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ON CONTINUOUS RINGS WITH CHAIN CONDITIONS

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Dedicated to Professor Nguyen Dinh Tri on his sixtieth birthday

Abstract. *In this paper, results on artinian rings, especially on QF rings, are obtained and presented.*

1. INTRODUCTION

For a right or left self-injective ring R the following conditions are equivalent:
i) R is quasi-Frobenius
ii) R has ACC on right annihilators
iii) R has ACC on essential right ideals
(see C. Faith [10] and Dinh Van Huynh, Nguyen V. Dung and R. Wisbauer [7]).

Inspired by this result, several authors investigated chain conditions in continuous rings, e.g. [3], [4], [11],... In this paper we follow this line and prove some more results on continuous rings satisfying weaker forms of ACC on annihilators or on essential right ideals. We also obtain results of rings with the restricted minimum condition on left ideals.

2.PRELIMINARIES

All rings R considered here are associative with identity and all modules are unitary. Let M be a left R-module. Then the socle of M is denoted by $Soc(M)$. A submodule N of M is essential in M (denoted by $N \overset{e}{\hookrightarrow} M$) if for each non-zero submodule L of M, $L \cap M \neq 0$. M has finite uniform dimension if M does not contain an infinite direct sum of non-zero submodules. For a subset A of a ring R,

$r(A)$ and $l(A)$ denote the right and left annihilators of A in R, respectively. For a module M, we denote by $E(M)$, $J(M)$ and $Z(M)$ the injective hull, the Jacobson radical and the singular submodule of M, respectively.

A module M is called a CS module if for every submodule A of M there exists a direct summand A^* (denoted by $A^* \overset{\oplus}{\hookrightarrow} M$) containing A such that $A \underset{e}{\hookrightarrow} A^*$. M is called a continuous module if M is a CS module and for every submodule A and B of M with $A \cong B$ and $B \overset{\oplus}{\hookrightarrow} M$ implies $A \overset{\oplus}{\hookrightarrow} M$. A ring R is called left (right) continuous if R is as a left (right, respectively) R-module continuous.

A ring R is said to be *orthogonally finite* if there is no infinite set of orthogonal idempotents in R, and R is called a ring *of enough idempotents* if the identity of R can be written as a sum of a finite number of orthogonal primitive idempotents of R. We have the implication:

Orthogonally finite $\Longrightarrow$ enough idempotents.

However, the converse is not true in general, see for example [5, p. 112].

The following results are used repeatedly in our paper:

Lemma 2.1 ([16, Theorem 1.2]).*Any left continuous ring R satisfies the following conditions:*

i) For any idempotent e and left ideal A contained in Re, there exists an idempotent f in Re such that Rf is an essential extension of A in R.

ii) If $Rg \cap Rh = 0$ for idempotents g and h, then $Rg + Rh$ is generated by an idempotent of R.

Lemma 2.2 ([16, Theorem 4.6]).*If R is a left continuous ring, then $Z(_RR) = J(R)$; $R/J(R)$ is a regular left continuous ring and idempotents modulo $J(R)$ can be lifted.*

Lemma 2.3. *If R is a left CS ring having enough idempotents, then R is a direct sum of indecomposable uniform left ideals.*

P r o o f. By definition we have $R = Re_1 \oplus \ldots \oplus Re_n$ where each Re_i is an indecomposable left ideal and $\{e_i\}_{i=1}^n$ is a set of primitive orthogonal idempotents. Since every Re_i is again a CS module, it follows that all Re_i are uniform.

Lemma 2.4 ([12, Theorem 13]). *Let $M = \bigoplus_{i=1}^{k} M_i$. Then M is continuous if and only if each M_i is continuous and M_j - injective for $j \neq i$.*

Lemma 2.5 ([16, Theorem 7.10]).*Suppose R is a two-sided continuous, two-sided artinian ring. Then R is a quasi-Frobenius ring.*

Lemma 2.6 ([4, Lemma 6]).*Let R be a semiprimary ring with ACC on left annihilators such that $Soc(_RR) = Soc(R_R)$ is finitely generated as a right ideal. Then R is right artinian.*

3. RESULTS

First we consider continuous rings with restricted chain conditions on annihilators. Motivated by [2, Theorem 1], we get

Theorem 3.1. *Suppose R is a left continuous ring and H is an ideal of R. If H has a decomposition as a left ideal*

$$H = \bigoplus_{i \in I} H_i$$

such that each H_i is indecomposable and the ring R/H is orthogonally finite, then I is finite.

