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RENEWAL PROCESSES ON TOPOLOGICAL SPACES WITH UNIFORM ACTION GROUPS

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Dedicated to Professor Nguyen Dinh Tri on his sixtieth birthday

Abstract. *In this paper we investigate some basic questions of renewal processes, induced by uniform actions of locally compact groups on topological spaces.*

INTRODUCTION

Renewal processes on topological groups were studied in depth by many authors (see, for example, [1], [2]). This article is devoted to the renewal theory on topological spaces with transformation groups. An essential obstacle in establishing and proving basic results, concerning the finiteness of the renewal function, is the possible appearance of noncompact stationary subgroups at points of the space. Generally, the study of random processes on topological spaces with transformation groups (in particular, on homogeneous spaces) is much more complicated than on topological groups and some open problems still exist in this area. For instance, the Loynes dichotomy theorem is not true for induced random walks on homogeneous spaces in general, although it holds under certain assumptions ([3], [5], [8]). Some reasons of this observation were discussed in [8], [9].

In this paper we consider uniform actions of locally compact groups on topological spaces and investigate some basic questions of renewal processes induced by these actions on the corresponding topological spaces.

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1. UNIFORM ACTIONS OF GROUPS

Let M be a topological space, G a topological group with the unit element e. We say that G acts (continuously) on M if given a continuous map $f : G \times M \to M$ satisfying the following conditions:

i) For any $g \in G$ the map $g : M \to M$, sending each point $x \in M$ to the point $f(g,x)$, is a homeomorphism. In this case we write $gx = f(g,x)$.

ii) $gh(x) = g(Hx)$ for any $g, h \in G$ and any $x \in M$.

iii) $ex = x$ for any $x \in M$.

For each point $x \in M$ the set

$$H_x = \{g \in G | gx = x\}$$

is a closed subgroup of G, called the stationary subgroup at the point x. It is easy to check that if $Y = gx$ then $H_y = gH_xg^{-1} = \{ghg^{-1} | h \in H_x\}$.

The set

$$G_x = \{gx \in M | g \in G\}$$

is called the orbit of the point x under the action $G.G_x$ is a closed subset of M, homeomorphic to the homogeneous space G/H_x.

Definition 1.1. *An action of a topological group G on a topological space M is called uniform at a point $x \in M$ if for any neighborhood U of x there exists a neighborhood U' of x such that $h(U') \subset U$ for all $h \in H_x$. An action G on M is called uniform if it is uniform at each point of M.*

Proposition 1.2. *If an action of a topological group G on a topological space M is uniform at a point $x \in M$, then it is uniform at each point of the orbit G_x.*

P r o o f. Assume that $x \in M$ is uniform point of the action G, that is for any neighborhood U of x there exists a neighborhood U' of x such that $h(U') \subset U$ for every $h \in H_x$. Letting $y = gx \in G_x$, we have $H_y = gH_xg^{-1}$. Suppose now that V is an arbitrary neighborhood of y. Set $U = g^{-1}(V)$. According to the assumption one can choose a neighborhood U' of x such that $H_x(U') \subset U$. Put $V' = g(U')$. We have $H_y(V') = gH_xg^{-1}(V') = gH_x(U') \subset g(U) = V$. This completes the proof.

Corollary 1.3. *Suppose the action G on M is transitive. Then it is uniform if and only if it is uniform at a point of M.*

For any two points $x, y \in M$ consider the set

$$H(x,y) = \{g \in G | gx = y\}.$$

Clearly, $H(x,y) \neq \emptyset$ if and only if x and y belong to the same orbit, i.e. $G_x = G_y$. Assume that $H(x,y) \neq \emptyset$ and $y = gx$. Then $h \in H(x,y)$ if and only if $hx = gx$ or $g^{-1}hx = x$. This means that $g^{-1}h \in H_x$ or $h \in gH_x$. Thus, $H(x,y) = gH_x$. Similarly, $H(x,y) = H_y g$. From the definition it follows that

$$H(x,y)^{-1} = H(y,x) \text{ for any } \quad x,y \in M. \tag{1.1}$$

Now, let U and V be subsets of M and put

$$H(U,V) = \bigcup_{(x,y)\in U\times V} H(x,y).$$

We note that if U is a neighborhood of x or V is a neighborhood of y, then $H(U,V)$ is a neighborhood of $H(x,y)$ in G. From (1.1) it follows that

$$H(U,V)^{-1} = H(V,U) \text{ for any } \; U,V \subset M. \tag{1.2}$$

Theorem 1.4. *Let a topological group G act uniformly on a topological space M. Then for any $x \in M$ and any open subsets V, V' such that $\overline{V} \subset V'$ and $\overline{V}$ is compact, there exists a neighborhood U of x such that $H(U,V) \subset H(x,V')$.*

