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ON GLOBALIZATION OVER U(1)-COVERING OF ZUCKERMAN $(\mathfrak{g}, K)$-MODULES

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Dedicated to Professor Nguyen Dinh Tri on his sixtieth birthday

Abstract. *Using the metaplectic representation in terms of the Bergmann-Segal model we lift the construction suggested by W.Schmid and J.A.Wolf to the case of $U(1)$-covering by using the technique of P.L.Robinson and J.H.Rawnsley. Our purpose is to give an algebraic version of the multidimensional quantization with respect to $\mathbf{Z}_2$-covering (as a special case) and $U(1)$-covering. By lifting to $U(1)$-covering, from a basic datum we firstly describe in terms of local cohomology the maximal globalization of the Harish - Chandra modules in the case of maximally real polarizations. Then we use the change of polarization to extend the indicated results to the general case.*

INTRODUCTION

In 1979-1980, Do Ngoc Diep [1-2] has proposed the procedure of multidimensional quantization for general case, starting from arbitrary irreducible bundles. This procedure could be viewed as a geometric version of the construction of M.Duflo [4] . Recently, in 1988, W.Schmid and J.A.Wolf [3] described in terms of local cohomology the maximal globalization of the Harish-Chandra modules to realize the discrete series representations of semi-simple Lie groups by using the geometric quantization and the derived Zuckerman functor modules.

In this paper, we modified the construction suggested by W.Schmid and J.A.Wolf to the case of $U(1)$-covering by using the technique of P.L.Robinson andJ.H.Rawnsley [5]. Our purpose is to give an algebraic version of the multidimentional quantization with respect to Z_2-covering (as a special case) and $U(1)$-covering. By lifting to $U(1)$-covering, from a basic datum as in [3], we shall describe in terms of local cohomology the maximal real polarizations (Theorem 1). Then we use the change of polarization to extend the indicated results to the general case (Theorem 2). Our work is in much influenced from the loc.cit. work of W.Schmid and J.A.Wolf. The main results can be considered as some $U(1)$-analogues of Schmid-Wolf's ones. The key moment in $U(1)$-lifting is the usage of $G^{U(1)} \approx K.B^{U(1)}$ as the fibered product of $B^{U(1)} \to B$ and $G \to B$ over B. Other very useful fact is that $\chi^{U(1)}$ works out for $U(1)$-covering so good as χ in Schmid-Wolf's situation. So we keep the same notation and exposition as in [3].

1. CLASSICAL CONSTRUCTIONS AND THREE GEOMETRIC COMPLEXES

1.1. Classical constructions. Let G be a connected, linear, semi-simple Lie group. Denote by $\mathcal{G}$ the Lie algebra of G and $\mathcal{G}^*$ its dual space. The group G acts on $\mathcal{G}$ by the adjoint representation Ad, and on $\mathcal{G}^*$ by K-representation. Let $F \in \mathcal{G}^*$ and G_F be the stabilizer of this point. Denote by $\mathcal{G}_F$ its Lie algebra. Fix a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}_{\mathbf{C}} = \mathcal{G} \otimes \mathbf{C}$. We shall only consider $F \in \mathcal{G}^*$ such that $\mathcal{G}_F \otimes \mathbf{C} = \mathcal{H}$. Then $H = G_F$ is a Cartan subgroup of G. We now assume that K-orbit Ω_F passing F is $U(1)$-admissible [8], i.e., there exists a unitary character $\chi_F^{U(1)} : H^{U(1)} \to S^1$ such that

$$d\chi_F^{U(1)}(X,\varphi) = \frac{i}{h}[F(X) + \varphi], \quad \text{where } (X,\varphi) \in \mathcal{H} \otimes U(1)_{\mathbf{C}}.$$

Let $\mathcal{B}$ be a closed positive polarization in $\mathcal{G}_{\mathbf{C}}$, we know that $\mathcal{B}$ is a Borel subalgebra of $\mathcal{G}_{\mathbf{C}}$ with $\mathcal{H} \subset \mathcal{B}$. Let $\tilde{\sigma}$ be some fixed irreducible unitary representation of H in a separable Hilbert space such that the restriction of $\chi^{U(1)} = (\tilde{\sigma}.\sigma_j).\chi_F^{U(1)}$ to $(H^0)^{U(1)}$ is a multiple of the character $\chi_F^{U(1)}$, where σ_j is the homomorphism defined in ([8],§2). Let B_0 the corresponding analytic subgroup in G of $\mathcal{B} \cap \mathcal{G}$ and $B = H.B_0$. We see that $B^{U(1)} = H^{U(1)} \times B_0$ is the $U(1)$-covering of B and there exists a unique irreducible representation $\sigma : B^{U(1)} \to U(V)$ such that

$$\sigma|_{H^{U(1)}} = \chi^{U(1)}, \quad \text{(see [8], §2)}.$$

By virtue of the representation $\sigma : B^{U(1)} \to U(V)$, denote by $E_{\sigma,\rho} = G^{U(1)} \underset{B^{U(1)},\sigma}{\times} V$ the vector bundle on $B \setminus G$ associated with σ. Then the inverse image bundle $\pi^* E_{\sigma,\rho}$ is a vector bundle on $\Omega = H \setminus G$, where π is the natural projection from $H \setminus G$ to $B \setminus G$. In the category of smooth vector bundles $\pi^* E_{\sigma,\rho}$ and $G^{U(1)} \underset{H^{U(1)},\sigma|H^{U(1)}}{\times} V$ are equivalent.

Then we as in [8] obtain a homogeneous vector bundle

$$\mathbf{E}^{U(1)} = G^{U(1)} \underset{H^{U(1)},\chi^{U(1)}}{\times} V \longrightarrow H \setminus G, \tag{1.1}$$

on $H \setminus G$ associated with the representation $\chi^{U(1)} = (\tilde{\sigma}.\sigma_j).\chi_F^{U(1)}$. In the view of [3], we can say the bundle (1.1) associated to the basic datum $(H, \mathcal{B}, \chi^{U(1)})$.