P r o o f. Suppose that I is an infinite set. Since R is left continuous, there exists an idempotent e of R such that $H \underset{e}{\hookrightarrow} Re$. Since I is infinite, $e \notin H$. We can find a set of orthogonal idempotents in R/H as follows.

Step1. Write $I = \Lambda_1 \dot\cup \Gamma_1$ with $|\Lambda| = |\Gamma_1| = |\Lambda_1|$, where $\dot\cup$ denotes disjoint union and $|\cdot|$ denotes the cardinality. Let $B = \bigoplus_{\Lambda_1} H_\lambda$ and $C_1 = \bigoplus_{\Gamma_1} H_\lambda$. Then $H = B_1 \oplus C_1$. By Lemma 2.1, there exists B_1', C_1' contained in Re such that $B_1 \underset{e}{\hookrightarrow} B_1' \overset{\oplus}{\hookrightarrow} R$ and $C_1 \underset{e}{\hookrightarrow} C_1' \overset{\oplus}{\hookrightarrow} R$. Hence $B_1 + C_1 = B_1' + C_1'$. Since R is left continuous, $B_1' \oplus C_1' \overset{\oplus}{\hookrightarrow} R$, say $B_1' \oplus C_1' \oplus T = R$. Note that $H \hookrightarrow B_1' \oplus C_1' \hookrightarrow Re$. It follows that $B_1' \oplus C_1' \underset{e}{\hookrightarrow} Re$ and whence $Re = B_1' \oplus C_1' \oplus (Re \cap T)$, i.e. $B_1' \oplus C_1'$ is a direct summand of Re, however $B_1' \oplus C_1' \underset{e}{\hookrightarrow} Re$ hence $Re = B_1' \oplus C_1'$. So there are elements $e_1 \in B_1'$, $f_1 \in C_1'$ and $e = e_1 + f_1$. Whence $e_1 = re$ and then $e_1 = e_1 e$. Similarly, $f_1 = f_1 e$. Now we are going to prove $Re_1 = B_1'$. In fact, $Re_1 \hookrightarrow B_1'$. If $x \in B_1'$ then $x = r'e$ for some $r' \in R$, hence $x(1-e) = 0$ and $x = xe = x(e_1 + f_1) = xe_1 + xf_1$. Thus $xf_1 = x - xe_1 \in B_1' \cap C_1' = 0$, i.e. $x = xe_1 \in Re_1$. So $Re_1 = B_1'$. Similarly, $Rf_1 = C_1'$.

We claim that e_1, f_1 are orthogonal idempotents. We have $e_1 = e_1 e = e_1(e_1 + f_1) = e_1^2 + e_1 f_1$. It follows that $e_1 f_1 = e_1^2 - e_1 \in B_1' \cap C_1' = 0$. Thus $e_1 f_1 = 0$ and $e_1^2 = e_1$. Similarly, $f_1 e = 0$ and $f_1^2 = f_1$.

Since B_1, C_1 are not finitely generated, $e_1 \notin B_1$ and $f_1 \notin C_1$. We prove that $e_1 \notin H$, $f_1 \notin H$. In fact, if not, suppose $e_1 \in H$ then $Re_1 \hookrightarrow H = B_1 \oplus C_1$. Since $B_1 \hookrightarrow Re_1$, it follows that $Re_1 = B_1 \oplus (C_1 \cap Rf)$, i.e. B_1 is a direct summand of Re_1, however since $B_1 \underset{e}{\hookrightarrow} Re_1$, it follows that $B_1 = Re_1$, a contradiction. Similarly, $g \notin H$.

Step 2. Repeat the above argument on C_1, writing $\Gamma_1 = \Lambda_2 \cup \Gamma_2$ with $|\Gamma_1| = |\Lambda_2| = |\Gamma_2|$. As in step 1, let $B_2 = \oplus_{\Lambda_2} H$ and $C_2 = \oplus_{\Gamma_2} H$. Then $C_1 = B_2 \oplus C_2$ and $Rf_1 = Re_2 \oplus Rf_2$ where e_2 and f_2 are orthgonal idempotents. Now we claim that e_1, e_2, f_2 are orthogonal. Indeed, $e_2 = e_2 f_1$, $f_2 = f_2 f_1$ and $e_1 f_1 = f_1 e_1 = 0$, so $e_2 e_1 = e_2 f_1 e_1 = 0$ and $f_2 e_1 = f_2 f_1 e_1 = 0$. Also $0 = e_1 f_1 = e_1(e_2 + f_2) = e_1 e_2 + e_1 f_2$, it follows that $-e_1 e_2 = e_1 f_2 \in Re_2 \cap Rf_2 = 0$. Hence $e_1 e_2 = e_1 f_2 = 0$. Of course, $e_2, f_2 \notin H$.