From Theorem 1.4 and the equality (1.2) it follows immediately the following

Corollary 1.5. *Given a uniform action of a topological group G on a topological space M. Then, for any $y \in M$ and open subsets U, U' such that $\overline{U} \subset U'$ is compact, there exists a neighborhood V of y such that $H(U,V) \subset H(U',y)$.*

To prove Theorem 1.4 we need the following lemmas:

Lemma 1.6. *Let the action G on M be uniform. Suppose $x,y \in M$ are arbitrary points. Then for any neighborhood V of y there exists a neighborhood U of x such that $h(U) \subset V$ for all $h \in H(x,y)$.*

P r o o f. If $H(x,y) = \emptyset$, then the statement is obvious. Assume that $H(x,y) \neq \emptyset$ and let $y = gx$. As noted above we have $H(x,y) = gH_x$. Suppose now that V is a neighborhood of y. We put $U = g^{-1}(V)$. By definition there exists a neighborhood U' of x such that $H_x(U') \subset U$. Hence $H(x,y)(U') = gH_x(U') \subset g(U) = V$, completing the proof.

Lemma 1.7. *Given a uniform action G on M. Let $x \in M$ be an arbitrary point, V and V' be open subsets in M such that $\overline{V} \subset V'$ and $\overline{V}$ is compact. Then there exists a neighborhood U of x such that $h(U) \subset V'$ for any $h \in H(x,V)$.*

P r o o f. If $V \cap G_x = \emptyset$, i.e. $H(x,V) = \emptyset$, then the statement is obvious. Assume that $V \cap G_x \neq \emptyset$. Suppose $z \in \overline{V} \cap G_x$. Each element $h \in H(x,V)$ can be expressed in the form $h = gh'$, where $h' \in H(x,z)$, $g \in H(z,V)$. From the continuity of the action G on M it follows that there exists a neighborhood V_z of z in M and a neighborhood K_z of the unit e in G such that $g(V_z) \subset V'$ for any $g \in K_z$. Choose a neighborhood W_z of z in M such that $W_z \subset V_z$ and $G_x \cap W_z \subset \{gz | g \in K_z\}$. By Lemma 1.6 there exists a neighborhood U_z of x in M such that $h(U_z) \subset W_z$ for any $h \in H(x,z)$. Since $\overline{V} \cap G_x$ is compact (because $\overline{V}$ is compact), one can select a finite covering $\{W_{z_i}\}_{1\leq i\leq k}$ of $\overline{V}\cap G_x$. Set $U=\bigcap_{i=1}^{k} U_{z_i}$. We prove that U is a desired neighborhood of x. Namely, suppose $h\in H(x,\overline{V})$ $(=H(x,\overline{V}\cap G_x))$ and let $h\in H(x,W_{z_i})$. As noted above, h has the form $h=gh'$, where $h'\in H(x,z_i), g\in K_{z_i}$. Then $h(U)=gh'(U)\subset gh'(U_{z_i})\subset g(W_{z_i})\subset g(V_{z_i})\subset V'$. The proof is complete.

P r o o f of Theorem 1.4. A neighborhood $U\subset M$ is said to be symmetric at a point $x\in U$ if $gx\in U$ implies $g^{-1}x\in U$ for any $g\in G$. Clearly, for any neighborhood U of x in M one can find a neighborhood $U'\subset U$, symmetric at x (such neighborhoods can be obtained from symmetric neighborhoods of the unit e in the group G).

Now, suppose V, V' are open subsets in M such that $V\subset\overline{V}\subset V'$ and $\overline{V}$ is compact. According to Lemma 1.7 one can choose a neighborhood U of x such that $h(U)\subset V'$ for any $h\in H(x,V)$. Moreover, by virtue of the above remark U can be supposed to be symmetric at the point x. Let $y\in U, y=gx$. Then $g^{-1}x\in U$. For any $\tilde{g}\in H(y,V)$ we have $\tilde{g}=hg^{-1}$, where $h\in H(x,V)$. Hence, $\tilde{g}x=hg^{-1}x\in h(U)\subset V'$, i.e. $\tilde{g}\in H(x,V')$. Consequently, $H(U,V)\subset H(x,V')$. The proof is complete.