Suppose that $\dim_{\mathbb{C}} \Omega_F = m$. Let $\mathcal{C}^q(\mathbf{E}^{U(1)})$ denote the sheaf of differential forms of type $(0,q)$ on $H \setminus G$ with coefficients in $\mathbf{E}^{U(1)}$. We know that each differential form of this type is a section of the bundle $\mathbf{E}^{U(1)} \otimes \lambda^q \mathbf{N}^*$ on $H \setminus G$, where $\mathbf{N} \to H \setminus G$ is the homogeneous vector bundle with fibre $\mathcal{N} \cong \mathcal{B}/\mathcal{H}$ and $\mathbf{N}^*$ is its dual. Denote by $\mathcal{O}_{\mathcal{N}}(\mathbf{E}^{U(1)})$ the sheaf of germs of partially holomorphic C^∞ sections of $\mathbf{E}^{U(1)}$ that are annihilated by $\mathcal{N} \oplus \mathcal{U}(1)_{\mathbb{C}}$. Then we have

$$0 \to \mathcal{O}_{\mathcal{N}}(\mathbf{E}^{U(1)}) \xrightarrow{i} \mathcal{C}^0(\mathbf{E}^{U(1)}) \to \dots \xrightarrow{\overline{\partial}_{\mathbf{E}^{U(1)}}} \mathcal{C}^m(\mathbf{E}^{U(1)}) \to 0 \tag{1.2}$$

where the mapping i is induced by inclusion of the space of partially invariant partially holomorphic sections of $\mathbf{E}^{U(1)}$ into the space of smooth sections, and the mappings $\overline{\partial}_{\mathbf{E}} U(1)$ are induced by the usual operator, mapping a form of type (o,q) to a form of type $(o, q+1)$.

By taking global sections, (1.2) induces a sequence of the form

$$0 \to C^\infty(H \setminus G; \mathcal{O}_{\mathcal{N}}(\mathbf{E}^{U(1)})) \to C^\infty(H \setminus G;\ \mathcal{C}^0(\mathbf{E}^{U(1)})) \to \dots$$
$$\to C^\infty(H \setminus G;\ \mathcal{C}^m(\mathbf{E}^{U(1)})) \to 0 \tag{1.3}$$

and this sequence of abelian groups forms a cochain complex

$$C^\infty(H \setminus G;\ \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot} \mathbf{N}^*),\ \overline{\partial}_{\mathbf{E}^{U(1)}} \tag{1.4}$$

Denote by $H^p\ (C^\infty(H \setminus G;\ \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot} \mathbf{N}^*))$ the $p-th$ cohomology group of the cochain complex (1.4) we have

Proposition 1.1. *There exists a canonical isomorphism* $H^p(C^\infty(H \setminus G;\ \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot} \mathbf{N}^*)) \cong H^p(H \setminus G;\ \mathcal{O}_{\mathcal{N}}(\mathbf{E}^{U(1)}))$, $p \geq 0$ *where* $H^p(H \setminus G;\ \mathcal{O}_{\mathcal{N}}(\mathbf{E}^{U(1)}))$ *is the sheaf cohomology group of the space* $H \setminus G$ *of degree* p *with coefficients in* $\mathcal{O}_{\mathcal{N}}(\mathbf{E}^{U(1)})$.

P r o o f. We know that $H \setminus G \cong \Omega_F$ is the paracompact almost complex manifold. So (1.2) is a fine resolution of $\mathcal{O}_{\mathcal{N}}(\mathbf{E}^{U(1)})$,and then it is acyclic. Thus, our assertion is an analogue of Dolbeault's theorem.□

1.2. Three geometric complexes. We see that the differential $\overline{\partial}_{\mathbf{E}^{U(1)}}$ of (1.4) extends naturally to hyperfunction sections, so we have a complex

$$C^{-\omega}(H \setminus G;\ \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot} \mathbf{N}^*),\ \overline{\partial}_{\mathbf{E}^{U(1)}} \tag{1.5}$$

Let X denote the flag variety of Borel subalgebras of $\mathcal{G}_{\mathbb{C}}$. Since H normalizes $\mathcal{B}$, there exists a natural G-invariant fibration $H \setminus G \to S = G.\mathcal{B} \subset X$, where S is the G-orbit passing $\mathcal{B}$ in X, and as homogeneous real analytic submanifold of the complex manifold X,S has the structure of CR-manifold.

Then the bundle $\mathbf{E}^{U(1)} \to H \setminus G$ push down to a G-homogeneous bundle $\mathbf{E}^{U(1)} \to S \cong B \setminus G$ and we obtain as in [3] the Cauchy-Riemann complex

$$C^{-\omega}(S;\ \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}_S^*),\ \bar{\partial}_S \tag{1.6}$$

where $\mathbf{N}_S = \mathbf{T}^{0,1}(S)$ is a G-homogeneous vector bundle based on $\mathcal{N}/\mathcal{N} \cap \overline{\mathcal{N}}$.

Denote by $X^{U(1)}$ the flag variety of $U(1)$-invariant Borel subalgebras of $\mathcal{G}_{\mathbb{C}} \oplus \mathcal{U}(1)_{\mathbb{C}}$. We obtain the natural projection $\pi_X : X^{U(1)} \to X$. By using the Gauss' decomposition $G = K.B$, where K is a fixed maximal compact subgoup in G, we have $B \setminus G \cong B^{U(1)} \setminus K.B^{U(1)}$. Note that $K.B^{U(1)}$ acts on the flag variety $X^{U(1)}$. Let $S^{U(1)} = (K.B^{U(1)}).(\mathcal{B} \oplus \mathcal{U}(1)_{\mathbb{C}})$ be the orbit passing $(\mathcal{B} \oplus \mathcal{U}(1)_{\mathbb{C}})$ in $X^{U(1)}$, it is easy to show that $B^{U(1)}$ is the stabilizer of $(\mathcal{B} \oplus \mathcal{U}(1)_{\mathbb{C}})$. Then there exists a diffeomorphism of $S^{U(1)}$ onto S. By the projection $\pi_X : X^{U(1)} \to X$ we obtain the homogeneous bundle $\pi_X^*\mathbf{E}^{U(1)} \to S^{U(1)}$, and have the complex

$$C^{-\omega}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}_{S^{U(1)}}^*),\ \bar{\partial}_{S^{U(1)}} \tag{1.7}$$

where $\mathbf{N}_{S^{U(1)}} = \pi_X^*\mathbf{N}_S$, and $\bar{\partial}_{S^{U(1)}}$ is induced by the CR- operator $\bar{\partial}_S$.

