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ON EMBEDDING OF SOBOLEV SPACES OF INFINITE SMOOTHNESS

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Dedicated to Professor Nguyen Dinh Tri on his sixtieth birthday

Abstract. *Embedding theorems are traditional for Sobolev spaces of finite smoothness. We give necessary and sufficient conditions for the non-triviality and existence of the embedding of Sobolev spaces of multivariate functions of infinite smoothness using the Convex Analysis technique.*

1. INTRODUCTION

Let $A \subset \mathbf{R}^n \times (0,1]$. The Sobolev space $W_2(A)$ consists of all measurable functions on $\mathbf{R}^n$ for which the seminorm

$$\|f\|_{W_2(A)} := \sup_{(r,a)\in A} a\|f^{(r)}\|_2$$

is finite, where $\|\cdot\|_2$ denotes the norm of $L_2(\mathbf{R}^n)$; $r \in \mathbf{R}^n$, $a \in (o,1]$; and $f^{(r)}$ is the Weyl-Liouville fractional derivative of order r (see Section 2 for definition). If A is unbounded, then fuctions of $W_2(A)$ have a common infinite smoothness.

In this paper we study necessary and sufficient conditions of the existence of the embedding $W_2(A) \hookrightarrow W_2(B)$ for preassigned A and B, i.e. of the validity of the inequality

$$\|f\|_{W_2(B)} \le M\|f\|_{W_2(A)}, \; f \in W_2(A), \tag{1}$$

with some positive constant M.

Dubinskii [3] considered multidimensional Sobolev spaces of infinite smoothness. We refer to [1,2] for surveys and bibliography on embedding theorems for unidimensional Sobolev spaces of infinite order. In particular, a necessary and sufficient condition for the existence of the embedding the unidimensional Sobolev space $W_p(A)$ into $W_p(B)$ was obtained in [1].

Let

$$F_A(t) := \sup_{(r,a)\in A} a|t|^r$$

where $|t|^r = \prod_{j=1}^{n} |t_j|^{rj}$, t_j denotes the $j-th$ coordinate of $t \in \mathbf{R}^n$.

The purpose of this paper is to show that with certain restrictions on A the embedding (1) is equivalent to the inequality

$$F_B(t) \leq MF_A(t),\ \forall t \in \mathbf{R}^n \quad (2)$$

with the same constant M given in (1).

2.PRELIMINARIES

By a certain reason the fractional derivatives can not be defined for distributions of the Schwarz space $S'(\mathbf{R}^n)$ (cf., e.g., [4]). We give a definition of the Weyl-Liouville fractional derivative for a special class of distributions, introduced by Lizorkin [4].

Let $X \subset S(\mathbf{R}^n)$ be the space of all test functions φ such that

$$\int_{-\infty}^{+\infty} t_j^k \varphi(t)dt_j = 0,\ j = 1,\ldots,n;\ k = 0,1,2,\ldots$$

and let $Y = \mathcal{F}(X)$ where $\mathcal{F}$ is the Fourier transform. Both spaces X and Y are non-trivial closed subspaces of $S(\mathbf{R}^n)$. Let X' and Y' be the spaces of distributions defined as the sets of continuous functionals on X and Y, respectively. The Fouried transform $\mathcal{F} : X' \to Y'$ and its inverse $\mathcal{F}^{-1} : Y' \to X'$ are defined in a way similar to those for distributions from the Schwarz space. The space Y possesses the following property: if φ belongs to Y, then so does $(E_r\varphi)(t) := (it)^r\varphi(t)$ for any $r \in \mathbf{R}^n$, where $(it)^r = \prod_{j=1}^{n}(it_j)^{rj}$, $(it_j)^{rj} = |t_j|^{rj}\exp(\frac{i\pi}{2}r_j \mathrm{sign} t_j)$. (The space $S(\mathbf{R}^n)$ does not possess this property.) This allows us to define the fractional derivative $f^{(r)}$ for a distribution $f \in X'$ by putting

$$f^{(r)} := \mathcal{F}^{-1} \circ E_r \circ \mathcal{F} f$$

where the operator $E_r : Y' \to Y'$ is defined as follows:

$$<E_r f, \varphi> = <f, E_r \varphi>, \ f \in Y', \ \varphi \in Y.$$

Note that the space $L_2(\mathbf{R}^n)$ may be considered as a subspace of X' or Y'or Y'. If $f \in X'$ such that $f^{(r)} \in L_2(\mathbf{R}^n)$, $r \in \mathbf{R}^n$, then we have the Plancherel equality

$$\|f^{(r)}\|_2^2 = \int_{\mathbf{R}^n} |t|^{2r} |\mathcal{F} f|^2 dt \tag{3}$$

In what follows, as usual, we identify measurable functions f and g on R^n if the set $\{x|_{f(x)\neq g(x)}\}$ has zero measure.