2. THE RENEWAL FUNCTIONS OF THE ACTION

Suppose now that M is a locally compact normal topological space and G is a locally compact normal topological group, acting uniformly on M. Consider the σ- fields on M and G, consisting of Borel subsets (that is the σ- fields generated by compact subsets on M and G, respectively).

Let p be a Radon measure on M and q a Randon measure on G. The convolution of p and q is defined to be a Radon measure $p*g$ on M given by the formula

$$p*g(X)=\int_G p(g^{-1}X)q(dg) \tag{2.1}$$

for any Borel subset X on M. In particular, if $M=G$ then we have the convolution of two Radon measures on G. It is easy to verify that

$$(p*q_1)*q_2=p*(q_1*q_2) \tag{2.2}$$

for any Radon measure p on M and any Radon measures q_1 and q_2 on G. In particular, the equality (2.2) makes it possible to define the convolution powers $q^{*n}=q*q*\ldots*q$ (n times) of a Radon measure q on G.

Now, suppose that q is a normalized positive measure (i.e. probability distributions). Then so are its convoltion powers q^{*i}. The sums

$$Q = \sum_{i=0}^{\infty} q^{*i}, \qquad Q_n = \sum_{i=0}^{n} q^{*i},$$

where $q^{*0} = 1$ denotes the normalized measure concentrated at e, are called *the renewal functions* of the action G, associated to q.

Definition 2.1. *Suppose $z \in M$. A point $x \in M$ is said to be finite with respect to (z,q) (or simply, (z,q)-finite) if there exists a neighborhood W_z of z in M such that $Q(H(W_z,x)) < \infty$.*

Remark 2.2. If $z \notin G_x$ then there exists a neighborhood W_z of z such that $W_z \cap G_x = \emptyset$, i.e. $H(W_z,x) = \emptyset$. This means that x is (z,q)-finite.

Theorem 2.3. *If $x \in M$ is a (z,q)-finite point, then there exists a neighborhood U of x in M, consisting of (z,q)-finite points.*

P r o o f. Assume that x is (z,q)-finite. By definition there exists a neighborhood W of z such that $Q(H(W,x)) < \infty$. Choose a neighborhood W' of z, $\overline{W}' \subset W$ and $\overline{W}'$ is compact. According to Corollary 1.5 one can find a neighborhood U of x, satisfying the condition $H(W',U) \subset H(W,x)$. For any $y \in U$ we have $H(W',y) \subset H(W',U) \subset H(W,x)$. Consequently, $Q(H(W',y) \leq Q(H(W,x) < \infty$. This means that y is a (z,q)-finite point. Thus, the theorem has been proved.

Definition 2.4. *Suppose $z \in M$. A point $x \in M$ is called infinite with respect to (z,q) (or simply (z,q)-infinite) if it is not (z,q)-finite.*

Definition 2.5. *A (z,q)-finite point $x \in M$ is called (z,q)-positive if $Q(H(W,x)) > 0$ for any neighborhood W of the point z. In the converse case x is called (z,q)-trivial.*

Clearly, M splits into the (z,q)-positive, (z,q)-trivial and (z,q)-infinite points.

Theorem 2.6. *Suppose $z \in M$. A point $x \in M$ is (z,q)-trivial if and only if $x \notin \{x = gz | g \in Supp(Q)\}$.*

P r o o f. First of all we note that $\mathrm{Supp}(Q)$ is the closure of $\bigcup_{i=0}^{\infty} \mathrm{Supp}(q^{*i})$. Setting

$$S_i = \{x = gz | g \in \mathrm{Supp}(q^{*i})\}$$
$$S = \{x = gz | g \in \mathrm{Supp}(Q)\}$$