The bundle $\pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot p}\mathbf{N}_{S^{U(1)}}^* \to S^{U(1)} \cong B \setminus G$ pull back to trivial bundless on G, so the comples (1.7) is isomorphic to the complex

$$\{C^{-\omega}(G) \otimes V \otimes \Lambda^{\cdot}(\mathcal{N}/\mathcal{N} \cap \overline{\mathcal{N}})^*\}^{\mathcal{N}\cap\overline{\mathcal{N}},B^{U(1)}},\ \delta_{\mathcal{N},\mathcal{N}\cap\overline{\mathcal{N}}} \tag{1.8}$$

for relative Lie algebra cohomology of $(\mathcal{N}, \mathcal{N} \cap \overline{\mathcal{N}})$ and hyperfunction coefficients.

A section $\tilde{s} \in C^{-\omega}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot p}\mathbf{N}_{S^{U(1)}}^*)$ is said to be $H^{U(1)}$-equivariant iff $\tilde{s}(hx) = \chi^{U(1)}(h).\tilde{s}(x)$, $\forall h \in H^{U(1)}$, $x \in S^{U(1)} \cong B \setminus G$.

Denote by $C_{H^{U(1)}}^{-\omega}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot p}\mathbf{N}_{S^{U(1)}}^*)$ the space of $H^{U(1)}$-equivariant partially holomorphic C^∞ sections of $C^{-\omega}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot p}\mathbf{N}_{S^{U(1)}}^*)$, we have

Proposition 1.2. *There exists a canonical isomorphism*

$$C_{H^{U(1)}}^{-\omega}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^p\mathbf{N}_{S^{U(1)}}^*) \cong C^{-\omega}(S; \mathbf{E}^{U(1)} \otimes \Lambda^p\mathbf{N}_S^*) \tag{1.9}$$

as vector spaces.

P r o o f. The assertion follows from the definition of $H^{U(1)}$-equivariant sections and the construction of homogeneous bundles, (see [7, §3]). □

We see that $H \setminus G \to S^{U(1)}$ has Euclidean space fibres. By applying the Poincare' Lemma to those fibres we see that inclusion of the complex (1.8) in the complex (1.5) induces an isomorphism of cohomology. Then as in [3] we obtain the following results.

Proposition 1.3. *There are canonical isomorphisms*

$$H^p(C^{-\omega}(H \setminus G; \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*)) \cong H^p(C^{-\omega}_{H^{U(1)}}(S^{U(1)};\ \pi^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*_{S^{U(1)}}))$$
$$\cong H^p(\{C^{-\omega}(G) \otimes V \otimes \Lambda^{\cdot}(\mathcal{N}/\mathcal{N} \cap \overline{\mathcal{N}})^*\}^{\mathcal{N}\cap\overline{\mathcal{N}}, B^{U(1)}}).$$

Let $\tilde{S}$ denote the germ of neighborhoods of S in X. Then $\mathbf{E}^{U(1)} \to S$ has a unique holomorphic $\mathcal{G}$-equivariant extension $\tilde{\mathbf{E}}^{U(1)} \to \tilde{S} \subset X$, and we obtain the Dolbeault complex

$$C^{-\omega}(\tilde{S}; \tilde{\mathbf{E}}^{U(1)} \otimes \Lambda^{\cdot} T_X^{0,1*}), \overline{\partial} \tag{1.10}$$

with coefficients that are hyperfunctions on $\tilde{S}$ with support in S.

Similarly, we have $\pi_X^*\tilde{\mathbf{E}}^{U(1)} \to \tilde{S}^{U(1)} \subset X^{U(1)}$ and then obtain the complex

$$C^{-\omega}(\tilde{S}^{U(1)};\ \pi_X^*\tilde{\mathbf{E}}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{T}^{0,1*}_{X^{U(1)}}),\ \overline{\partial}_{U(1)} \tag{1.11}$$

where $\mathbf{T}^{0,1}_{X^{U(1)}} = \pi_X^*\mathbf{T}^{0,1}_X$, and $\overline{\partial}_{U(1)}$ is induced by $\overline{\partial}$.

By using the canonical isomorphism $C^{-\omega}_{H^{U(1)}}(\tilde{S}^{U(1)};\ \pi_X^*\tilde{\mathbf{E}}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{T}^{0,1*}_{X^{U(1)}}) \cong C^{-\omega}(S;\ \tilde{\mathbf{E}}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{T}_X^{0,1*})$, we see as in [3] that

$$H^p(C^{-\omega}_{H^{U(1)}}(\tilde{S}^{U(1)};\ \pi_X^*\tilde{\mathbf{E}}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{T}^{0,1*}_{X^{U(1)}})) \cong H^p_S(\tilde{S}, \mathcal{O}(\tilde{\mathbf{E}}^{U(1)})) \tag{1.12}$$

where the right hand side of (1.12) is local cohomology along S. □

2. ISOMORPHISMS OF THE COHOMOLOGIES AND THE INDUCED FRÉCHET TOPOLOGY

2.1. Isomorphisms of the cohomologies. We fix a basic datum $(H, \mathcal{B}, \chi^{U(1)})$ as in Section 1.1. Let $S = G.\mathcal{B} \subset X$. Denote by Y the variety of ordered Cartan subalgebras. As homogeneous $G_{\mathbf{C}}$-space, we have $Y \cong H_{\mathbf{C}} \setminus G_{\mathbf{C}}$, where $G_{\mathbf{C}}$ is the adjoint group of $\mathcal{G}_{\mathbf{C}}$, and $H_{\mathbf{C}}$ is the connected subgroup with Lie algebra $\mathcal{H}$. Since $H_{\mathbf{C}}$ normalizes $\mathcal{B}$, there is a natural projection $p : Y \to X$ with fibre $p^{-1}(\mathcal{B}) = \exp(\mathcal{N})$. Let $S_Y = G.\mathcal{H} \subset Y$ be the G-orbit through the base point in Y, we have $p : S_Y \to S$, with fibre $\exp(\mathcal{N} \subset \mathcal{G}) = \exp(\mathcal{N} \cap \overline{\mathcal{N}} \cap \mathcal{G})$. Then S_Y is a real form of the complex manifold Y and $u = codim_{\mathbf{R}}(S)$ is the fibre dimension of the natural projection $p : S_Y \to S$.