To formulate and prove the results we need terminology and some facts from Convex Analysis. We recall some definitions and refer to the book [5] for more details. For $f : \mathbf{R}^n \to [-\infty, +\infty]$ let epi $f := \{(x,y) \in R^{n+1} | f(x) \leq y\}$; dom $f := \{x \in \mathbf{R}^n | f(x) < \infty\}$. A function f is called convex if epi f is a convex set in $\mathbf{R}^{n+1}$. The function

$$f^*(t) := \sup_x (< x, t > - f(x))$$

is called the conjugate function of f, where $< x, t >= \prod_{j=1}^{n} x_j t_j$. For $C \subset \mathbf{R}^n$ denote by co C and cl C the convex and closed hull of C, respectively. A vector $z \in \mathbf{R}^n$ is called receding direction of C if $x - mz \in C$ for any $x \in C$ and $m \geq 0$.

3. NON-TRIVIALITY

First we note the following property of F_A:

$$F_A \text{ is continuous on int dom} F_A. \tag{4}$$

Indeed, let

$$G_A(x) := \sup_{(r,a)\in A} (< r, x > + \ln a).$$

Then,

$$F_A(t) = \exp G_A(\ln|t_1|, \dots, \ln|t_n|) \tag{5}$$

and G_A is a convex function. Thus, G_A is continuous on int dom G_A (cf. [5, Theorem 10.1]). This and (5) imply (4).

Theorem 1 . *For any $f \in W_2(A)$ the support of $\mathcal{F} f$ is almost contained in dom F_A i. e.*

$$meas\{supp\mathcal{F} f \backslash domF_A\} = 0. \tag{6}$$

Moreover, the space $W_2(A)$ is non-trivial, i. e. $W_2(A) \neq \{0\}$ iff

$$int\ domF_A \neq \emptyset.$$

P r o o f. Since the convex function G_A is closed as the upper bound of a selection of affine functions, by (5) so is F_A also, and therefore, dom F_A is closed. Assume that there exists a non-zero function $f \in W_2(A)$ such that meas $\{\mathrm{supp}\mathcal{F} f \backslash \mathrm{dom}F_A\} \neq 0.$. Then $\mathcal{F} f$ is non-zero, too. Thus, there exists a closed ball V such that $V \cap \mathrm{dom}F_A = \emptyset$ and

$$\int_V |\mathcal{F} f|^2 dt > 0.$$

By (3) we have

$$\|f\|^2_{W_2(A)} = \sup_{(r,a)\in A} a \int_{R^n} |t|^{2r} |\mathcal{F} f|^2 dt$$

$$\geq \sup_{(r,a)\in A} a \int_V |t|^{2r} |\mathcal{F} f|^2 dt \geq \sup_{(r,a)\in A} \inf_{x\in V} (a^2 |x|^{2r}) \int_V |\mathcal{F} f|^2 dt.$$

Note that $a^2|x|^{2r}$ are lower semicontinuous on the compact set V. Hence, by virtue of the relation $V \cap \mathrm{dom}F_A = \emptyset$ it is not hard to verify that

$$\sup_{(r,a)\in A} \inf_{x\in V} (a^2|x|^{2r}) = \infty.$$

Thus, we obtain $\|f\|_{W_2(A)} = \infty$. This contradiction shows that if $f \in W_2(A)$, then (6) holds.

Now let int dom $F_A = \emptyset$ and $f \in W_2(A)$. Since the measure of the boundary of dom F_A is zero, from (6) it follows that meas(supp$\mathcal{F} f$) = 0. This means that only the zero function belongs to $W_2(A)$.

On the contrary, assume that int dom $F_A \neq \emptyset$. Then there exists a closed ball $U \subset$ int domF_A. Let $g = \mathcal{F}^{-1}\chi_U$ where χ_U is the charateristic function of U. Obviously, g is non-trivial. Using (3) and (4) we have

$$\|g\|_{W_2(A)} \leq \mathrm{Vol}\ U \max_{t\in U} F_A(t),$$

proving $g \in W_2(A)$. □

4. EMBEDDING THEOREMS

Theorem 2. *Let int dom $F_A \neq \emptyset$. Then the embedding (1) implies (2).*

P r o o f. Let (1) hold. This is equivalent to the fact that for any $(s,b) \in B$ and $f \in W_2(A)$

$$b\|f^{(s)}\|_2 \leq M \sup_{(r,a)\in A} a\|f^{(r)}\|_2. \tag{7}$$

We first transform this condition into a form more suitable for use. In view of (3) from (7) we have

$$b^2 \int_{R^n} |t|^{2s}|\mathcal{F}f|^2 dt \leq M^2 \sup_{(r,a)\in A} a^2 \int_{R^n} |t|^{2r}|\mathcal{F}f|^2 dt$$

for any $(s,b) \in B$ and $f \in W_2(A)$. By replacing $x = (\ln|t_1|, \dots, \ln|t_n|)$, from the latter inequality it is easy to verify that

$$b^2 \int_{R^n} \exp < 2s, x > f(x)dx \leq M^2 \int_{R^n} \exp 2G_A(x) f(x)dx \tag{8}$$

for any $(s,b) \in B$ and for all non-negative functions f for which the right side of (8) is finite.