Step 3. Assuming $e_1, e_2, \ldots, e_n, f_n$ are orthogonal idempotents obtained by writing $\Lambda = \Lambda_1 \dot\cup \Lambda_2 \dot\cup \ldots \dot\cup \Lambda_n \dot\cup \Gamma_n$ with $|\Lambda_i| = |\Gamma_i| = |\Lambda|$, then $\Gamma_n = \Lambda_{n+1} \dot\cup \Gamma_{n+1}$ with $|\Lambda_{n+1}| = |\Gamma_{n+1}| = |\Lambda|$ as above yields orthogonal idempotents $\{e_1, \ldots, e_n, e_{n+1}, f_{n+1}\}$. As in step 1, each $e_i \notin H$ and $f_{n+1} \notin H$. Then the set $\{e_1 + H, e_2 + H, \ldots\}$ gives an infinite set of orthogonal idempotents of R/H, a contradiction. Hence I is a finite set.

The following corollary extends a result of Armendariz and Park [2, Theorem 1].

Corollary 3.2. *Suppose R is a left continuous ring and the ring $R/Soc(_RR)$ is orthogonally finite. Then $Soc(_RR)$ is a finitely generated left R - module.*

Corollary 3.3. *If R is a left continuous ring and $R/Soc(_RR)$ is orthogonally finite, then R is a semiperfect ring.*

P r o o f. By Corollary 3.2, $Soc(_RR)$ contains no infinite family of orthogonal idempotents. By Lemma 2.2, R/J is a left continuous regular ring. R/J also has no infinite set of orthogonal idempotents, if not, by lifting of idempotents we can find an infinite family $\{e_i\}$ of orthogonal idempotents of R. Then $\{e_i + Soc(_RR)\}$ is a family of orthogonal idempotents in $R/Soc(_RR)$. By above, $Soc(_RR)$ contains a finite family of orthogonal idempotents, hence $\{e_i + Soc(_RR)\}$ is infinite, this contradicts the orthogonal finiteness of $R/Soc(_RR)$. Thus R/J is semisimple. By Lemma 2.2, R is a semiperfect ring.

Corollary 3.3 extends results of S. K. Jain, López- Permouth and S. T. Rizvi [11], V. Camillo; M. F. Yousif [3, Lemma 13], and Armendariz, Park [2,Corollary 2].

Theorem 3.4. *If R is a left continuous ring and $R/Soc(_RR)$ has ACC on left annihilators, then R is semiprimary.*

P r o o f. Note that if a ring R has ACC on left annihilators then it is orthogonally finite, because if not, then there exists an infinite chain of annihilators:

$$l(e_1, e_2, \ldots) \underset{\neq}{\hookrightarrow} l(e_2, e_3, \ldots) \underset{\neq}{\hookrightarrow} l(e_3, e_4, \ldots) \underset{\neq}{\hookrightarrow} \ldots$$

a contradiction. Hence by Corollary 3.3, R is semiperfect. We use a technique of [2, Theorem 3] to show that J is nilpotent. Put $S = Soc(_RR)$. Let $\{a_1, a_2, \ldots\}$ be a subset of J. Let $(a_1R + S)/S \underline{\hookleftarrow} (a_1a_2R + S)/S \underline{\hookleftarrow} \ldots$ be a descending chain of subsets of R/S. Then $l((a_1R + S)/S) \underline{\hookrightarrow} l((a_1a_2R + S)/S) \underline{\hookrightarrow} \ldots$. Since R/S has ACC on left annihilators, there exists a positive integer t such that:

$$l((a_1a_2\dots a_tR+S)/S)=l((a_1a_2\dots a_t\dots a_{t+k}R+S)/S),\ \forall k=0,1,\dots(*).$$