By the projection $p : Y \to X$ there exists a homomorphism

$$p^{U(1)}: Y^{U(1)} \longrightarrow X^{U(1)}$$

such that the following diagram

$$\begin{tikzcd} Y^{U(1)} \arrow[r, "p^{U(1)}"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & X^{U(1)} \arrow[d, "\pi_X"] \\ Y \arrow[r, "p"'] & X \end{tikzcd} \tag{2.1}$$

is commutative, where π_X and π_Y are natural projections. We let $\mathbf{T}_{Y|X}$ denote the complexified relative tangent bundle of the fibration $p: Y \to X$, and $\mathbf{T}^{1,0}_{Y|X}$, $\mathbf{T}^{0,1}_{Y|X}$ the subbundle of holomorphic, respectively antiholomorphic, relative tangent vectors. Then, there exists a $G_{\mathbf{C}}$-invariant isomorphism

$$p^*\mathbf{T}_X \oplus \mathbf{T}_{Y|X} \cong \mathbf{T}_Y,$$

which is compatible with the complex structure and the Lie bracket.

By a similar way as in [3], we obtain the complex

$$C^{-\omega}(S_Y;\ p^*\tilde{\mathbf{E}}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{T}^{1,0*}_{Y|X}), \tag{2.2}$$

Let $S_Y^{U(1)} = (K.B^{U(1)}).(\mathcal{H}\oplus\mathcal{U}(1)_{\mathbf{C}}) \subset Y^{U(1)}$ be the orbit passing $(\mathcal{H}\oplus\mathcal{U}(1)_{\mathbf{C}})$ we see that $S_Y^{U(1)} \cong S_Y$. By using the diagram (2.1), we obtain

$$C^{-\omega}(S_Y^{U(1)};\ p^{U(1)*}\pi_X^*\tilde{\mathbf{E}}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{T}^{U(1)1,0*}_{Y|X}),\ \bar{\partial}_{U(1)} \tag{2.3}$$

where $\mathbf{T}^{U(1)1,0}_{Y|X} = \pi_X^*\mathbf{T}^{1,0}_{Y|X}$, and $\bar{\partial}_{U(1)}$ is induced by $\bar{\partial}$.

We see that $\mathbf{T}^{1,0}_{Y|X}$ is modeled on $\mathcal{N} = \mathcal{B} \oplus \mathcal{U}(1)_C/\mathcal{H} \oplus \mathcal{U}(1)_{\mathbf{C}}$ and $\pi_X^*\tilde{\mathbf{E}}^{U(1)}$ is also modeles on $H^{U(1)}$-module V, so the subcomplex of the complex (2.3)

$$C^{-\omega}_{H^{U(1)}}(S_Y^{U(1)};\ p^{U(1)*}\pi_X^*\tilde{\mathbf{E}}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{T}^{U(1)1,0*}_{Y|X}), \tag{2.4}$$

coincides with the complex (1.5). By a similar argument as in [3, §6], we obtain

Proposition 2.1. *There are canonical isomorphisms*

$$\begin{aligned} H^p(C^{-\omega}(H \setminus G; \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*)) &\cong H^p(C^{-\omega}_{H^{U(1)}}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}_{S^{U(1)}})) \\ &\cong H^{p+u}_S(\tilde{S}; \mathcal{O}(\tilde{\mathbf{E}}^{U(1)})) \end{aligned}$$

as G-modules without topology, where $u = codim_{\mathbf{R}}(S \subset X)$.

2.2. The induced Fréchet topology. We fix a Cartan involution θ of G with $\theta K = K$. Then $H = T \times A$, where $\mathcal{H} = \mathcal{T} + \mathcal{A}$ are the ± 1-eigenspaces of $\theta|_{\mathcal{H}}$, $A = \exp(\mathcal{A} \cap \mathcal{G})$. Consider the orbit $S = G.\mathcal{B} \subset X$, where $\mathcal{H} \subset \mathcal{B}$.

It follows from Proposition 7.1 in [3] that we can define a relative orbit $S_{\max} = G.\mathcal{B}_{\max}$, where $\mathcal{H} \subset \mathcal{B}_{\max}$ and $\mathcal{B}_{\max}$ is maximally real for that condition.

Then G has a cuspidal parabolic subgroup $P = MA.N_H$, where $\theta M = M$ and $\mathcal{B}_{\max} \subset \mathcal{P}$, $\mathcal{P} = \text{Lie } P$. Moreover, the fibrations $S \to S_{\max}$ and $S_{\max} \to P \setminus G$ induce a fibration $S \to P \setminus G$. Since $S^{U(1)} \approx S$, we obtain the fibration $S^{U(1)} \to P \setminus G$. Let $\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)})$ be the sheaf of germs of hyperfunctions on $S^{U(1)}$ that are C^{∞} along the fibres of $S^{U(1)} \to P \setminus G$. Then $\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)})$ defines a complex of sheaves $\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)}; \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^p \mathbf{N}^*_{S^{U(1)}})$ of germs of $H^{U(1)}$- equivariant sections of the bundle $\pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^p \mathbf{N}^*_{S^{U(1)}} \to S^{U(1)}$ coefficients in $\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)})$.

