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INFORMATION SEMIMODULES AND ABSORBING SUBSEMIMODULES

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Dedicated to Professor Nguyen Dinh Tri on his sixtieth birthday

Abstract. *The purpose of this short survey is to introduce the notion of an information semimodule and indicate some of the applications such structures in theoretical computer science. Also, we will indicate some of the mathematical theory available for information semimodules by bringing Poyatos' construction of a version of the Jordan - Hölder theorem for information semimodules. To this end, we make use of the notion of an absorbing subsemimodule of a semimodule, studied independently by Poyatos and Takahashi.*

1. THERE ARE PLENTY OF INFORMATION ALGEBRAS OUT THERE

A *semiring* is a nonempty set R on which operations of addition and multiplication have been defined such that the following conditions are satisfied:

1) $(R,+)$ is a commutative monoid with identity element 0_R;
2) $(R,.)$ is a monoid with identity element $1_R \neq 0_R$;
3) Multiplication distributes over addition from either side;
4) $0_R r = 0_R = r0_R$ for all $r \in R$.

We will usually write 0 instead of 0_R and 1 instead of 1_R if there is no room for confusion. Also, multiplication will normally be written as concatenation. We will always denote $R \setminus \{0\}$ by R^*. We will follow the standard conventions: if a is an element of a semiring R and k is a positive integer then the sum $a+\ldots+a$ (k summands) will be denoted by ka and the product $a\ldots a$ (k factors) will be denoted by a^k. We also set $a^0 = 1_R$ for all $a \in R^*$. For a detailed introduction to the theory of semirings, as well as many examples and applications of this theory, refer to [Golan, 1991].

A semiring R is *zerosumfree* if it satisfies the condition that $r+s \in R^*$ for all $r \in R^*$ and all $s \in R$. Countably complete semirings are always zerosumfree, as are those partially-ordered semirings in which 0 is the unique minimal element. See [Golan, 1991] for details. Thus, for example, the semiring of linear relational operators introduced in [Ioannidis & Wong, 1990] to describe the opertors performed on the relations in a database systems is zerosumfree. Note that a zerosumfree semiring is as far from being a ring as possible: for every nonzero element r of a ring R there exists an element s such that $r+s=0$. A semiring R is *entire* if it satisfies the condition that $rs \in R^*$ for all $r \in R^*$ and all $s \in R^*$. It is clear that a sufficient condition for a semiring R to be entire is that it be *left multiplicatively cancellative*, namely if $sr = rs'$ implies that $s = s'$ for all $r \in R^*$. Similarly, a sufficient condition for R to be entire is that it be right multiplicatively cancellative. Thus, in particular, a *division semiring*, i.e. a semiring in which every nonzero element has a (two-sided) multiplicative inverse, is surely entire.

Entire zerosumfree semirings arise naturally in graph theory, especially in the consideration of path problems, and prove considerable information about the structure of graphs. See, for example, [Gondran & Minoux, 1984]. With this in mind, Kuntzmann [1972] called such structures *information algebras (algèbres de renseignement)*. The semiring $\mathbf{N}$ of nonnegative integers, the semiring $\mathbf{Q}^+$ of nonnegative rational numbers, and the semiring $\mathbf{R}^+$ of nonnegative real numbers are all natural examples of information algebras. With these examples obviously in mind, in [Eilenberg, 1974], information algebras are called as "positive semirings" and many other authors follow this terminology. However, "positive" is also used in a different sense in semiring theory, and so it is best to avoid its use in this context. Iwano and Steiglitz [1990] have defined the structure of an information algebra on the set of all convex polygons, with the sum of two polygons being taken to be the convex hull of their vector summation. Another important class of information algebras consists of those semirings of the form (R, max, min), where R is some bounded totally-ordered set having minimal element 0 and maximal element 1. In particular, we can take R to be $\mathbf{B} = \{0,1\}$ or $\mathbf{I} = [0,1]$, obtaining the *boolean semiring* and the *fuzzy semiring* respectively. If R is an integral domain then the semiring of all ideals of R is an information algebra. A division semiring is either an information algebra or a division ring [Mitchell & Sinutoke, 1982].

One of the most widely studied and applicable semirings is an information algebra: if A is a nonempty set then the *free monoid* $\mathbf{F}A$ is the set of all finite strings $a_1a_2\dots a_n$ of elements of A. (Note: this set is usually denoted A^*; here we have deliberately chosen a nonstandard notation in order not to cause sonfusion with the use of * given above.) This can be turned into a monoid by taking multiplication to be concatenation of strings. The identity of this monoid is just the empty string, which we denote by $\square$. The elements of A are called *symbols or letters* and the elements of $\mathbf{F}A$ are called *words*. Subsets of $\mathbf{F}A$ are called *(formal) languages* on A. The set sub($\mathbf{F}A$) of all formal languages on A has the structure of an information algebra in which addition and multiplication are defined by

$L + L' = L \cup L'$ and $LL' = \{ww' | \ w \in L$ and $w' \in L'\}$. The additive identity of this semiring is ϕ while its multiplicative identity is $\{\Box\}$. This information algebra was first considered as part of Kleene's algebraic formulation of the theory of machines [Kleene, 1956] and lies at the heart of modern algebraic automata theory. For further details refer to [Berstel & Reutenauer, 1988], [Eilenberg, 1974], [Lallement, 1979] or [Salomaa & Soittola, 1978]. If $S = \{L \subseteq \mathbf{F}A | L = \phi$ or $\Box \in L\}$ then S is an information subalgebra of sub($\mathbf{F}A$).

Another important example of an information algebra is the following. Let $R = \mathbf{R} \cup \{-\}$ and define operations $\oplus$ and $\otimes$ on R by setting $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$. Then $(R, \oplus, \otimes)$ is a zerosumfree division semiring and so is an information algebra, called the *schedule algebra*, having very important applications in optimization theory, graph theory, and the modeling of industrial processes. Refer, for example, to [Cuninghame-Greene, 1979]. Similarly, $(\mathbf{R} \cup \{\infty\}, \min, +)$ is an information algebra with important applications in graph theory and optimization, such as the solution of the shortest-path problem [Gondran & Minoux, 1984]. This semiring has a subsemiring $(\mathbf{N} \cup \{\infty\}, \min, +)$ which is an information algebra as well and has important applications in the theory of formal languages and automata theory, including the study of the nondeterministic complexity of a finite automaton; it is also used for cost minimization in operations research. See [Mascle, 1986] and [Simon, 1988]. For a generalzation of this construction in which $\mathbf{R}$ is replaced by an arbitrary linearly-ordered abelian group, refer to [Butkovic, 1985].

If R is a semiring which is not a ring then $1 + r \neq 0$ for all $r \in R$. In this case, we set $P(R) = \{0\} \cup \{1 + r | r \in R\}$ and note the following result.