By Lemma 2.2, $J(R)=Z(_RR)$, hence $S.J=0$, i.e. $S\hookrightarrow l(J)$.
Thus for every n, $S\hookrightarrow l(J)\hookrightarrow l(a_1,a_2,\dots,a_n)$. We will prove that

$$l((a_1\dots a_nR+S)/S)\hookrightarrow l(a_1a_2\dots a_na_{n+1})/S\hookrightarrow l((a_1\dots a_na_{n+1}R+S)/S).$$

In deed, let $b+S\in l((a_1\dots a_nR+S)/S)$, then $ba_1a_2\dots a_n\in S\hookrightarrow l(J)\hookrightarrow l(a_{n+1})$. It follows that $ba_1a_2\dots a_na_{n+1}=0$, i.e. $b\in l(a_1a_2\dots a_na_{n+1})$, hence $b+S\in l(a_1a_2\dots a_{n+1})/S$. It is clear that

$$l(a_1\dots a_{n+1})/S\hookrightarrow l((a_1\dots a_{n+1}R+S)/S)$$

and by (*), it follows that $l(a_1\dots a_ta_{t+1})/S=l(a_1a_2\dots a_ta_{t+1}\dots a_{t+k})/S$, $k=0,1,\dots$ therefore $l(a_1\dots a_ta_{t+1})=l(a_1a_2\dots a_ta_{t+1}\dots a_{t+1+k})$, $k=0,1,\dots$ Particularly $l(a_1\dots a_ta_{t+1})=l(a_1a_2\dots a_ta_{t+1}a_{t+2})$. We shall prove that $a_1a_2\dots a_ta_{t+1}=0$. In fact, note that $l(a_{t+2})\underset{e}{\hookrightarrow}R$ because $a_{t+2}\in J=Z(_RR)$. Take $y\in l(a_{t+2})\cap Ra_1a_2\dots a_ta_{t+1}$. Then $ya_{t+2}=0$ and $y=xa_1a_2\dots a_ta_{t+1}$, for some $x\in R$. Thus $0=ya_{t+2}=xa_1a_2\dots a_ta_{t+1}a_{t+2}$, so $x\in l(a_1a_2\dots a_ta_{t+1}a_{t+2})$ $=l(a_1a_2\dots a_ta_{t+1})$. Thus $y=xa_1a_2\dots a_ta_{t+1}=0$. We have $l(a_{t+2})\cap Ra_1a_2\dots a_ta_{t+1}=0$, it follows that $Ra_1a_2\dots a_{t+1}=0$, especially $a_1a_2\dots a_ta_{t+1}=0$. Hence J is left T-nilpotent and the ideal $(J+S)/S$ of the ring R/S is also left T-nilpotent. Since R/S has ACC on left annihilators and by [5, Lemma 1.33], $(J+S)/S$ is nilpotent, there exists a positive integer m such that $J^m\hookrightarrow S$. Thus $J^{m+1}\hookrightarrow S.J=0$, i.e. J is nilpotent. This proves that R is semiprimary.

Corollary 3.5 (Jain, López-Permouth and Rizvi [11, Theorem 3] and Camillo and Yousif [3, Corollary 7]). *Let R be a left continuous ring with ACC on essential left ideals. Then R is left artinian.*

P r o o f. By [7, Lemma 2] $R/soc(_RR)$ is left noetherian. Using Lemma 3.17 we see that R is then left noetherian. Hence Theorem 3.4 shows that R is semiprimary. Thus R is left artinian.

By using the technique of proving Lemma 2.6, we can show the following:

Lemma 3.6. *If R is a semiprimary ring such that R has ACC on left annihilators and R satisfies the following conditions:*

i) $Soc(_RR)\hookrightarrow Soc(R_R)$ and

ii) $(SocR_R)_R$ is finitely generated.

Then R is right artinian.