Taking global sections, we arrive at a subcomplex of (1.7)

$$\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot} \mathbf{N}^*_{S^{U(1)}}), \quad \overline{\partial}_{S^{U(1)}}. \tag{2.5}$$

Proposition 2.2. *The inclusion* $\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot} \mathbf{N}^*_{S^{U(1)}}) \to \mathcal{C}^{-\omega}_{H^{U(1)}}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot} \mathbf{N}^*_{S^{U(1)}})$ *induces isomorphisms of cohomology.*

P r o o f. By applying the usual Dolbeault Lemma and the (standard) argument on hyperfunction, we see [3] that the sheaves $\mathcal{C}^{-\omega}_{H^{U(1)}}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^p \mathbf{N}^*_{S^{U(1)}})$, and $\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^p \mathbf{N}^*_{S^{U(1)}})$ are soft, and the inclusions of $\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)}; \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^p \mathbf{N}^*_{S^{U(1)}})$ into $\mathcal{C}^{-\omega}_{H^{U(1)}}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^p \mathbf{N}^*_{S^{U(1)}})$ induce isomorphisms of cohomology sheaves.

On the other hand, it follows from [6] that the inclusion of sheaves induces an isomorphism of hypercohomology. Since both complexes consist of soft sheaves, the hypercohomology is just the cohomology of associated complex of global section. This completes the proof of our proposition. □

Remark that the theory of hyperfunctions with values in a reflexive Banach space is developed exactly in the same way as for complex valued hyperfunctions. So by a similar argument as in ([3], §7) we obtain:

Proposition 2.3. *The* $\mathcal{C}^{-\omega}_{P\setminus G}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^p \mathbf{N}^*_{S^{U(1)}})$ *have natural Préchet topologies. In those topologies,* $\overline{\partial}_{S^{U(1)}}$ *is continuous and the actions of* G *are Fréchet representations.*

3. THE TENSORING ARGUMENT AND MAXIMALLY REAL POLARIZATIONS

3.1. The tensoring argument. Recall some notions from [3]:

An admissible Fréchet G-module has property (MG) if it is the maximal globalization of its underlying Harish-Chandra module.

A complex $(C^{\cdot}, d)$ of Fréchet G-modules has propecty (MG) if d has closed range, the cohomologies $H^p(C^{\cdot}, d)$ are admissible and of finite length, and each $H^p(C^{\cdot}, d)$ has property (MG).

Given a basic datum $(H, \mathcal{B}, \chi^{U(1)})$, we say that the corresponding homogeneous vector bundle $\mathbf{E}^{U(1)} \to S^{U(1)}$ has property (MG) if the partially smooth Cauchy-Riemann complex (2.5) has property (MG).

Denote $H^p(S^{U(1)};\ \mathbf{E}^{U(1)}) = H^p(C^{-\omega}_{H^{U(1)}}(S^{U(1)};\ \pi_X^* \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot} \mathbf{N}^*_{S^{U(1)}}))$

Proposition 2.2 shows that $H^p(S^{U(1)};\ \mathbf{E}^{U(1)})$ is calculated by a Fréchet complex, then $H^p(S^{U(1)};\ \mathbf{E}^{U(1)})_{(K)}$ is calculated by the subcomplex of K-finite forms in that Fréchet complex. In particular, we can define morphisms

$$H^p(S^{U(1)};\ \mathbf{E}^{U(1)})_{(K)} \longrightarrow A^p(G, H, \mathcal{B}, \chi^{U(1)}), \tag{3.1}$$

where $A^p(G, H, \mathcal{B}, \chi^{U(1)}) \cong H^p(C^{\text{for}}(H\backslash G; \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot} \mathbf{N}^*)_{(K)})$ are Harish-Chandra modules for G. Then $H^p(S^{U(1)};\ \mathbf{E}^{U(1)})$ will be the globalization of $A^p(G, H, \mathcal{B}, \chi^{U(1)})$ if (3.1) are isomorphisms. (See [3], §3). □

Recall that as in [3], the bundle $E^{U(1)} \to S^{U(1)}$ has property (Z) if the maps (3.1) are isomorphisms. Note that $f \in \mathcal{H}^*$ can be identified with $F \in (\mathcal{H} \oplus \mathcal{U}(1)_{\mathbf{C}})$ such that $F|_{\mathcal{U}(1)_{\mathbf{C}}} = 0$, then $\mathcal{H}^* \subset (\mathcal{H} \oplus \mathcal{U}(1)_{\mathbf{C}})^*$. So, we can consider the following condition for a pair $(H, \mathcal{B})$:

There exist a positive root system Φ^+ and a number $C > 0$ such that:

$$\left\{\begin{array}{l} \textit{If } E^{U(1)} \to S^{U(1)} \textit{ is irreducible,} \\ \lambda = d\chi^{U(1)}|_{\mathcal{H}} \in \mathcal{H}^*, \\ \lambda_{\mathbf{R}} \textit{ is the restriction of } \lambda \textit{ to the real form } \mathcal{H}_{\mathbf{R}} \\ \qquad\qquad \textit{on which roots take real values,} \\ <\lambda_{\mathbf{R}}, \alpha >> C, \textit{ for all } \alpha \in \Phi^+, \\ \textit{then } \mathbf{E}^{U(1)} \to S^{U(1)} \textit{ has both properties } (MG) \textit{ and } (Z). \end{array}\right. \tag{3.2}$$

Proposition 3.1. *Fix $(H, \mathcal{B})$. If (3.2) is true, then for arbitrary basic datum $(H, \mathcal{B}, \chi^{U(1)})$, the bundle $\mathbf{E}^{U(1)} \to S^{U(1)}$ has both properties (MG) and (Z).*

P r o o f. Assume (3.2). Applying Corollary 8.12 and Lemma 8.13 in [3] we see that if Proposition 3.1 fails, then it must fail for a basic datum $(H, \mathcal{B}, \chi^{U(1)})$

with $\mathbf{E}^{U(1)} \to S^{U(1)}$ irreducible. So we can suppose that $\mathbf{E}^{U(1)} \to S^{U(1)}$ is irreducible, $\lambda = d\chi^{U(1)}|_{\mathcal{H}} \in \mathcal{H}^*$. Let C be as in (3.2), r_0 as in Lemma 8.18 in [3]. Suppose $\lambda_0 \in \mathcal{H}^*_{\mathbf{R}}$ such that $< \lambda_0, \alpha >> C$ for all $\alpha \in \Phi^+$. Thus, by (3.2), the bundle $\mathbf{E}^{U(1)} \to S^{U(1)}$ has both (MG) and (Z).