1.1 PROPOSITION: *If R is a semiring which is not a ring then $P(R)$ is a subsemiring of R which is an information algebra.*

Let R be a semiring and let $(X, ^*)$ be a *finite factorization monoid*, namely a monoid satisfying the condition that for each $x \in X$ there set $\{(x', x'') \in X \times X | x'^* x'' = x\}$ is finite. On the set $R << X >>$ of all functions from X to R define operations of addition and multiplication as follows:

1) $(f + g)(x) = f(x) + g(x)$;

2) $(fg)(x) = \sum\{f(x')g(x'') | x'^* x'' = x\}$;

for all $x \in X$ and $f, g \in R << X >>$. This is a semiring called the *convolution semiring* in X over R. The semiring $R << X >>$ has a subsemiring $R < X > = \{f \in R << X >> | f(x) \neq 0$ for only finitely-many elements x of $X\}$, called the *monoid semiring* in X over R. Of course, $R < X > = R << X >>$ if X is finite. If $X = \mathbf{F}A$ for some nonempty set A then $R << \mathbf{F}A >>$ is the *semiring of formal power series* in A over R and $R < \mathbf{F}A >$ is the *semiring of formal polynominals* in A over R. If $A = \{t\}$ we follow the usual convention and write $R[t]$ instead of $R < \mathbf{F}\{t\} >$. Let A be a nonempty set and let θ be a symmetric and transitive relation on A. The *partially commutative free monoid* $\mathbf{M}(A,\theta)$ is the quotient of the monoid $\mathbf{F}A$ by the congruence relation generated by the set of all pairs of the form (ab, ba) with $(a,b) \in \theta$. Then $R < \mathbf{M}(A,\theta) >$ is entire if and only if R is entire [Duchamp & Thibon, 1988] and so we see that $R < \mathbf{M}(A,\theta) >$ is an information algebra if and only if R is.

1.2 PROPOSITION [Hebisch & Weinert, 1990b]: *If R is an information algebra and X is a finite factorization monoid then any subsemiring of $R << X >>$ containing $R < X >$ is an information algebra.*

The converse of Proposition 1.2 is not necessarily true. Conditions under which it is have been investigated in [Hebisch & Weinert, 1990b]. In particular, if X is a right - absorbing monoid and R is a semiring then $R < X >$ is an information algebra if and only if R is. Similarly, if X is the free abelian group generated by an arbitrary set then $R < X >$ is an information algebra if and only if R is. (Note that, in this case, X does not have the finite factorization property so $R << X >>$ cannot be defined.)

The conditions of being zerosumfree and entire are independent and there are plenty of semirings satisfying one of these conditions but not the other. For example, if R is a commutative integral domain then R is entire but not zerosumfree. Entire semirings which are not zerosumfree have been studied in [Hebisch & Weinert, 1990]. On the other hand, if R is a commutative ring which is not an integral domain then the semiring of all ideals of R is zerosumfree but not entire. Countably complete semirings, which have important applications in theoretical computer science, are zerosumfree but not necessarily entire and hence not necessarily information algebras [Krob, 1987]. A linkage between the two conditions does exist, however, for finite semirings.

1.3 PROPOSITION [Hebisch & Weinert, 1990a]: *Every finite entire semiring which is not a ring is an information algebra.*

Any semiring R can be embedded in an information algebra R^θ constructed in the following manner: let u be an element not in R and set $R^\theta = R \cup \{u\}$. Extend the operations of addition and multiplication on R to R^θ by setting $r+u = u+r = r$ for all $r \in R^\theta$ and $ru = ur = u$ for all $r \in R^\theta$. It is straightforward to verify that R^θ is an information algebra satisfying having additive identity equal to u. Actually, in order for R^θ to be an information algebra we can even relax the conditions on a semiring and insist only that $(R,+)$ be a commutative semigroup rather than a monoid or than it is a monoid but that condition (4) in the definition of a semiring is not completely satisfied. Such a situation occurs in the following examples coming from theoretical computer science.

To begin with, let us consider the following model of nondeterministic computer programs defined in [Main & Black, 1989]. We are given a nonempty set D of "states" in which the computer can be, one of which is the distinguished state

$\perp$ of being in an unending loop. Let R be the family of all relations r on D (i.e. nonempty subsets of $D \times D$) satisfying the condition that $(\perp, d) \in r$ if and only if $d = \perp$. The operations of addition and multiplication on R are defined as follows :

1) $r + s = r \cup s$;

2) $rs = \{(d, d'') \in R |$there exist an element $d' \in D$ such that $(d, d') \in r$ and $(d', d'') \in s\}$.

Then $(R, +)$ is a commutative monoid with identity element $0 = (\perp, \perp)$ and $(R, .)$ is a monoid with identity element $1 = \{(d, d) | d \in D\}$. Moreover, multiplication distributes over addition from either side while $0r = 0$ for al $r \in R$. It is not true, however, that $r0 = 0$ for all $r \in R$. Therefore R is not a semiring. Nonetheless, it can be embedded in an information algebra R^θ by the above construction. We also note that $R' = \{r \in R | r0 = 0\}$ is a subsemiring of R.

The elements of R correspond to nondeterministic programs on D. Addition corresponds to a nondeterministic choice ("either r or s") and multiplication corresponds to sequential composition ("first r then s").

Related to this example is the notion of a *command algebra* defined in [Hesselink, 1990], which is in turn a special case of the process algebras studied in [Baeten, Bergstra & Klop, 1987]. Such an algebra consists of a set R on which operations of addition (nondeterministic choice) and multiplication (sequential composition) are defined such that $(R, +)$ is an idempotent commutative semigroup, $(R, .)$ is a semigroup, and multiplication distributes over addition from either side. Such algebras lack both an additive and a multiplicative identity so that R^θ, defined as above, still lacks a multiplicative identity. This can be remedied by the process of *Dorroh extension*: embed R^θ in $S = R^\theta \times \mathbb{N}$ via the map $a \mapsto (a, 0)$ and define operations of addition and multiplication on S by $(a, n) + (a', n') = (a + a', n + n')$ and $(a, n)(a', n') = (na' + n'a + aa', nn')$. This turns S into an information algebra having multiplicative identity $(u, 1)$, where u is the additive identity of R^θ.

An interpretation of the element 0 in a general process algebra is given in [Baeten & Bergstra, 1990], where it is understood to mean the process of "predicatable failure", which the system under consideration will try to avoid if at all possible (as opposed to "deadlock", which the system may enter but cannot try to avoid).

If R and S are semirings then a function $\gamma : R \to S$ is a *morphism of semirings* if and only if :

1) $\gamma(0_R) = 0_s$;

2) $\gamma(1_R) = 1_s$;

3) $\gamma(r + r') = \gamma(r) + \gamma(r')$ for all $r, r' \in R$;

4) $\gamma(rr') = \gamma(r)\gamma(r')$ for all $r, r' \in R$.

A morphism of semirings which is both injective and surjective is called an *isomorphism*. If there is an isomorphism between semirings R and S we write

$R \cong S$.