P r o o f. We prove the lemma by induction on the index of nilpotency of J. Suppose $J^{n-1}\neq o$ and $J^n=0$ for some positive integer n. If $n=1$, it is clear that the lemma holds. Suppose the result is true for every $k < n$. Since R has ACC on left annihilators, there exists a finite subset $\{j_1, \dots, j_m\}$ of J such that $r(J) = r(\{j_1, \dots, j_m\})$. Since R/J is semisimple, $Soc(_RR) = r(J)$ and $Soc(R_R) = l(J)$. Let $\overline{R} = R/Soc(_RR)$. Clearly $\overline{R}$ is a semiprimary ring with $J(\overline{R})^{n-1} = 0$ and $\overline{R}$ retains the ACC on left annihilators. Now, if $\overline{J}\overline{x} = \overline{0}$ and $x \in \overline{x}$, then $Jx \subseteq r(J) \subseteq l(J)$, i.e. $0 = (Jx)J = J(xJ)$, i.e. $xJ \subseteq r(J)$. Thus $\overline{x}\overline{J} = \overline{0}$, i.e. $Soc(_{\overline{R}}\overline{R}) \subseteq Soc(\overline{R}_{\overline{R}})$. Consider the R-homomorphism f from $\overline{R}$ to $\bigoplus_{i=1}^{m} j_iR$ defined by $f(\overline{x}) = (j_1x, \dots, j_mx)$ (f is well-defined because $Soc(_RR) = r(J)$). Moreover, f is a monomorphism because $r(j_1, \dots, j_m) = Soc(_RR)$. Since $f(Soc(\overline{R}_R)) \subseteq Soc(R_R)$ which is finitely generated, it follows that $Soc(\overline{R}_R)$ and hence $Soc(\overline{R}_{\overline{R}})$ is finitely generated. Now by induction hypotheses, $\overline{R}$ is right artinian. Since $Soc(_RR) \subseteq Soc(R_R)$, it follows that $Soc(_RR)$ is right artinian, hence R is right artinian. The lemma is proved.

Theorem 3.7. *Let R be a left continuous ring. If R has ACC on left annihilators and $(SocR_R)_R$ is finitely generated, then R is right artinian.*

P r o o f. First R is semiperfect by Lemma 2.3. On the other hand, since R/J is semisimple we have $l(J) = Soc(R_R)$, however, since R is left continuous, $Soc(_RR).J = 0$. Hence $Soc(_RR) \subseteq Soc(R_R)$. Note that $R/Soc(_RR)$ has also ACC on left annihilators. By Theorem 3.4, R is semiprimary. From this and Lemma 3.6, it follows that R is right artinian.

Corollary 3.8 ([4, Theorem 1]). *If R is left and right continuous and R has ACC on left annihilators, then R is a QF ring.*

P r o o f. We can directly apply Theorem 3.7. But we can also prove as follows: Since R satisfies ACC on left annihilators, R is orthogonally finite. By Lemma 2.3, R is a direct sum of uniform right ideals and uniform left ideals, especially $(SocR_R)_R$ is finitely generated. By Theorem 3.7, R is right artinian. Moreover, since R is right and left continuous, it follows that

$$Soc(R_R) = Soc(_RR).$$

By [15, Theorem 3.5], R is a QF ring.

The condition "$Soc(R_R)$ is a finitely generated right R-module" in Theorem 3.7 is not superfluous as we can see from the following example:

Example: (Faith [9,7.11'. 1]). Let $R = \mathbb{Q}(x_1, \dots, x_n, \dots)$ the rational function field in infinitely many indeterminates, and let $S = \mathbb{Q}(x_1^2, x_2^2, \dots, x_n^2, \dots)$, let $f(x_i) = x_i^2, f(a) = a \quad \forall a \in \mathbb{Q}, \forall i$. Thus f is a ring epimorphism, and $\dim R_S = \infty$.

Let

$$A = \begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$$

then $\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$ is a right ideal of A, for any S-subspace V of R. Consider the subring (A, f) constructed via the homomorphism $f : R \to S$, then we cut down to just three left ideals:

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}, \text{ and } (A, f).$$

(A, f) is clearly a left continuous ring satisfying ACC on left annihilators. Note that $Soc(A, f)$ is not a finitely generated right (A, f)-module. (A, f) is also not a right artinian ring because the right ideals

$$\begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$$

of A is also right ideals of (A, f).

Now we obtain the following theorem which shows that with some additional conditions a continuous ring can become quasi-Frobenius.

Theorem 3.9. *Let R be a left CS right continuous ring. If R satisfies ACC on essential right ideals, then R is a QF ring.*

P r o o f. By Corollary 3.5, R is right artinian. In parlicular, R is orthogonally finite. By Lemma 2.3, R is a direct sum of uniform right ieals and uniform left ideals. Moreover, since R is right continuous, it follows that

$$Soc(R_R) \subseteq Soc({}_RR).$$

By [15, Theorem 3.5], R is a QF-ring.