we have $S = \overline{(\bigcup_{i=0}^{\infty} S_i)}$. Suppose $x \in S_i$, i.e. $x = gz$, where $g \in \mathrm{Supp}(q^{*i})$. Then for any neighborhood W of z the set $H(W,x)$ contains a neighborhood of g in G and therefore $q^{*i}(H(W,x)) > 0$. Consequently, $Q(H(W,x)) > 0$. Suppose $x \in S$ and let W be a neighborhood of z: By force of Corollary 1.5, $H(W,x) \supset H(W',U)$ for a neighborhood W' of z and a neighborhood U of x. On the other hand, U contains a point $x' \in \bigcup_{i=0}^{\infty} S_i$, say, $x' \in S_i$. Obviously, $H(W',U) \supset H(W',x')$. By using the fact proved above we have $Q(H(W,x) \geq Q(H(W',x')) > 0$. Thus, all the points of S are not (z,q)-trivial. Suppose now $x \notin S$. There exists a neighborhood U of x such that $U \cap S = \emptyset$. This means that $H(z,U) \cap \mathrm{Supp}(Q) = \emptyset$, i.e. $Q(H(z,U)) = 0$. According to Theorem 1.4, $H(W,x) \subset H(z,U)$ for a neighborhood W of z. Hence, $Q(H(W,x)) = 0$. Consequently, x is (z,q)-trivial. The proof is complete.

Remark 2.7. It is easy to see that $\mathrm{Supp}(Q)$ coincides with the closed semigroup in G, generated by $\mathrm{Supp}(q)$.

Theorem 2.8. *Suppose that a point $z \in M$ is (z,q)-finite. Then every point of G_z is (z,q)-finite.*

P r o o f. The assumption of the theorem means that $Q(H(W,z)) < \infty$ for a neighborhood W of z. Replacing W by a smaller neighborhood if necessary one can assume, by virtue of Corollary 1.5, that $Q(H(W,W)) < \infty$. Consider a symmetric neighborhood U of z, satisfying the conditions: $U \subset \overline{U} \subset W$, $\overline{U}$ is compact. According to the Urysohn's Lemma, there exists a continuous function $\varphi(x)$ on M such that $0 \leq \varphi(x) \leq 1$, $\varphi(x) = 1$ on $\overline{U}$ and $\varphi(x) = 0$ on $M \setminus W$. Consider the functions:

$$\varphi_n(x) = \int_G \varphi(g^{-1}x) Q_n(dg), \quad n = 0, 1, 2, \ldots.$$

We have

$$\varphi_n(x) = \int_{H(W,x)} \varphi(g^{-1}x) Q_n(dg) + \int_{G \setminus H(W,x)} \varphi(g^{-1}x) Q_n(dg)$$
$$= \int_{H(W,x)} \varphi(g^{-1}x) Q_n(dg) \leq Q_n(H(W,x)).$$

Hence, if $x \in W$ then

$$\varphi_n(x) \leq Q_n(H(W,W)) \leq Q(H(W,W)) < \infty. \tag{2.3}$$

On the other hand,

$$\varphi_n(x) = \int\limits_G \varphi(g^{-1}x)Q_n(dg) =$$

$$= \int\limits_G \varphi(g^{-1}x)q^{*0}(dg) + \int\limits_G \varphi(g^{-1}x)Q_{n-1} * q(dg)$$

$$= \varphi(x) + \int\limits_G \varphi(g^{-1}x)(\int\limits_G Q_{n-1}(h^{-1}dg)q(dh)).$$

Putting $s = h^{-1}g$, we have $g = hs$, $g^{-1} = s^{-1}h^{-1}$; and the expression above has the form

$$\varphi_n(x) = \varphi(x) + \int\limits_G (\int\limits_G \varphi(s^{-1}(h^{-1}x))Q_{n-1}(ds))q(dh)$$

$$= \varphi(x) + \int\limits_G \varphi_{n-1}(h^{-1}x)q(dh)$$

$$= \varphi(x) + \int\limits_G \varphi_{n-1}(g^{-1}x)q(dg).$$

In partitular, if $x \notin W$ then $\varphi(x) = 0$, and we have

$$\varphi_n(x) = \int\limits_G \varphi_{n-1}(g^{-1}x)q(dg). \tag{2.4}$$

Now we prove by induction that

$$\varphi_n(x) \leq c = \max\{Q(H(W,W)), 1\}. \tag{2.5}$$

Really, for $n = 0$ we have $\varphi_0(x) = \varphi(x) \leq 1$. Assume that (2.5) is true for $n-1$. If $x \in W$ then $\varphi_n(x) \leq Q(H(W,W)) \leq c$ by force of (2.3). If $x \notin W$ then from (2.4) and the induction hypothesis it follows that $\varphi_n(x) \leq c$. By that way (2.5) is true for n.