We remark that the embedding of R into R^θ is not a morphism of semirings since it does not preserve additive identities. Also, note that a morphic image of an information algebra needs not be an information algebra, which suffices to show that the class of information algebra is not a variety . We also note that this class is, clearly, not closed under taking direct products as well. If $B = \{0, 1\}$ is the boolean semiring (with idempotent addition and multiplication) and R is an arbitrary semiring then the function $\gamma : R \to B$ defined by $\gamma(0) = 0$ and $\gamma(r) = 1$ for $r \neq 0$ is a morphism of semirings if and only if R is an information algebra.

Finally, for the record, we mention some rather obvious algebraic properties of information algebras. Clearly an information algebra can have no nonzero nilpotent elements nor any nonzero nilpotent ideals. An element a of a semiring R is *complemented* if and only if there exists an element b of R satisfying $ab = ba = 0$ and $a+b = 1$. Complemented elements are important in the study of semirings, and central complemented elements are used to determine direct-sum decompositions. See [Golan, 1991] for details. If R is an information algebra, however, R cannot have any complemented elements other than 0 and 1.

2. OVER INFORMATION ALGEBRA WE CONSTRUCT INFORMATION SEMIMODULES

If R is a semiring then a commutative monoid $(M, +)$ with additive identity 0_M is a *left R-semimodule* if there exists a function $R \times M \to M$, denoted by $(r, m) \mapsto rm$ and called *scalar multiplication*, which satisfies the following conditions for all elements r and r' of R and all elements m and m' of M :

1) $(rr')m = r(r'm)$;
2) $r(m + m') = rm + rm'$;
3) $(r + r')m = rm + r'm$;
4) $1_R m = m$;
5) $r0_M = 0_M = 0_R m$.

Right R - semimodules are defined in an analogous manner. We will usually work with left semimodules, with the corresponding results for right semimodules taken as proven without explicit mention. Any semiring is clearly both a left and right semimodules over itself. When we speak of a semiring as a semimodule over itself, we will mean a left semimodule unless the contrary is specifically indicated. Again, we will denote $M \setminus \{0_M\}$ by M^*.

We begin with an example : A *left seminearring* is a structure $(P, +, .)$ satisfying all of the axioms of a semiring exept that distributivity of mutiplication over addition from the right may not hold. That is to say, if a, b, and c are elements of P then $a(b + c) = ab + ac$ but $(b + c)a$ is not necessarily equal to $ba + ca$. Such structures arise, inter alia, in the study of shift maps on semimodules, as seen in [Golan, 1991]. They also arise naturally in the study of process algebras in the sense of [Baeten, Bergstra & Klop, 1987] or [Baeten & Weijland, 1990]. In this situation we have a set of "processes" on which we have defined two operations: + ("alternative composition") and . ("sequential composition"). If $(P, +, .)$ is a left seminearring then $P_0 = \{k1 | k \in \mathbb{N}\}$ is a subsemiring of P since it is easy to verify that in P_0 both distributive laws hold. Moreover, if $\{S_i | i \in \Omega\}$ is a chain of subsemirings of P then so is $\cup\{S_i | i \in \Omega\}$. Therefore, by Zorn's Lemma, every seminearring has a maximal subsemiring R. Moreover, P is clearly a left R-semimodule.

A similar situation occurs for the quemirings introduced in [Elgot, 1976] to study computation using abstract automata. A *quemiring* is a structure of the form $R \times M$, where R is a semiring and M is a left R-semimodule, on which addition is defined componentwise and multiplication is given by $(a, m).(a', m') = (aa', am' + m)$. This is not a semiring since $(a, m).(0, 0) = (0, 0)$ only when $m0 = 0$. Also, while right distributivity of multiplication over addition always holds, left distributivity holds only sometimes. However, the quemiring $R \times M$ certainly contains $R' = \{(a, 0) | a \in R\}$ as a subsemiring and can profitably be studied as a left R'-semimodule.

A left R-semimodule having more than one element is *nontrivial.* An R-semimodule M is *zerosumfree* if and only if it satisfies the condition that $m + m' \in M^*$ for all $\in M^*$ and all $m' \in M$. It is *entire* if and only if it satisfies the condition that $rm \in M^*$ for all $r \in R^*$ and all $m \in M^*$. A semimodule which is both zerosumfree and entire is an *information semimodule.*

2.1 PROPOSITION: *A semiring R is an information algebra if and only if there exists a nontrivial left information R-semimodule.*

If M is a left R-semimodule then a nonempty subset N of M is a *subsemimodule* of M if and only if it is closed under taking sums and scalar products. Clearly $\{0\}$ is always a subsemimodule of a left R-semimodule M. A left R - semimodule which has no subsemimodules other than $\{0\}$ and itself is *simple.* If N and N' are subsemimodules of a left R-semimodule M then $N + N' = \{n + n' | n \in N$ and $n' \in N'\}$ is a subsemimodule of M containing both N and N'. If $\mathcal{A}$ is a nonempty family of subsemimodules of a left R - semimodule M then $\cap\mathcal{A}$ is a subsemimodule of M, as is $\sum \mathcal{A} = \{m_1 + \ldots + m_t | m_i \in \cup\mathcal{A}$ for each $i\}$.

Let R be a semiring and let M be a left R-semimodule. Furthermore, let ideal(R) be the set of all ideals of R and let $ssm(M)$ be the set of all subsemimodules of M. Then ideal(R) has a natural semiring structure (see [Golan, 1991] for details) and $(ssm(M), +)$ is a left ideal(R)-semimodule, with addition in $ssm(M)$ being defined as above and scalar multiplication being given by $IN = \{a_1x_1 + \ldots + a_nx_n | a_i \in I;\ x_i \in N\}$ for all $I \in$ ideal(R) and $N \in ssm(M)$.

If A is a nonempty subset of a left R-semimodule M then the set RA of all elements of M of the form $a_1m_1 + \ldots + a_nm_n$ with $a_i \in R$ and $m_i \in A$,is a subsemimodule of M and, indeed, is the intersection of all subsemimodules of M containing A, called the subsemimodule of M *generated* by A. By convention, we set $R\emptyset = \{0\}$. If $RA = M$ then A is a *set of generators* for M. A nonempty subset A of a left R-semimodule M is *weakly linearly independent* if and only if for any finite subset $\{x_1, \ldots, x_n\}$ of A and finite subset $\{a_1, \ldots, a_n\}$ of R we have $\sum a_ix_i$ only when $a_i = 0$ for all i; it is *linearly independent* if and only if for any finite subset $\{x_1, \ldots, x_n\}$ and A and finite subsets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ of R, we have $\sum a_ix_i = \sum b_ix_i$ only when $a_i = b_i$ for all $1 \leq i \leq n$. If M is an information semimodule then any nonempty subset of M^* is weakly linearly independent. A [weakly] linearly independent set of generators for a left R-semimodule M is a *[weak] basis* for M. These notions are discussed in [Takahashi, 1984b].