In order to prove (2) it suffices to show that for any $(s,b) \in B$

$$b|t|^s \leq MF_A(t), \ \forall t \in \text{dom } F_A. \tag{9}$$

Let t° be an arbitrary point of dom F_A. Put $x^\circ = (\ln|t_1^\circ|, \dots, \ln|t_n^\circ|)$. Then $x^\circ \in \text{dom}F_A$. Since G_A is a closed convex function, dom G_A is a closed convex set. Moreover, int dom $G_A \neq \emptyset$ because int dom $F_A \neq \emptyset$. Hence it follows that there exists a n-dimensional simplex $S \subset \text{dom}G_A$ such that $x^\circ \in S$. Let $S_h = hS + (1-h)x^\circ$, $0 < h \leq 1$. Clearly, $x^\circ \in S_h \subset \text{dom}G_A$. As a closed convex function G_A is continuous on every locally simplicial subset of dom G_A, in particular, on S_h (cf. [5, Theorem 10.2]). Applying the characteristic function of S_h we have

$$b^2(\text{Vol}S_h)^{-1} \int_{S_h} \exp < 2s, x > dx \leq M^2(\text{Vol}S_h)^{-1} \int_{S_h} \exp 2G_A(x)dx.$$

Using the mean value theorem and then, letting h tend to zero in this inequality, we obtain

$$b^2 \exp < 2s, x^\circ > \leq M^2 \exp 2G_A(x^\circ).$$

This is equivalent to (9) with arbitrary $t = t^\circ$.□

Theorem 3 . *Let int dom $F_A \neq \emptyset$ and let co $\{(r, -\ln a) \mid (r, a) \in A\}$ be a closed set in $\mathbf{R}^{n+1}$. Then (2) implies the embedding (1).*

P r o o f. Let (2) hold. For the sake of simplicity we put $M = 1$ in (2). Thus, (2) is equivalent to

$$< s, x > + \ln b \leq G_A(x), \ \forall x \in \mathrm{dom} G_A \tag{10}$$

for any $(s, b) \in B$. To prove (1) it is sufficient to check (7). From the definition of the conjugate function it follows that (10) holds if and only if for any $(s, b) \in B$

$$(s, -\ln b) \in \text{epi } G_A^*. \tag{11}$$

It is not hard to verify that

$$\text{epi } G_A^* = cl(Q + H) \tag{12}$$

where $Q = co\{(r, -\ln a) \mid (r, a) \in A\}$ and $H = \{x \in R^{n+1} \mid x_1 = x_2 = \cdots = x_n = 0;\ x_{n+1} = h,\ h \geq 0\}$. We have cl $(Q + H) =$ cl $Q +$ cl H because H does not have any receding direction opposite to the receding directions of Q (cf. [5, Corollary 9. 1. 2]). Therefore, cl $(Q + H) = Q + H$ because both sets Q and H are closed. Hence by (11) and (12) we have $(s, -\ln b) \in Q + H$. It means that there exist elements $(r^j, a^j) \in A$ and non-negative numbers h_0, m_j, $j = 1, \ldots, k$, such that

$$\sum_{j=1}^{k} m_j = 1, \tag{13}$$

$$s = \sum_{j=1}^{k} m_j r^j, \tag{14}$$

$$\ln b = \sum_{j=1}^{k} m_j \ln a^j - h_0. \tag{15}$$

Let $f \in W_2(A)$. By (3), (13) and (14) we have

$$\|f^{(s)}\|_2^2 = \int_{R^n} \Big(\prod_{j=1}^{k} |t|^{2r^j} |\mathcal{F} f|^2 \Big)^{m_j} dt.$$

On the other hand, the Holder inequality gives

$$\int_{R^n} \Big(\prod_{j=1}^{k} |t|^{2r^j} |\mathcal{F} f|^2 \Big)^{m_j} dt \leq \prod_{j=1}^{k} \Big(\int_{R^n} |t|^{2r^j} |\mathcal{F} f|^2 dt \Big)^{m_j}.$$

Therefore,

$$\|f^{(s)}\|_2^2 \le \prod_{j=1}^{k} \|f^{(r^j)}\|_2^{2m_j}.$$

Hence by (13) - (15) we obtain

$$b\|f^{(s)}\|_2 \le \exp\Big(\prod_{j=1}^{k} m_j \ln a^j - h_0\Big) \prod_{j=1}^{k} \|f^{(r^j)}\|_2^{m_j} \le$$

$$\le \prod_{j=1}^{k} \big(a^j \|f^{(r^j)}\|_2\big)^{m_j} \le \|f\|_{W_2(A)}. \square$$

Combining Theorems 2 and 3 we have finally:

Theorem 4 . *Let int dom $F_A \neq \emptyset$ and let co $\{(r, -\ln a) \mid (r,a) \in A\}$ be a closed set. Then (2) is a necessary and sufficient condition for the existence of the embedding (1).*

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