bolletino Un. Matematica Italiana, 3, (1970), 670-677.
4. Fasano A., Primicerio M., Su un problema unidimensionale di diffusione in un mezzo a contorno mobile con condizioni ai limiti non lineari, Annali Mat. Pura Appl., 93, (1972), 333-357.

5. Friedman A., Partial Differential Equations of Parabolic Type, Prentice Hall, Inc., Englewood Cliffs, N. J., 1964.
6. Tarzia D. A., Villa L. T., On the free boundary problem in the Wen-Langmuir shringking core model for noncatalytic gas-solid reactions, Mechanica, 24, (1989), 86-92.
7. Wen C. Y., Noncatalytic heterogeneous solid fluid reaction models, Industrial Eng. Chem., 60 n. 9, (1968), 33-54

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