Now, we fix one such λ_0 and let as in [3]:

$s_1 = \mathrm{Sup}\{r > 0 : \|\lambda_{\mathbf{R}} - \lambda_0\| < r$ implies (MG) and (Z) for $\mathbf{E}^{U(1)} \to S^{U(1)}\}$. We see that $s_1 \geq r_0$. Suppose s_1 is finite. It follows from Lemma 8.18 in [3] that there exists $s_2 > s_1$ such that the open ball $B(s_2)$, radius s_2 and center 0, contains $B(s_1)U$ {accessible from $B(s_1)$}. Then, by applying Lemmas 8.15 and 8.16 in [3] we see that properties (MG) and (Z) carry over from $B(s_1)$ to $B(s_2)$. That contradicts the choice of s_1. So s_1 is infinite. This completes the proof of our proposition. □

3.2. Maximally Real Polarization. Fix a basic datum $(H, \mathcal{B}, \chi^{U(1)})$, let $S = G.\mathcal{B} \subset X$ and $u = Codim_{\mathbf{R}} S$.

Proposition 3.2. *If $\mathcal{B}$ is maximally real then the (Z) part of (3.2) is true.*

P r o o f. Recall that G has a cuspidal parabolic subgroup $P = MA.N_H$ associated to H, such that $\mathcal{B} \subset \mathcal{P}$. Here $H = T \times A$ with $T = H \cap K, A = \exp(\mathcal{A} \cap \mathcal{G})$. Then $S^{U(1)} \cong H.N_H \backslash G$ and $S^{U(1)}$ fibres over $P \backslash G$ with holomorphic fibres $T \backslash M$. Let $\mathbf{E}^{U(1)} \to S^{U(1)}$ be irreducible, $\lambda = d\chi^{U(1)}|_{\mathcal{H}} \in \mathcal{H}^*$ and $\chi_T^{U(1)} = \chi^{U(1)}|_{T^{U(1)}}$, where $T^{U(1)}$ is the inverse image of T in $H^{U(1)}$ under the $U(1)$-covering projection. We see that $d\chi_T^{U(1)}|_{\mathcal{T}} = \lambda_{|\mathcal{T}}$. Suppose that $\mathbf{E}^{U(1)}|_{T\backslash M}$ is sufficiently negative. Then, it follows from Lemma 9.3 in [3] that the maps

$$H^p(T \backslash M, \mathbf{E}^{U(1)}|_{T\backslash M})_{(K\cap M)} \to A^p\left(M, T, \mathcal{B} \cap \mathcal{M}, \chi_T^{U(1)}\right) \tag{3.3}$$

are isomorphisms of $(\mathcal{M}, K \cap M)$-modules, these modules nonzero just for $p = dim_{\mathbf{C}}[T \backslash (K \cap M)]$. Let Z^p and B^p denote the corresponding spaces of closed and exact smooth $(K \cap M)$-finite $\mathbf{E}^{U(1)}$-valued $(0,p)$-forms on $T \backslash M$, and ${}^0Z^p$ and ${}^0B^p$ denote the corresponding spaces with "smooth" replaced by "formal power series" for the coefficients. Then as in [3], it follows from (3.3) that $B^p \backslash Z^p \cong {}^0B^p \backslash {}^0Z^p$ as $(\mathcal{M}, K \cap M)$-modules, where the isomorphism is induced by $Z^p \to {}^0Z^p$, Taylor series at 1.T.

Applying the Poincare' Lemma to the fibres $\cong N_H$ of the fibration $H \backslash G \to H.N_H \backslash G \cong S^{U(1)}$ we see that $A^p(G, H, \mathcal{B}, \chi^{U(1)})$ can be computed from the complex of left K-finite, right $K \cap M$ invariant, functions from K to the Zuckerman complex for $T \backslash M$. So, as in [3], we see that

$$H^p(S^{U(1)}; \mathbf{E}^{U(1)})_{(K)} \to A^p(G, H, \mathcal{B}, \chi^{U(1)})$$

is surjective. Since here each side is induced from the corresponding side of (3.3), then the surjection is an isomorphism. □

Proposition 3.3. *If $\mathcal{B}$ is maximally real then the (MG) part of (3.2) is true.*

P r o o f. Suppose that $\mathbf{E}^{U(1)} \to S^{U(1)}$ is irreducible, $\lambda = d\chi^{U(1)}|_{\mathcal{H}} \in \mathcal{H}^*$. Let $\lambda = \nu + i\sigma$, $\nu \in i\,(\mathcal{T} \cap \mathcal{G})^*$ deep in the negative Weil chamber of $\Phi(\mathcal{M}, \mathcal{T})$. We see that $H^p(S^{U(1)};\, \mathbf{E}^{U(1)}) = H^p(C^{-\omega}_{P\backslash G}(S^{U(1)};\, \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*_{S^{U(1)}}))$ vanishes except in degree $p_0 = \dim_{\mathbf{C}}(T \setminus K \cap M)$ and $H^{p_0}(S^{U(1)};\, \mathbf{E}^{U(1)}) = W$ is the $C^{-\omega}$ induced representation $Ind^G_{MAN_H}(\eta \otimes e^{i\sigma})$. The induced module $H^{p_0}(S^{U(1)}; \mathbf{E}^{U(1)})$ has finite length because η is irreducible. Then, as in the proof of Lemma 9.8 in [3], we see that W is (MG) and the operator $\bar{\partial}_{S^{U(1)}}$ has closed range. In particular, W inherits a Fréchet topology from the space $C^{-\omega}_{P\backslash G}(S^{U(1)};\, \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{p_0}\mathbf{N}^*_{S^{U(1)}})$. This completes the proof of our proposition. □

Now, we suppose that $\mathcal{B}$ is maximally real polarization. Propositions 3.1, 3.2 and 3.3 thus show that $\mathbf{E}^{U(1)} \to S^{U(1)}$ satisfies both (MG) and (Z). Combining this with Propositions 2.1, 2.2 and 2.3 we obtain.