2.2 PROPOSITION: *Let R be an information algebra. A left R-semimodule M is an information semimodule if and only if it has a weak basis.*

A complete classification of all cyclic information semimodules over the information algebra $\mathbf{N}$ of natural numbers is given in [Takahashi, 1985].

If M is a left R-semimodule then $V(M) = \{m \in M | m + m' = 0$ for some element m' of $M\}$ is surely a subsemimodule of M. The semimodule M is zerosumfree precisely when $V(M) = \{0\}$. At the opposite extreme, a left R-semimodule M is an *R-module* if and only if $V(M) = M$.

A subsemimodule of a zerosumfree [resp. entire] left R-semimodule is again zerosumfree [resp. entire]. Thus the class of information semimodules is closed under taking subsemimodules. If N is an information subsemimodule of a left R-semimodule M then $N \S N' = \{0\} \cup \{n + n' | n \in N^*$ and $n' \in N'\}$ is a subsemimodule of $N + N'$ which contains N but does not necessarily contain N'. If $m \in M$ we set $T(m) = Rm \S M$.

An element w of a left R-semimodule M is *infinite* in M if and only if $w+m = w$ for all $m \in M$. If w and w' are infinite elements of M then $w = w + w' = w'$. Thus any left R-semimodule M can have at most one infinite element. The element w is *strongly infinite* in M if and only if $rw = w$ for all $r \in R^*$. If the semiring R is antisimple then every infinite element of a left R-semimodule is strongly infinite. Indeed, in this situation any element r of R^* is of the form $1 + r'$ for $r' \in R^*$ and so $rw = (1 + r')w = w + r'w = w$. If M has a strongly-infinite element w then the set $C(M) = \{0, w\}$ is the *crux* of M; if M has no strongly infinite elements we set $C(M) = \{0\}$. Note that $C(M)$ os always an information subsemimodule of M. We say that M is *crucial* if and only if $M = C(M)$. Otherwise it is *noncrucial.* If w is a strongly-infinite element of a left R-semimodule M and if N is a subsemimodule of M then $N \cup \{w\}$ is also a subsemimodule of M. Also, we note that if w is a strongly-infinite element of a left R-semimodule M then $T(w) = C(M)$. If w is a strongly-infinite element of a left N-semimodule M then for each element m of M the set $W(m) = \{m' \in M | m + m' = w\} \cup \{0\}$ is a subsemimodule of M. Clearly $C(M) \subseteq W(m)$ for each $m \in M$.

Let R be a semiring and let M be a left R-semimodule. If $(X, ^*)$ is a finite factorization monoid then, as we have seen in Section 1, we can define the semiring $R << X >>$ and its subsemiring $R < X >$. Moreover, if $M << X >>$ is the set of all functions from X to M then $M << X >>$ has the structure of a left $R << X >>$-semimodule if we define addition and scalar multiplication as follows: if $u, v \in M << X >>$ and $f \in R << X >>$ then:

1) $(u + v)(x) = u(x) + v(x)$;
2) $(fu)(x) = \sum\{f(x')u(x'') | x'^*x'' = x\}$.

Moreover, if $M < X >= \{u \in M << X >> | u(x) \neq 0$ for only finitely-many elements x of $X\}$ then $M < X >$ is a left $R < X >$-semimodule. Moreover, if R is an information algebra and M is an information left R-semimodule then $M << X >>$ is an information left $R << X >>$-semimodule and $M < X >$ is an information left $R < X >$-semimodule.

We have already seen that strongly-infinite elements can exist in R-semimodules only when R is an information algebra. A strongly - infinite element w of an R-semimodule M is *primitive* in M if and only if $m + m' = w$ for all elements m and m' of M^*. Thus, for example, if M has a strongly-infinite element w then w is a primitive element of $C(M)$.

We now note an example due to [Goldstern, 1985]. Let $R = \{a_0, a_1 \ldots, b_0, b_1, \ldots\}$ on which we have operations of addition and multiplication defined by:

1) $a_i + a_j = a_{i+j}$ for all $i, j \in$N;
2) $b_i + b_j = b_0$ for all $i, j \in$N;
3) $a_j + b_i = b_i + a_j = b_k$, where $k = i - j$ if $i > j$ and $k = 0$ if $i \leq j$;
4) $a_i a_j = a_{ij}$ for all $i, j \in$N;
5) $b_i b_j = b_0$ for all $i, j \in$N;
6) $b_i a_j = a_j b_i = b_0$ for all $i \in$N and all $j > 1$;
7) $b_i a_0 = a_0 b_i = a_0$ for all $i \in$N;
8) $b_i a_1 = a_1 b_i = b_i$ for all $i \in$N.

This is a semiring with additive identity a_0 and multiplicate identity a_1 having a strongly-infinite element b_0 which is not primitive.

If R is an information algebra then Poyatos [1972] has shown that one can always adjoin a strongly infinite element to any left R-semimodule. Indeed, if M is such a semimodule and if w is an element not in M one can define a semimodule structure on $M\{w\} = M \cup \{w\}$ by setting $m + w = w + m = w$ for all $m \in M \cup \{w\}, rw = w$ for all $r \in R^*$, and $0w = 0_M$. This construction can be iterated. Let M be a left R-semimodule and let $\{w_1, w_2, \ldots\}$ be a countably-infinite set of distinct elements not in M. Let $N_1 = M\{w_1\}$ and, for each $i > 1$, let $N_i = N_{i-1}\{w_i\}$. Then each N_i is a subsemimodule of N_{i+1}, and all of them are subsemimodules of $N = \cup N_i$. In N we note that:

1) $m + w_i = w_i$ for all $m \in M$ and all $i \geq 1$;

2) $w_i + w_j = w_{\max\{i,j\}}$ for all $i, j \geq 1$;

3) $rw_i = w_i$ for all $r \in R^*$ and all $i \geq 1$.

If N is a nonzero subsemimodule of a left R-semimodule M, set $P(N,M) = \{m \in M | rm + n \in N^* \text{ for all } r \in R \text{ and all } n \in N^*\} = \{m \in M | Rm + N^* = N^*\}$. Clearly $0_M \in P(N,M)$ and $P(N,M)$ is a subsemimodule of M. Moreover, $P(M,M) = \{m \in M|$ no nonzero multiple of m has an additive inverse $\}$. Thus we surely have $V(M) \cap P(M,M) = \{0_M\}$. If $N \neq \{0_M\}$ and $P(N,M)$ has an infinite element then that element must belong to N. By convention, we also set $P(\{0_M\}, M) = M$ for every left R-module M.