Furthermore, for any $x \in G_z$ we have

$$\varphi_n(x) = \int\limits_G \varphi(g^{-1}x)Q_n(dg) \geq \int\limits_{H(U,x)} \varphi(g^{-1}x)Q_n(dg)$$

$$= \int\limits_{H(U,x)} Q_n(dg). \tag{2.6}$$

From (2.5) and (2.6) it follows that $Q(H(U,x)) \leq c < \infty$. Consequently, x is a (z,q)-finite point. The proof is complete.

Theorem 2.9. *Suppose that a point $z \in M$ is (z,q)-infinite. Then every point of $S = \{x = gz | g \in Supp(Q)\}$ is (z,q)-infinite.*

P r o o f. The assumption of the theorem means that $Q(H(W,W)) = \infty$ for any neighborhood W of z. Choose a neighborhood U of z such that $U \subset \overline{U} \subset W$ and $\overline{U}$ is compact. Consider an Urysohn's function φ defined as in the proof of Theorem 2.6. We have

$$\varphi_n(x) = \int_G \varphi(g^{-1}x)Q_n(dg) = \int_{H(W,x)} \varphi(g^{-1}x)Q_n(dg) \geq$$

$$\geq \int_{H(U,x)} \varphi(g^{-1}x)Q_n(dg) = Q_n(H(U,x)). \tag{2.7}$$

Now we construct a symmetric neighborhood V of z such that $H(z,V) \subset H(U,x)$ for any $x \in V$. Choose a symmetric neighborhood U_1 of z such that $H_z(U_1) \subset U.U_1 \cap G_z$ can be expressed in the form $\{gz | g \in A\}$, where A is a symmetric neighborhood of e in G. Further, take V so that $V \cap G_z = \{gz | g \in B\}$ for B being a symmetric neighborhood of e such that $B^2 \subset A$. We verify that V satisfies our requirement. Indeed, suppose that $x \in V \cap G_z$ and $g \in H(z,V)$. We have $x = b_1 z$ $(b_1 \in B)$ and $gz = b_2 z \in V$ $(b_2 \in B)$. Therefore, $g = b_2 h$ $(h \in H_z)$ and $g^{-1}x = h^{-1}b_2^{-1}b_1 z \in U$ because $b_2^{-1}b_1 z \in U_1$. Consequently, $g \in H(U,x)$. By using Theorem 1.4 one can find a neighborhood V' of z such that $H(V',V') \subset H(z,V)$. Thus, from (2.7) it follows that

$$\varphi_n(x) \geq Q_n(H(V',V')) \to \infty \quad \text{for} \quad x \in V. \tag{2.8}$$

We use the following formula

$$\varphi_n(x) = \varphi(x) + \int_G \varphi_{n-1}(g^{-1}x)q(dg), \tag{2.9}$$

derived in the proof of Theorem 2.6. For each $x \in S_1 = \{x = gz | g \in \text{Supp}(q)\}$ we have $q(H(V,x)) > 0$. Then

$$\varphi_n(x) \geq \int_{H(V,x)} \varphi_{n-1}(g^{-1}x)q(dg) \to \infty \quad (n \to \infty).$$

From (2.9) one can obtain the following formula

$$\varphi_n(x) = \varphi(x) + \int_G \varphi(g^{-1}x)q(dg) + \int_G \varphi_{n-2}(g^{-1}x)q^{*2}(dg). \qquad (2.10)$$

For each $x \in S_2 = \{x = gz | g \in \mathrm{Supp}(q^{*2})\}$ we have $q^{*2}(H(V,x)) > 0$. Therefore,

$$\varphi_n(x) \geq \int_{H(V,x)} \varphi_{n-2}(g^{-1}x)q^{*2}(dg) \to \infty \quad (n \to \infty).$$

Similarly, one can prove that $\varphi_n(x) \to \infty$ for any $x \in S_i = \{x = gz | g \in \mathrm{Supp}(q^{*i})\}, i = 0, 1, 2, \ldots$. Consequently, $\varphi_n(x) \to \infty$ for any $x \in \bigcup_{i=0}^{\infty} S_i$.