Theorem 1. *For any maximally real polarization $(H, \mathcal{B}, \chi^{U(1)})$, there are topological isomorphisms between Fréchet G-modules*

$$H^p(C^{-\omega}(H \setminus G; \mathbf{E}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*)) \cong H^p(C^{-\omega}_{H^{U(1)}}(S^{U(1)};\, \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*_{S^{U(1)}})) \cong$$

$$\cong H^{p+u}(\tilde{S}; \mathcal{O}(\tilde{\mathbf{E}}^{U(1)}))$$

which are canonically and topologically isomorphic to the action of G on the maximal globalization of $A^p(G, H, \mathcal{B}, \chi^{U(1)})$. □

4. CHANGE OF POLARIZATION

In this section, as in [3] we formulate a dual statement on change of polarization and then show that Theorem 1 holds also for arbitrary basic datum $(H, \mathcal{B}, \chi^{U(1)})$. Suppose that $H = G_F$ is fixed. Let $\mathcal{B} \subset \mathcal{G}_{\mathbf{C}}$ be a polarization such that $\mathcal{H} \subset \mathcal{B}$ and $\mathcal{B}$ is not maximal real. Lemma 7.2 in [3] gives us a complex simple root α such that $\overline{\alpha} \notin \Phi^+$. Denote $\Phi_0^+ = S_\alpha\Phi^+$, $\mathcal{B}_0 = S_\alpha\mathcal{B}$ and $S_0 = G.\mathcal{B}_0$.

Given $\gamma \in \Phi(\mathcal{G}_{\mathbf{C}}, \mathcal{H})$, we can view γ as an element of $(\mathcal{H} \oplus \mathcal{U}(1)_{\mathbf{C}})^*$. Since $\mathcal{H}$ is the Cartan subalgebra of $\mathcal{G}_{\mathbf{C}}$, we obtain a representation $e^\gamma : H^{U(1)} \to \mathbf{C}^*$. Then we have vector bundles $\mathbf{L}_\gamma^{U(1)} \to S_0^{U(1)}$ and $\mathbf{L}_\gamma^{U(1)} \to S^{U(1)}$. Applying Lemma 10.6 in [3] we obtain G-equivariant morphisms of complexes.

$$C^{-\omega}_{H^{U(1)}}(S_0^{U(1)};\, \pi_X^*\mathbf{E}^{U(1)} \otimes \Lambda^p\mathbf{N}^*_{S_0^{U(1)}}) \to$$

$$C^{-\omega}_{H^{U(1)}}(S^{U(1)};\, \pi_X^*\mathbf{E}^{U(1)} \otimes \mathbf{L}_{-\alpha}^{U(1)} \otimes \Lambda^{p+1}\mathbf{N}^*_{S^{U(1)}}) \tag{4.1}$$

and (4.1) restricts to a morphism of subcomplexes

$$C^{-\omega}_{P\backslash G}(S_0^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\Lambda^p\mathbf{N}^*_{S_0^{U(1)}})\to$$
$$C^{-\omega}_{P\backslash G}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\mathbf{L}^{U(1)}_{-\alpha}\otimes\Lambda^{p+1}\mathbf{N}^*_{S^{U(1)}}) \tag{4.2}$$

Let $C^{-\omega}_{S_0^{U(1)}}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\mathbf{L}^{U(1)}_{-\alpha}\otimes\Lambda^{\cdot}\mathbf{N}^*_{S^{U(1)}})$ be the subcomplex of the complex $C^{-\omega}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\mathbf{L}^{U(1)}_{-\alpha}\otimes\Lambda^{\cdot}\mathbf{N}^*_{S^{U(1)}})$ consisting of forms ω and $\overline{\partial}_{S^{U(1)}}\omega$ vanish on (0,1) vectors tangent to the fibres of $S^{U(1)}\to S_0^{U(1)}$.

Applying the Dolbeault Lemma we see that the inclusion

$$C^{-\omega}_{S_0^{U(1)}}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\mathbf{L}^{U(1)}_{-\alpha}\otimes\Lambda^{\cdot}\mathbf{N}^*_{S^{U(1)}})\hookrightarrow$$
$$C^{-\omega}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\mathbf{L}^{U(1)}_{-\alpha}\otimes\Lambda^{\cdot}\mathbf{N}^*_{S^{U(1)}}) \tag{4.3}$$

induces isomorphisms on cohomology. On the other hand, we obtain the morphism of complexes

$$C^{-\omega}(S_0^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\Lambda^p\mathbf{N}^*_{S_0^{U(1)}})\to$$
$$C^{-\omega}_{S_0^{U(1)}}(S^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\mathbf{L}^{U(1)}_{-\alpha}\otimes\Lambda^{p+1}\mathbf{N}^*_{S^{U(1)}}). \tag{4.4}$$

Let $\mathcal{B}_\alpha=\mathcal{B}+\mathcal{G}_\alpha=\mathcal{B}_0+\mathcal{G}_{-\alpha}$. Denote by X_α the flag manifold of parabolic subalgebras of $\mathcal{G}_C$ which are $\mathrm{Int}(\mathcal{G}_C)$-conjugate to $\mathcal{B}_\alpha$ and consider the orbit $S_\alpha=G.\mathcal{B}_\alpha\subset X$. The natural projection $p_\alpha: X\to X_\alpha$ is holomorphic, and there exists a homomorphism $p_\alpha^{U(1)}: X^{U(1)}\to X_\alpha^{U(1)}$ such that $p_\alpha\circ\pi_X=\pi_{X_\alpha}\circ p_\alpha^{U(1)}$, where $\pi_{X_\alpha}: X_\alpha^{U(1)}\to X_\alpha$ is the natural projection.