2.3 PROPOSITION: *If N is a subsemimodule of a left R- semimodule M then:*

1) $N \subseteq P(N,M)$ if and only if N is zerosumfree;

2) $P(N, P(N,M)) = P(N,M)$ for any subsemimodule N of a left R-semimodule M;

3) If M' is a subsemimodule of M then $P(N,M') = P(N,M) \cap M'$;

4) If $\{M_i | i \in \Omega\}$ is a family of R-semimodules and N is a subsemimodule of $M = \cap\{M_i | i \in \Omega\}$ then $P(N,M) = \cap\{P(N,M_i) | i \in \Omega\}$;

5) If $\{N_i | i \in \Omega\}$ is a family of subsemimodules of M satisfying $\cap N_i = N$ then $\cap P(N_i, M) \subseteq P(N,M)$;

6) If M' is a subsemimodule of $P(N,M)$ then $N + M' = N \cup M'$ and N § $M' = N$.

If M and N are left R-semimodules then a map $\alpha : M \to N$ is an *R-homomorphism* if and only if $(m + m')\alpha = m\alpha + m'\alpha$ and $(rm)\alpha = r(m\alpha)$ for all $m, m' \in M$ and $r \in R$. Note that we follow the standard convention of writing homomorphism as acting on the side opposite scalar multiplication so that homomorphism of right R-semimodules will be written as acting on the left. A bijective R-homomorphism is an *R-isomorphism*. The *kernel* of an R-homomorphism $\alpha : M \to N$ of left R-semimodules is $ker(\alpha) = \{m \in M | m\alpha = 0_N\}$. This is surely a subsemimodule of M. Indeed, this subsemimodule of M which is *subtractive*, i.e. it satisfies the condition that if $m' \in ker(\alpha)$ and $m' + m \in ker(\alpha)$ for some $m \in M$ then $m \in ker(\alpha)$. Subtractive subsemimodules play an important part in the development of semimodule theory. See [Golan, 1991] for details. If α and β are R-homomorphisms from a left R-semimodule M to a left R-semimodule N then the map $\alpha + \beta$ from M to N defined by $\alpha + \beta : m \mapsto m\alpha + m\beta$ is also an R- homomorphism. Indeed, it is easy to see that the set $Hom_R(M,N)$ of all R-homomorphisms from M to N is a monoid (=N-semimodule) under this operation, the identity element of which is the constant map $m \mapsto 0$.

Let M be a left R-semimodule. An R-homomorphism from M to itself is called an *R-endomorophism* of M. The set $End_R(M)$ of all R-endomorphisms of M is a semiring under the operations of addition and multiplication defined by setting $m(\alpha+\beta) = m\alpha + m\beta$ and $m(\alpha\beta) = (m\alpha)\beta$ for all $m \in M$ and all $\alpha, \beta \in End_R(M)$. The additive identity of $End_R(M)$ is the map given by $m \mapsto 0$ for all $m \in M$ and its multiplicative identity is the map given by $m \mapsto m$ for all $m \in M$.

If M is a left R-semimodule and $S = End_R(M)$ then M is a right S-semimodule, with scalar multiplication defined by $m.\alpha = m\alpha$ for all $m \in M$ and $\alpha \in S$.

For example, let A be a nonempty set and as before, let $S = sub(\mathbf{F}A)$ be the semiring of all formal languages on A. Following the terminology of [Abramsky & Vickers, 1990] we say that a *transition system* $(P, \rightarrow)$ over A consits of a nonempty set P together with a subset $\rightarrow$ of $P \times A \times P$, where we write $p \xrightarrow{a} q$ instead of $(p, a, q) \in \rightarrow$. The elements of A are the *atomic actions* of the transition system, while the elements of P are the *processes* of the system.

Each element a of A defines a function $\theta_a : sub(P) \rightarrow sub(P)$ given by $m\theta_a = \{q \in P | p \xrightarrow{a} q$ for some $p \in m\}$. We can expand this notion by defining θ_w for each $w \in \mathbf{F}A$ recursively as follows:

1) If $w = \square$ then θ_w is the identity map;

2) If $w = va$ for $v \in \mathbf{F}A$ and $a \in A$ then $m\theta_w = (m\theta_v)\theta_a$.

Furthermore, if $L \in S$ then we can define a function $\theta_L : \mathrm{sub}(P) \rightarrow \mathrm{sub}(P)$ by setting $m\theta_L = \cup\{m\theta_w | w \in L\}$. Moreover, $(\mathrm{sub}(P), \cup)$ is a left $\mathbf{B}$- semimodule. If $L \in S$ then θ_L is a $\mathbf{B}$-endomorphism of $\mathrm{sub}(P)$ for each $L \in S$ and, indeed, $\{\theta_L | L \in S\}$ is a subsemiring of the semiring of $\mathbf{B}$-endomorphisms of $\mathrm{sub}(P)$. Moreover, the map $L \mapsto \theta_L$ is a morphism of semirings from S to the semiring of $\mathbf{B}$-endomorphisms of $\mathrm{sub}(P)$. Thus $\mathrm{sub}(P)$ is canonically a right S-semimodule, where, for each $m \in \mathrm{sub}(P)$ and each $L \in S$, we have $mL = \{q \in P | p \xrightarrow{w} q$ for some $p \in M$ and $w \in L\}$. This semimodule is zerosumfree but not necessarily entire. If S' is the subsemiring of S satisfied by $S' = \{L \in S | L = \emptyset$ or $\square \in L\}$ then $\mathrm{sub}(P)$ is an information S'-semimodule.

Let R be an information algebra and let $\{(M_i, +_i) | i \in \Omega\}$ be a family of information left R-semimodules each of which has a strongly-infinite element w_i. Without loss of generality, we can assume that the M_i are disjoint as sets. Set $M = \cup\{M_i \setminus C(M_i) | i \in \Omega\} \cup \{0, w\}$, where 0 and w do not belong to any of the M_i. Define addition and scalar multiplication of elements of M as follows:

1) $0 + m = m + 0 = m$ and $r0 = 0$ for all $m \in M$ and all $r \in R$;

2) $w + m = m + w = w$ and $rw = w$ for all $m \in M$ and all $r \in R$;

3) If $m, m' \in M \setminus \{0, w\}$ then $m + m'$ is defined to be $m +_i m'$ if both m and m' belong to $M_i \setminus C(M_i)$ for some $i \in \Omega$; otherwise, $m + m' = w$;

4) If $m \in M_i \setminus C(M_i)$ then rm is the same as the corresponding value in M_i.

It is straightforward to check that M, as definde above, is an information semimodule having a strongly-infinite element w. Moreover, for each $i \in \Omega$ we have an injective R- homomorphism $\lambda_i : M_i \to M$ which sends the zero-element 0_i of M_i to 0, sends w_i to w, and sends each element of $M_i \setminus C(M_i)$ to itself. We will denote the semimodule M constructed in this way by $\sqcup_{i \in \omega} M_i$.