On the other hand we have

$$\varphi_n(x) = \int_{H(W,x)} \varphi(g^{-1}x)Q_n(dg) + \int_{G\setminus H(W,x)} \varphi(g^{-1}x)Q_n(dg) =$$

$$= \int_{H(W,x)} \varphi(g^{-1}x)Q_n(dg) \leq Q_n(H(W,x)).$$

Thus, for each $x \in \bigcup_{i=0}^{\infty} S_i$ we have $Q_n(H(W,x)) \to \infty$ for any neighborhood W of z.

Consequently, x is a (z,q)-infinite point. If $x \in S = \bigcup_{i=1}^{\infty} S_i$, then any neighborhood U_x of x intersects $\bigcup_{i=1}^{\infty} S_i$ at a point x'. Therefore, $Q_n(H(W,U_x)) \geq Q_n(H(W,x')) \to \infty$ for any neighborhood W of z. Choosing a suitable neighborhood $W' \subset W$ we have $Q_n(H(W,x)) \geq Q_n(H(W',U_x)) \to \infty$ for any W. Hence x is a (z,q)-infinite point.

Now let us sum up the results of this section. It turns out that from Theorems 2.6 - 2.9 it follows a complete description of trivial, positive and infinite points.

Theorem 2.10. *a) If a point $z \in M$ is (z,q)-finite (i.e. $Q(H(W,W)) < \infty$ for a neighborhood W of z), then every point $x \in M$ is (z,q)-finite. Moreover, the points of $S = \{x = gz | g \in Supp(Q)\}$ are (z,q)-positive and the points of $M \setminus S$ are (z,q)-trivial.*

b) If a point $z \in M$ is (z,q)-infinite (i.e. $Q(H(W,W)) = \infty$ for any neighborhood W of z), then every point of $S = \{x = gz | g \in Supp(Q)\}$ is (z,q)-infinite and every point of $M \setminus S$ is (z,q)-trivial.

3. THE RENEWAL FUNCTIONS OF THE SPACE

Let p be a positive measure on M and q a normalized positive measure on G. Then the measures $p * q^{*i}$ $(i = 0, 1, 2, \dots)$ are positive. The sums

$$p * Q = \sum_{i=0} p * q^{*i}, \quad p * Q_n = \sum_{i=0}^{n} p * q^{*i}$$

are called *the renewal functions of the space M*, associated to p and q.

Definition 3.1. *A point $x \in M$ is said to be finite with respect to (p,q) (or simply (p,q)-finite) if it is (z,q)-finite for any $z \in Supp(p)$. In the converse case x is called infinite with respect to (p,q) (or simply (p,q)-infinite).*

Remark 3.2. If $z \notin G_x$ then there exists a neighborhood W_z of z such that $W_z \cap G_x = \emptyset$. Consequently, x is (z,q)-finite. This means, in particular, that for a point $x \in M$ to be (p,q)-finite it suffices that x is (z,q)-finite for each $z \in G_x \cap \mathrm{Supp}(p)$.

Theorem 3.3. *Suppose all the points of $G_x \cap Supp(p)$ are (p,q)-finite. Then every point of G_x is (p,q)-finite.*

P r o o f. Let $y \in G_x$ and $z \in G_x \cap \mathrm{Supp}(p)$. Since z is (p,q)-finite it is, in particular, (z,q)-finite. Then by Theorem 2.10 y is (z,q)-finite too. Taking Remark 3.2 into account we can conclude that y is a (p,q)-finite point. The proof is complete.

Theorem 3.4. *Suppose that $G_x \cap Supp(p)$ contains (p,q)-infinite points. Then every point of the set $R = \bigcup_z \{y = gz | g \in Supp(Q)\}$, where the sum runs through the set $\{z \in G_x \cap Supp(p) | z \quad is \quad (z,q) - infinite\}$, is (p,q)-infinite. The points of $G_x \setminus R$ are (p,q)-finite.*

P r o o f. According to Theorem 2.10 for each (z,q)-infinite point $z \in G_x \cap \mathrm{Supp}(p)$ the points of $\{x = gz | g \in \mathrm{Supp}(Q)\}$ are (z,q)-infinite. Therefore, all the points of R are (p,q)-infinite. Suppose $x \notin R$. Applying Theorem 2.10 again we see that x is (z,q)-trivial for any (z,q)-infinite point $z \in \mathrm{Supp}(p)$. Consequently, x is a (p,q)-finite point and that completed the proof.