Let $U_\alpha\subset S_\alpha$ be an S_α-open subset whose $\overline{U}_\alpha$ is compact and has an X_α-open neighborhood over which $p_\alpha: X\to X_\alpha$ holomorphically trivial. Let $U_0^{U(1)}=S_0^{U(1)}\cap(p_\alpha^{U(1)})^{-1}\pi^{-1}_{X_\alpha}(U_\alpha)$ and $U^{U(1)}=S^{U(1)}\cap(p_\alpha^{U(1)})^{-1}\pi^{-1}_{X_\alpha}(U_\alpha)$ we see that (4.4) localizes to maps

$$C^{-\omega}_{H^{U(1)}}(U_0^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\Lambda^p\mathbf{N}^*_{S_0^{U(1)}})\to$$
$$C^{-\omega}_{H^{U(1)},S_0^{U(1)}}(U^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)}\otimes\mathbf{L}^{U(1)}_{-\alpha}\otimes\Lambda^{p+1}\mathbf{N}^*_{S^{U(1)}}) \tag{4.5}$$

Let $Cl(U_0^{U(1)})^\sim$ and $\mathcal{B}d(U_0^{U(1)})^\sim$ denote germs of neighborhood of $Cl(U_0^{U(1)})$ and $\mathcal{B}d(U_0^{U(1)})$ in $S^{U(1)}US_0^{U(1)}$. The analogue of (A.9) in [3] for $S_0^{U(1)}$ is

$$C^{-\omega}(U_0^{U(1)};\pi_X^*\mathbf{E}^{U(1)}\otimes\Lambda^{\cdot}\mathbf{N}^*_{S_0^{U(1)}})=\frac{C^\omega(Cl(U_0^{U(1)});\mathbf{F}^{U(1)}\otimes\Lambda^{c-p}\mathbf{N}^*_{S_0^{U(1)}})'}{C^\omega(\mathcal{B}d(U_0^{U(1)});\mathbf{F}^{U(1)}\otimes\Lambda^{c-p}\mathbf{N}^*_{S_0^{U(1)}})'} \tag{4.6}$$

where $c = \dim_{CR} S_0$ and $\mathbf{F}^{U(1)} = \mathbf{E}^{U(1)} \otimes \mathbf{L}^{U(1)}_{-2\rho+2\alpha}$.

Similarly, we have

$$C^{-\omega}_{S_0^{U(1)}}(U^{U(1)};\ \pi_X^*\mathbf{E}^{U(1)} \otimes \mathbf{L}^{U(1)}_{-\alpha}\Lambda^{p+1}\mathbf{N}^*_{S^{U(1)}}) = \frac{C^{\omega}_{S_0^{U(1)}}(Cl(U_0^{U(1)})^{\sim};\ \tilde{\mathbf{F}}^{U(1)} \otimes \Lambda^{c-p}\mathbf{N}^*_{S^{U(1)}})'}{C^{\omega}_{S_0^{U(1)}}(\mathcal{B}d(U_0^{U(1)})^{\sim};\ \tilde{\mathbf{F}}^{U(1)} \otimes \Lambda^{c-p}\mathbf{N}^*_{S^{U(1)}})'} \tag{4.7}$$

where $c+1 = \dim_{CR} S_\alpha$, $\mathbf{F}^{\ U(1)} = (\pi_X^*\mathbf{E}^{U(1)} \otimes \mathbf{L}^{U(1)}_{-\alpha}) \otimes \mathbf{L}^{U(1)}_{-2\rho+\alpha}$, (see [3,§13]). Then, by a similar argument in [3, §13] we obtain the dual statement, as follows:

$$\left\{\begin{array}{l} \textit{The restriction maps} \\ C^{\omega}_{H^{U(1)},S_0^{U(1)}}(Cl(U_0^{U(1)})^{\sim};\ \tilde{\mathbf{F}}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*_{S^{U(1)}}) \to \\ \qquad\qquad C^{\omega}_{H^{U(1)}}(Cl(U_0^{U(1)});\ \mathbf{F}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*_{S_0^{U(1)}}) \\ C^{\omega}_{H^{U(1)},S_0^{U(1)}}(\mathcal{B}d(U_0^{U(1)})^{\sim};\ \tilde{\mathbf{F}}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*_{S^{U(1)}}) \to \\ \qquad\qquad C^{\omega}_{H^{U(1)}}(\mathcal{B}d(U_0^{U(1)});\ \mathbf{F}^{U(1)} \otimes \Lambda^{\cdot}\mathbf{N}^*_{S_0^{U(1)}}) \\ \textit{induce isomorphisms in cohomology.}\ \square \end{array}\right. \tag{4.8}$$

We know as in [3] that the restriction maps (4.8) are continuous and surjective, and are dual via (4.6) and (4.7) to the maps of (4.5). Thus, it follows from (4.8) that we obtain the following statement

$$\left\{\begin{array}{l} \textit{Suppose that } \chi^{U(1)} \textit{is irreducible}, \lambda = d\chi^{U(1)}|_{\mathcal{H}} \in \mathcal{H}^*, \\ \textit{suppose further that } 2 < \lambda + \rho - \alpha, \alpha > / < \alpha, \alpha > \\ \qquad\qquad\qquad\qquad \textit{is not a positive integer.} \\ \textit{Then } (4.1) \textit{ induces an isomorphism of cohomology groups.}\ \square \end{array}\right. \tag{4.9}$$

Theorem 2. *Fix H, and suppose that $\mathcal{B}$ is not maximal real. Then, for arbitrary basic datum $(H, \mathcal{B}, \chi^{U(1)})$, the bundle $\mathbf{E}^{U(1)} \to S^{U(1)}$ has both properties (MG) and (Z). In other words, Theorem 1 holds for arbitrary basic data of the form $(H, \mathcal{B}, \chi^{U(1)})$.*

P r o o f. By using Propositions 3.2 and 3.3 we may assume by induction on $\dim S^{U(1)} - \dim S^{U(1)}_{\max}$ that every $\mathbf{E}^{U(1)} \to S_0^{U(1)}$ has both (MG) and (Z). Since the cohomologies and maps that occur in Theorem 1 all are compatible with coherent continuation, we may as in [3] assume that $2 < \lambda + \rho - \alpha, \alpha > / < \alpha, \alpha >$ is not a positive integer, where $\chi^{U(1)}$ is irreducible and $\lambda = d\chi^{U(1)}|_{\mathcal{H}} \in \mathcal{H}^*$. Then, by applying Lemma 10.15 in [3], it follows from (4.9) that

$$C^{-\omega}_{P\backslash G}(S^{U(1)};\ \pi^*_X \mathbf{E}^{U(1)} \otimes \mathbf{L}^{U(1)}_{-\alpha} \otimes \Lambda^{\cdot} \mathbf{N}^*_{S^{U(1)}})$$

has both properties (MG) and (Z). This completes the proof of our theorem. □

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