An important method of constructing surjective R-homomorphisms of semimodules is via R-congruence relations. An equivalence relation $\equiv$ on a left R-semimodule M is an *R-congruence relation* if and only if $m \equiv m'$ and $n \equiv n'$ imply that $m + n \equiv m' + n'$ and $rm \equiv rm'$ for all $r \in R$. If $\equiv$ is an R- congruence relation on M and if $m \in M$ then we denote the equivalence class of m with respect to this relation by $m/\equiv$ and set $M/\equiv = \{m/\equiv \mid m \in M\}$. We can define operations of addition and scalar multiplication on $M/\equiv$ by setting $m/\equiv + n/\equiv = (m+n)/\equiv$ and $r(m/\equiv) = (rm)/\equiv$ for all $m, n \in M$ and all $r \in R$. These operations turn $M/\equiv$ into a left R-semimodule, called the *factor semimodule* of M by $\equiv$. Moreover, we have a surjective R-homomorphism $M \to M/\equiv$ defined by $m \mapsto m/\equiv$ for all $m \in M$. For further details and examples of congruence relations on semimodules, refer to [Golan, 1991].

If R is an information algebra then any nontrivial information subsemimodule of a left R-semimodule M can be "contracted" to a strongly-infinite element of a factor semimodule of a subsemimodule of M. Indeed, let R be an information algebra and let N be a nontrivial information subsemimodule of a left R-semimodule M. Let M' be a subsemimodule of $P(N, M)$ properly containing N. Define a relation $\sim_N$ on M' by setting $m \sim_N m'$ if and only if $m = m'$ or $\{m, m'\} \subseteq N^*$. It is straightforward to check that this is an R-congruence relation on M' and so we can define the factor semimodule $M'/\sim_N$. Indeed, if $m \in M' \setminus N$ then $m/\sim_N = \{m\}$ so $M'/\sim_N$ is just $[M' \setminus N^*] \cup \{w\}$, where w is a strongly-infinite element of $M'/\sim_N$. If $\alpha : M' \to M'/\sim_N$ is the canonical surjection deficned by $\alpha : m \mapsto m/\sim_N$ then α is not injective unless N has precisely two elements. On the other hand, we always have $\ker(\alpha) = \{0\}$.

For notational convenience, we will denote the semimodule $M'/\sim_N$ by $M'//N$ and, for each element m of M', we will write $m//N$ instead of $m/\sim_N$. Thus $m//N = \{m\}$ for all $m \in M' \setminus N$ and $m//N = \{w\}$ for all $n \in N^*$. Following [Takahashi, 1984], we will call $M'//N$ the *Rees factor semimodule* of M' by N.

3. INFORMATION SEMIMODULES HAVE ABSORBING SUBSEMIMODULES

We now introduce a constrution first given by Poyatos [1972, 1973a, 1973b] in his construction of a version of the Jordan - Hölder theorem which would hold for semimodules, and later studied independently in [Takahashi, 1984a]. A subsemimodule N of a left R-semimodule M is ***absorbing*** if and only if it is entire and $M = P(N, M)$. In this case we write $N \sqsubseteq M$. That is to say, $N \sqsubseteq M$ if and only if N is entire and $M + N^* = N^*$. This surely implies that N is zerosumfree. Indeed, an entire left R-semimodule M is an information semimodule if and only if $M \sqsubseteq M$. Thus we see that absorbing subsemimodules are information subsemimodules. The converse need not be true. By Proposition 2.1 we see that it is meaningful to talk about the existence of nontrivial absorbing subsemimodules only for semimodules over information algebras. If $n \in \mathbf{N}$ then $\{0\} \cup \{i \in \mathbf{N} \mid i > n\}$ is an absorbing subsemimodule of $\mathbf{N}$.

Trivially, $\{0_M\} \sqsubseteq M$ for every left R-semimodule M. If w is an infinite element of a semimodule M then w is strongly infinite if and only if $\{0, w\} \sqsubseteq M$. Let M be an entire subsemimodule. If N is a subtractive subsemimodule of M satisfying the condition that $N = (M \setminus M') \cup \{0\}$ is a subsemimodule of M then surely $N \sqsubseteq M$. Thus, for example, if M is a left $\mathbf{N}$-semimodule which is not an abelian group then $V(M)$ satisfies the given condition and so $N = (M \setminus V(M)) \cup \{0\}$ is an absorbing subsemimodule of M.

3.1 PROPOSITION: *For subsemimodules N, N', M' of a left R-semimodule M we have:*

1) If $N \sqsubseteq M$ and $N \subseteq M'$ then $N \sqsubseteq M'$;

2) If $N \sqsubseteq M'$ then $M' \subseteq P(N, M)$;

3) If N is zerosumfree then $N \sqsubseteq P(N, M)$;

4) If $\mathcal{A}$ is a nonempty family of absorbing subsemimodules of M having intersection N then $N \sqsubseteq M$;

5) If $\mathcal{A}$ is a nonempty family of absorbing subsemimodules of M having union N then $N \sqsubseteq M$;

6) If $N, N' \sqsubseteq M$ then $N \cap N' \neq \{0\}$ and $N + N' \sqsubseteq M$, where in this situation we in fact have $N + N' = N \cup N'$;

7) If $N \sqsubseteq M$ then $N \cap M' \sqsubseteq M'$;

8) If M is entire and $N, N' \sqsubseteq M$ then $N \S N' \subseteq N \cap N'$;

9) If $N \sqsubseteq M$ then $N \cup N'$ is a subsemimodule of M.

Thus, in particular, from (2) and (3) of Proposition 3.1 we see that if N is an information subsemimodule of a left R- semimodule M then $P(N, M)$ is the largest subsemimodule of M containing N as an absorbing subsemimodule.

We also note that, by (4) and (5) of Proposition 3.1, the family $\operatorname{assm}(M)$ of all absorbing subsemimodules of M is a sublattice of the lattice $\operatorname{ssm}(M)$ of all subsemimodules of M. Indeed, this lattice is distributive by Proposition 3.1 (6).

The following example is in [Takahashi, 1984a]: Let R be an antisimple semiring. If M is a left R-semimodule which is not a group then $N = \{0\} \cup [M \setminus V(M)]$ is an absorbing subsemimodule of M. Indeed, it is clearly closed under addition. If $n \in N^*$ and $r \in R^*$ then $r = 1 + s$ for some $s \in R$. Thus $rn + n' = 0$ implies that $n + (sn + n') = 0$, which is impossible, proving that $rn \in N$. Finally, let $m \in M$ and $n \in N^* = M \setminus V(M)$. If $m + n \notin N^*$ then there exists an element m' of M satisfying $0 = m + n + m' = n + (m + m')$, contradicting the assumption that $n \in N^*$. Thus $N \sqsubseteq M$. We also note the converse: if $N \sqsubseteq M$ and N satisfies the condition that $M \setminus V(M) \subseteq N^*$ then $M \setminus V(M) = N^*$ for if $x \in V(M) \cap N^*$ then there exists an element y of $V(M)^*$ satisfying $y + x = 0$, contradicting the fact that $M + N^* = N^*$.