Note that from the proof of Theorem 3.3 it follows the following useful fact

Corollary 3.5. *All the points of $G_x \cap Supp(p)$ are (p,q)-finite if and only if each point $z \in G_x \cap Supp(p)$ is (z,q)-finite.*

Theorem 3.6. *Assume that $Supp(p)$ is compact and let x be a (p,q)-finite point. Then there exists a neighborhood W of $Supp(p)$ and a neighborhood U of x in M such that $Q(H(W,U)) < \infty$.*

P r o o f. Assume that x is (p,q)-finite. For each $z \in \mathrm{Supp}(p)$ one can find a neighborhood W_z of z such that $Q(H(W_z,x)) < \infty$. Choose a neighborhood W'_z

of z, $\overline{W}'_z \subset W_z$ and $\overline{W}'_z$ is compact. By Corollary 1.5 there exists a neighborhood U_z of x such that $H(W'_z, U_z) \subset H(W_z, x)$. Since Supp$(p)$ is compact one can select a finite covering $\{W'_{z_i}\}_{1\le i\le k}$ of Supp(p). Setting $W = \bigcup_{i=1}^{k} W'_{z_i}$ and $U = \bigcap_{i=1}^{k} U_{z_i}$, we have $H(W,U) = \bigcup_{i=1}^{k} H(W'_{z_i}, U) \subset \bigcup_{i=1}^{k} H(W'_{z_i}, U_{z_i}) \subset \bigcup_{i=1}^{k} H(W_{z_i}, x)$. Hence, $Q(H(W,U)) \le \sum_{i=1}^{k} Q(H(W_{z_i}, x)) < \infty$. The proof is complete.

From Theorem 3.6 it follows immediately the following result.

Corollary 3.7. *Assume that Supp(p) is compact and let x be a (p,q)-finite point. Then there exists a neighborhood U of x, consisting of (p,q)-finite points.*

Theorem 3.8. *Assume that Supp(p) is compact and let A be a compact subset of M, consisting of (p,q)-finite points. Then*

a) There exists a neighborhood W of Supp(p) and a neighborhood U of A such that $Q(H(W,U)) < \infty$.

b) $p * Q(A) \le p * Q(U) < \infty$.

P r o o f. By Theorem 3.6 each point $x \in A$ has a neighborhood U_x such that $Q(H(W_x, U_x)) < \infty$, where W_x is a neighborhood of Supp(p). Since A is compact one can cover it by a finite number of subsets U_{x_i} $(i = 1, 2, \dots, k)$. Setting $U = \bigcup_{i=1}^{k} U_{x_i}$ and $W = \bigcap_{i=1}^{k} W_{x_i}$ we have $H(W,U) \subset \bigcup_{i=1}^{k} H(W_{x_i}, U_{x_i})$. Hence $Q(H(W,U)) \le \sum_{i=1}^{k} Q(H(W_{x_i}, U_{x_i})) < \infty$, proving the statement a). Further, we have

$$p * Q(U) = \int_G p(g^{-1}U)Q(dg) = \int_{H(\mathrm{Supp}(p),U)} p(g^{-1}U)Q(dg)$$

$$\le \int_{H(\mathrm{Supp}(p),U)} Q(dg) = Q(H(\mathrm{Supp}(p), U) \le Q(H(W,U)) < \infty.$$

Thus, the proof is completed.

Remark 3.9. If G is a compact group then $Q(H(z, G_z)) = Q(G) = \infty$ for any $z \in M$. From Theorem 3.8 it follows immediately that there is no orbit, intersecting Supp(p) and consisting of (p,q)-finite points.

Theorem 3.10. *If x is a (p,q)-infinite point of M, then $p * Q(U) = \infty$ for any neighborhood U of x.*

P r o o f. Assume x is (p,q)-infinite and let $z \in$ Supp(p) such that x is (z,q)-infinite. Suppose U is a neighborhood of x. Choose a neighborhood U'

of x such that $\overline{U}' \subset U$. According to Lemma 1.7 one can find a neighborhood V of z so that $g(V) \subset U$ for any $g \in H(z, U')$. Since $z \in \mathrm{Supp}(p)$ we have $p(V) > 0$. Further we have $p * Q(U) = \int_G p(g^{-1}U)Q(dg) \geq \int_{H(z,U')} p(g^{-1}U)Q(dg)$, where $p(g^{-1}U) \geq p(V)$ because $V \subset g^{-1}U$ for any $g \in H(z, U')$. Therefore, $p * Q(U) \geq p(V)Q(H(z, U')) \geq p(V)Q(H(W, x)) = \infty$, where W is a neighborhood of z, completing the proof.