Theorem 3.10. *Let R be a left and right continuous ring such that $R/Soc({}_RR)$ has ACC on left annihilators. If $Soc(R/Soc({}_RR))$ is a finitely generated right $R/Soc({}_RR)$-module, then R is a QF ring.*

P r o o f. By the above proof, since R is left and right continuous, it follows that $S = Soc(R_R) = Soc({}_RR)$. By Corollary 3.2, S is a finitely generated left R-module and by Theorem 3.4, R is semiprimary. Thus S is also a finitely generated right R-module by Lemma 2.3. Let $\overline{R} = R/S$. Similar to the proof of Lemma 3.6, we have $Soc({}_{\overline{R}}\overline{R}) = Soc(\overline{R}_{\overline{R}})$. From this and Lemma 3.6, it follows that $\overline{R}$ is right artinian. By Theorem 3.9, R is a QF ring.

Example 3.11 (see [9, Example 7.11′.2,p. 338]). Two-sided continuousness in Theorem 3.10 is necessary. C. Faith gave an example as follows: Let R be a ring with only three left ideals $0, J(R)$ and R. R is left and right artinian, with the right composition length 3. Note that R is left continuous but not right continuous. R is not quasi-Frobenius. Thus a one-sided continuous two-sided artinian ring need not be quasi-Frobenius.

Now we obtain a result characterising QF rings by means of left continuous rings satisfying weaker conditions.

Theorem 3.12. *For a ring R the following conditions are equivalent:*
i) R is a QF ring.
ii) R is a left continuous right CS ring satisfying ACC on left annihilators such that $Soc(R_R)$is an artinian left R-module.
iii)R is a left continuous ring satisfying ACC on essential left ideals such that ${}_RR \oplus_R R$ or R_R is a CS module.

P r o o f. i) $\Rightarrow$ ii) is clear.

ii) $\Rightarrow$ iii). Assume now that R is a left continuous right CS ring satisfying ACC on left annihilators. Then R has only a finite set of orthogonal idempotents in R. It is easy to see that R is a direct sum of indecomposable uniform left ideals and uniform right ideals. By Theorem 3.7, R is a right artinian ring.

By a similar proof as that of Theorem 3.7, we obtain:

$$Soc({}_RR) \subseteq Soc(R_R). \qquad (1)$$

Now with the assumptions that R is right artinian, satisfying (1) and $Soc(R_R)$ is an artinian left R-module, we can prove that R is left artinian by induction on the index of nilpotency of J. This is similar to the proof of Lemma 3.6. Thus ii) $\Rightarrow$ iii).

iii) $\Rightarrow$ i). Assume now that R has ACC on essential left ideals and ${}_RR \oplus_R R$ is a CS module. Moreover R is left continuous. By [11, Theorem 3], R is left artinian. We also obtain (1). Further, R is also a direct sum of indecomposable uniform left ideals and a direct sum of uniform right ideals. By [15, Theorem 3.5], R is a left self-injective ring, proving that R is a QF ring.

For the case, when R is a left continuous right CS ring satisfying ACC on essential left ideals, see Theorem 3.9.

The proof of Theorem is complete.

Remark. This Theorem generalizes a recent result of V. Camillo and M. F. Yousif [4, Theorem 1].

Now we are going to consider a continuous ring with restricted minimum condition. Following [8], a ring R is called a left CPA ring if every cyclic left R-module is a direct sum of a projective module and an artinian module, and is called left RM ring (restricted minimum condition) if for each left essential ideal I of R, the module R/I is artinian.

Theorem 3.13. *If R is a left continuous left CPA ring, then R is left artinian.*

P r o o f. By [8, Theorem 2.1],R has a direct decomposition

$$_RR = A \oplus U^{(1)} \oplus \cdots \oplus U^{(t)},$$

where A is an ideal of R such that $_RA$ is artinian and each $U^{(i)}$ is a uniform left R-module with $Soc(_RU^{(i)}) = 0$. We will prove that $U^{(i)} = 0$ for every i. Assume on the contrary that $U^{(i)} \neq 0$ for some i. Take $0 \neq x \in U^{(i)}$, then $Rx =_R P \oplus_R B$ where $_RP$ is projective and $_RB$ is artinian; however $Soc(Rx) = 0$, it follows that $B = 0$, i.e. Rx is projective. Now,we consider the R-homomorphism φ from R onto Rx defined by $\varphi(r) = rx$. Then $Rx \cong R/Ker\varphi$. Since Rx is projective, it follows that $R = U^{(i)} \oplus Ker\varphi$. Since $U^{(i)} \cong Rx$ and R is left continuous, it follows that $R = Rx \oplus V$. Hence

$$U^{(i)} = R \cap U^{(i)} = (Rx \oplus V) \cap U^{(i)} = Rx \oplus (V \cap U^{(i)}).$$