Let M be a left R-semimodule containing an entire simple submodule N_0, let ssm(M) be the set of all subsemimodules of M, which we have already noted is a left semimodule over the semiring ideal(R) if all ideals of R. Let$U = \{N \in \text{ssm}(M) | N \supseteq N_0\}$. If $\{0\} \neq I \in \text{ideal}(R)$ and $N \in U$ then $IN \supseteq IN_0 = N_0$. Thus $U' = U \cup \{\{0\}\}$ is an absorbing subsemimodule of ssm(M).

Let R be an information algebra and let $\{M_i | i \in \Omega\}$ be a family of information left R-semimodules, each of which has a strongly-infinite element. Let $M = \sqcup_{i \in \Omega} M_i$ and, for each $i \in \Omega$,let $\lambda_i : M_i \to M$ be the injective R-homomorphism. Then $M_i\lambda_i \sqsubseteq M$ for each $i \in \Omega$.

Note that if R is an information algebra then it is zerosumfree and so $R \sqsubseteq R$ but it does not follow from this that if R is an information algebra then every nontrivial left R-semimodule has a nontrivial entire subsemimodule. Thus, for example, $\mathbf{N}$ is an information algebra but any nontrivial additive abelian group is a left $\mathbf{N}$-semimodule having no nontrivial absorbing subsemimodules.

If M and N are disjoint left R-semimodules then a *Takahashi extension* of M by N is a left R-semimodule T the underlying set of which is $M \cup N^*$ and the operations of addition and multiplication on which are so defined that $N \sqsubseteq T$. These extensions were first considered in [Takahashi, 1984a]. We will denote the family of all Takahashi extensions of M by N by Tak(M, N). By what we have already seen, a necessary condition for Tak(M, N) to be nonempty is that N be an information semimodule. If N is an information R-semimodule then a function ψ from N^* to N^* is a *translation* if and only if $\psi(n + n') = \psi(n) + n' = n + \psi(n')$ for all $n, n' \in N^*$. Denote the set of all translations of N^* by tr(N^*); this set is always nonempty since it surely contains the identity map. It is also closed under composition of functions and, indeed, is easily seen to be a monoid under composition.

An element m of a left R-semimodule M is *cancellable* if and only if $m + m' = m + m''$ implies $m' = m''$ for all $m', m'' \in M$. The semimodule M is *cancellative* if and only if every element of M is cancellable. See [Golan, 1991] for details. If N is an information semimodule having a nonzero cancellable element n_0 then tr(N^*) is an abelian monoid. Indeed, if n_0 is a cancellable element of N^* then for φ and ψ in tr(N^*) and $n \in N$ we have $\varphi\psi(n + n_0) = \varphi\psi(n) + n_0 = \psi(n) + \varphi(n_0)$ and $\psi\varphi(n + n_0) = n + \psi\varphi(n_0) = \psi(n) + \varphi(n_0)$ and so $\varphi\psi(n) + n_0 = \psi\varphi(n) + n_0$. Since

n_0 is cancellable, this implies that $\varphi\psi(n) = \psi\varphi(n)$ for all $n \in N^*$.

If $T \in \text{Tak}(M, N)$ then each element m of M induces a translation $\varphi_m \in \text{tr}(N^*)$ given by $\varphi_m(x) = m + x$. Thus we have a function $\varphi_T : M \to \text{tr}(N^*)$ given by $\varphi_T : m \mapsto \varphi_m$, and this is in fact a morphism of monoids since $\varphi_{m+m'} = \varphi_m\varphi_{m'}$, for all $m, m' \in M$. This morphism satisfies the additional condition:

(*) If $r \in R$, $m \in M$, and $n \in N^*$ then $r[\varphi_T(m)(n)] = \varphi_T(rm)(rn)$.

A morphism of monoids from M to $\text{tr}(N^*)$ with this property will be called *admissible*. Thus, for example, if M is any left R-semimodule and N is an information semimodule disjoint from M then the morphism $\varepsilon : M \to \text{tr}(N^*)$ defined by $\varepsilon(m)(n) = n$ is admissible. The set of all admissible morphisms from the monoid $(M, +)$ to $\text{tr}(N^*)$ will be denoted by $\text{Adm}(M, N)$. If $\oplus$ is the operation on $\text{Adm}(M, N)$ defined by setting $(\alpha \oplus \beta)(m) = \alpha(m)\beta(m)$ for all $m \in M$ then $(\text{Adm}(M, N), \oplus)$ is a monoid with identity element ε. If N has a nonzero cancellable element then, by what we have noted above, this monoid is abelian (i.e., it is a $\mathbb{N}$-semimodule).

Let φ be an admissible morphism of monoids from a left R- semimodule M to $\text{tr}(N^*)$, where N is an information left R- semimodule dissjoint from M. Set $T = M \cup N^*$ and define the operations of addition and scalar multiplication on T as follows:

1) If $m, m' \in M$ then $m + m'$ and rm are the same as in M;

2) If $n, n' \in N^*$ then $n + n'$ and rn are the same as in N;

3) If $m \in M$ and $n \in N^*$ then $m + n = n + m = \varphi(m)(n)$.

This turns T into a left R-semimodule having N as an absorbing subsemimodule and M as a subtractive subsemimodule; hence it is a Takahashi extension of M by N. Thus there is a bijective correspondence between the set of all Takahashi extensions of M by N and the set of all admissible morphisms of monoids from M to $\text{tr}(N^*)$. If $\varphi : M \to \text{tr}(N^*)$ is an admissible morphism of monoids, we will denote by $M \circledast_\varphi N$ the Takahashi extension of M by N defined by φ.

3.2 PROPOSITION: *Let R be a semiring and let $\alpha : M \to M'$ be an R-homomorphism of left R-semimodules satisfying the condition that $ker(\alpha) \sqsubseteq M$. If $N' \sqsubseteq M'$ then $N = N'\alpha^{-1} \sqsubseteq M$.*

If N is an absorbing subsemimodule of a left R-semimodule M then the left R-semimodule $N'//N$ is defined for any R -semimodule N' of M containing N. It is straightforward to verify that the map $N' \mapsto N'//N$ induces a bijective order - preserving correspondence between the family of all subsemimodules of M containing N and the family of all subsemimodules of $M//N$, which in turn restricts to a bijective correspondence between the family of all absorbing subsemimodules of M containing N and the family of all absorbing subsemimodules of $M//N$. The following result gives a "converse" of this construction for semimodules over information algebras.

3.3 PROPOSITION: *Let R be an information algebra and let $\alpha : M \to N$ be an R-homomorphism of left R-semimodules. If w' is a strongly-infinite element of N contained in $M\alpha$ then $M' = w'\alpha^{-1} \cup \{0\}$ is an absorbing subsemimodule of M.*

3.4. PROPOSITION [Poyatos, 1973a]: *Let N and N' be absorbing subsemimodules of a left R-semimodule M. Then:*

1) $N \sqsubseteq N \cup N'$;
2) $N \cap N' \sqsubseteq N'$;
3) $(N \cup N')//N$ *is R-isomorphic to* $N'//(N \cap N')$.