4. THE RENEWAL EQUATIONS

Let f be a continuous function on M. We define Radon measures F_n on M by setting

$$F_n(X) = \int_G [\int_X f(g^{-1}x)p(g^{-1}dx)]Q_n(dg) \tag{4.1}$$

for any Borel subset $X \subset M$. Since $Q_n = 1 + Q_{n-1} * q$ one can transform the formula (4.1) as follows

$$\begin{aligned} F_n(X) &= \int_G [\int_X f(g^{-1}x)p(g^{-1}dx)]1(dg) + \int_G [\int_X f(g^{-1}x)p(g^{-1}dx)]Q_n * q(dg) = \\ &= \int_X f(x)p(dx) + \int_G [\int_X f(g^{-1}x)p(g^{-1}dx)][\int_G Q_{n-1}(h^{-1}dg)q(dh)] \\ &= \int_X f(x)p(dx) + \int_G [\int_X f(g^{-1}x)p(g^{-1}dx)]Q_{n-1}(h^{-1}dg)q(dh)]. \end{aligned}$$

Substituting $h^{-1}g = s$ and therefore $g = hs$, $g^{-1} = s^{-1}h^{-1}$ we have

$$\begin{aligned} F_n(X) &= \int_X f(x)p(dx) + \int_G [\int_G [\int_X f(s^{-1}h^{-1}x)p(s^{-1}dh^{-1}x]Q_{n-1}(ds)]q(dh) \\ &= \int_X f(x)p(dx) + \int_G F_{n-1}(h^{-1}x)q(dh). \end{aligned}$$

Hence, we have the iterative formula

$$F_n(X) = \int_X f(x)p(dx) + \int_G F_{n-1}(g^{-1}X)q(dg). \tag{4.2}$$

The equation

$$F(X) = \int_X f(x)p(dx) + \int_G F(g^{-1}X)q(dg), \tag{4.3}$$

where F is a Radon measure on M and X is any Borel subset of M, *is called the renewal equation* with respect to the function f.

From Corollary 3.7 it follows that if Supp(p) is compact, then the union of all orbits, consisting of (p,q)-finite points, is an open subset in M. We denote it by M^*.

Theorem 4.1. *Assume that Supp(p) is compact and f is a continuous function, vanishing on $M \setminus M^*$. Then the measure F, given by setting*

$$F(X) = \int_G [\int_X f(g^{-1}x)p(g^{-1}dx)]Q(dg) \tag{4.4}$$

for any Borel subset $X \subset M$, satisfies the equation (4.3).

P r o o f. First of all we note that $F(X) = F(X \cap M^*)$ for any Borel subset $X \subset M$, because $f = 0$ on $M \setminus M^*$. Therefore, without loss of generality we may suppose that $X \subset M^*$. Let X be a compact subset of M^*. By Theorem 3.8, $Q(H(W,U)) < \infty$ for a neighborhood W of Supp(p) and a neighborhood U of X. The sequence $\{Q_n(H(W,U))\}$ is non-decreasing and bounded by $Q(H(W,U))$. Hence, $Q_n(H(W,U)) \to Q(H(W,U))$ when $n \to \infty$. On the other hand we have

$$|F(X) - F_n(X)| = F(X) - F_n(X) = \int_G [\int_X f(g^{-1}x)p(g^{-1}dx)]Q(dg)$$

$$- \int_G [\int_X f(g^{-1}x)p(g^{-1}dx)]Q_n(dg) =$$

$$= \int_{H(W,U)} [\int_X f(g^{-1}x)p(g^{-1}dx)](Q(dg) - Q_n(dg)) \leq$$

$$\leq c[Q(H(W,U)) - Q_n(H(W,U))] \to 0,$$

where $c = \max_{x \in \mathrm{Supp}(p)} f(x)$. Consequently, $F_n(X) \to F(X)$ for any compact subset X. This means that $F_n \to F$. Now, letting $n \to \infty$ in the formula (4.2) we obtain $F(X) = \int_X f(x)p(dx) + \int_G F(g^{-1}X)q(dg)$, completing the proof.

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