Since $Rx \neq 0$, $U^{(i)}$ is uniform, it follows that $V \cap U^{(i)} = 0$. Hence $U^{(i)} = Rx$ for every $x \neq 0$ of $U^{(i)}$, showing $U^{(i)}$ is simple, a contradiction to $Soc(U^{(i)}) = 0$. Therefore $U^{(i)} = 0$, and $_RR$ is artinian.

As a consequence of Theorem 3.13 we have:

Corollary 3.14. *If R is a left continuous left RM ring , then R is left artinian.*

P r o o f. Let A be a left ideal of R. Then there exists a direct summand A' of R such that $A \underset{e}{\hookrightarrow} A'$:

$$R = A' \oplus B.$$

Therefore $R/A \cong (A'/A) \oplus B$, with $_R(A'/A)$ artinian and $_RB$ projective. Hence R is a left CPA ring. By Theorem 3.13, R is left artinian.

As a consequence of Theorem 3.12 and Theorem 3.14 we obtain:

Corollary 3.15. *If R is a left continuous left RM right CS ring, then R is a QF ring.*

Corollary 3.16([13, Theorem 3.2]). *If R is a left self-injective left RM ring, then R is a quasi-Frobenius ring.*

The question whether or not a left continuous right RM ring is left artinian remains open. The following Theorem 3.18 answers this question in the semiprime case affirmatively.

Lemma 3.17. ([cf. 7, Lemma 1]). *Let M be a finitely generated CS left R-module. Suppose that M contains an infinite direct sum of nonzero submodules $H = \underset{\Lambda}{\oplus} H_\lambda$. Then the factor module M/H has infinite uniform dimension.*

Theorem 3.18. *Let R be a left continuous right RM semiprime ring. Then R is semisimple.*

P r o o f. Since R is semiprime, $S = Soc(_RR) = Soc(R_R)$. By Lemma 2.2, R/J is a regular left continuous ring and idempotents modulo $J(R)$ can be lifted.

It is clear that $\overline{R} = R/J$ is right RM. Let $\overline{S}_1$ be the right socle of $\overline{R}$. Then $\overline{R}/\overline{S}_1$ has finite uniform dimension as a right $\overline{R}$-module by [8, Lemma 2.4], hence $\overline{R}/\overline{S}_1$ is semisimple. Since $\overline{S}_1$ is also the left socle of $\overline{R}$. By Lemma 3.17, ${}_{\overline{R}}\overline{S}_1$ is finitely generated. By Corollary 3.2, $(\overline{S}_1)_{\overline{R}}$ is also finitely generated. Thus $\overline{R}$ is two-sided artinian. Therefore ${}_RS$ and S_R are finitely generated. By [8, Lemma 2.4], R/S has finite right uniform dimension, hence R has finite right uniform dimension, k say. It follows that R contains k independent uniform right ideals $U_1, \ldots, U_k$ such that:

$$U_R = U_1 \oplus \ldots \oplus U_k \underset{e}{\hookrightarrow} R_R;$$

hence $(R/U)_R$ is artinian. We also note that for each nonzero submodule V_i of $U_i (i = 1, \ldots, k)$, U_i/V_i is also artinian, then U_R has Krull dimension (at most 1). Hence R has right Krull dimension (at most 1). Since R is semiprime, it follows that R is right Goldie. By [5,Corollary 1.15], R has DCC on right annihilators. Therefore R has ACC on left annihilators. By Theorem 3.4, R is semisimple.

Theorem 3.18 generalizes a result of Dinh Van Huynh in [6, Proposition 2.2].

Remark. After finishing this paper we received a preprint of P. Ara and J. K. Park: On continuous semiprimary rings, in which Cor. 3.2, Cor. 3.3 and Theorem 3.4 are also obtained.

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