3.5 PROPOSITION [Poyatos, 1973b]: *Let R be an information algebra and let M be a left R-semimodule. If N, N', W and W' are subsemimodules of M satisfying $N' \sqsubseteq N$ and $W' \sqsubseteq W$, and if U, U', V and V' are the subsemimodules of M defined by:*

i) $U = N' \cup (N \cap W)$;
ii) $U' = N' \cup (N \cap W')$;
iii) $V = W' \cup (N \cap W)$;
iv) $V' = W' \cup (N' \cap W)$;

Then:

1) $U' \sqsubseteq U$ *and* $V' \sqsubseteq V$;
2) $U//U' \cong V//V'$.

From these results we can then conclude the following.

3.6. PROPOSITION [Poyatos, 1973a]: *If $N \subset N'$ are proper absorbing subsemimodules of a left R-semimodule M then $(M//N)//(M//N')$ is R-isomorphic to $M//N'$.*

Let M be a left R-semimodule having a nonempty family $\mathcal{A}$ of absorbing subsemimodules and let $N = \cup\mathcal{A}$. By Proposition 3.1(5) we see that N is again an absorbing subsemimodule of M and, indeed, is the unique maximal absorbing subsemimodule of M. We will denote this subsemimodule by $A(M)$.

4. COMPOSITION SERIES AND THE JORDAN - HÖLDER THEOREM

If M is a semimodule we see that $C(M) \sqsubseteq M$. An absorbing subsemimodule N of M is *quasiminimal* if and only if it properly contains $C(M)$ and there is no absorbing subsemimodule of M properly containing $C(M)$ and properly contained in N. That is to say, if M is a quasiminimal left R-semimodule then either M has no proper absorbing subsemimodules or it has precisely one such subsemimodule, namely its crux. A nontrivial left R-semimodule M is *quasisimple* if and only if it is quasiminimal and has no primitive elements. That is to say, a quasiminimal left R-module is quasisimple if it either has no strongly - infinite elements or has one which is not primitive.

If w is a strongly - infinite element of an information R- semimodule M which has no proper absorbing subsemimodules other than $C(M)$ then we know by Proposition 3.1(6) that $M^* + M^*$ equals either M^* or $\{w\}$. In the first case, M is quasisimple. In the second case, w is a primitive element of M.

Consider the following example [Poyatos, 1973a]: Let T be a nonempty set and let z, w be distinct elements not in T. Define the operation $+$ on $X = T \cup \{z, w\}$ by setting:

1) $t + t' = w$ for all $t, t' \in T$;

2) $x + z = z + x = x$ for all $x \in X$;

3) $x + w = w + x = w$ for all $x \in X$;

Moreover, for each $k \in \mathbf{N}$ and $x \in X$ define the element kx of X as follows:

4) $0x = z$;

5) $1x = x$;

6) $kx = w$ if $k > 1$ and $x \neq z$;

7) $kz = z$ for all $k \in \mathbf{N}$.

Then X becomes a left $\mathbf{N}$-semimodule with strongly- infinite element w. Indeed, if T' is any subset of T then $T' \cup \{z, w\}$ is an absorbing subsemimodule of X. Moreover, w is primitive in X.

From the definitions we see that if N is an absorbing subsemimodule of a left R-module of M and N' is an absorbing subsemimodule of M properly containing N then the following conditions are equivalent:

1) $N'//N$ is quasiminimal;

2) There is no absorbing subsemimodule of M properly containing N and properly contained in N'.

We also note that if $A(M) \neq M$ then, by Proposition 3.4, the R- semimodule $M//A(M)$ is quasisimple.

4.1 PROPOSITION [**Poyatos, 1972,1973a**]: *Let R be an information algebra. If M is a left R-semimodule and if m is an element of M^* then:*

1) $m \in A(M)$ if and only if $T(m) \sqsubseteq M$;

2) If $m \in A(M)$ then $Rm\ \S A(m)$ is the unique smallest absorbing subsemimodule of M containing m. Moreover, $Rm\ \S A(M) = T(m)$.

In particular, if $m \in A(M)^*$ then $T(m') \subseteq T(m)$ for all $m' \in T(m)^*$. Set $T'(m) = \{0\} \cup \{m' \in T(m)^* | T(m') \neq T(m)\}$.

4.2. PROPOSITION [**Poyatos, 1973b**]: *Let R be an information algebra and left M be a left R-semimodule having an absorbing subsemimodule. If $m \in A(M)^*$ then $T'(m)$ is a maximal proper absorbing subsemimodule of $T(m)$.*

We note that $w \in A(M)$ and if $m \in A(M)$ then $rm + w = w$ for each $r \in R$ and so $w \in T(m)$. Thus we conclude that $C(M) \subseteq T(m)$ for each $m \in A(M)$.

4.3 PROPOSITION [Poyatos, 1972]: *Let R be an information algebra and let M be a left R-semimodule having a strongly-infinite element w. Then:*

1) $L(M) = \{0\} \cup \{m \in A(M) | T(m) = C(M)\}$ is an absorbing subsemimodule of M which is the unique maximal subsemimodule N of $A(M)$ satisfying $n + m = w$ for all $n \in N^*$ and $m \in A(M)^*$.

2) If $m \in A(M) \setminus C(M)$ then $T(m) = C(M)$ if and onlly if $T(m)$ is quasiminimal.

4.4 PROPOSITION [Poyatos, 1973a]: *Let R be an information algebra and let M be a noncrucial information R- semimodule having a strongly - infinite element w. Then M is quasisimple if and only if $T(m) = M$ for all $m \in M \setminus C(M)$.*

4.5 COROLLARY: *Let R be an information semiring and let M be a noncrucial quasisimple information R-semimodule having a strongly - infinite element w. Then $T'(m') = C(M)$ for all $m' \in M \setminus C(M)$.*

4.6 PROPOSITION [Poyatos, 1973a]: *Let R be an information algebra and let M be a left R-semimodule having a strongly - infinite element w. If N is a quasiminimal absorbing subsemimodule of M then either N has a primitive element or it is quasisimple.*

If R is an information algebra and M is a left R- semimodule having an absorbing subsemimodule then, for every $m \in A(M)$, the left R-semimodule $\overline{T}(m) = T(m)//T'(m)$ is the *principal factor* of M at m. By what we have noted above, $\overline{T}(m)$ has a strongly - infinite element.

4.7 PROPOSITION: *Let R be an information algebra and let M be a left R-semimodule having an absorbing subsemimodule. If $m \in A(M)^*$ then $\overline{T}(m)$ either has a primitive element or is quasisimple.*

4.8 PROPOSITION [Poyatos, 1973b]: *Let R be an information algebra and let M be a left R-semimodule. If N' is a maximal proper absorbing submodule of a subsemimodule N of $A(M)$ then $N//N' \cong \overline{T}(m)$ for any $m \in N \setminus N'$.*

If M is a left R-semimodule then an *absorbing series* for M is a descending chain $M = N_0 \sqsupseteq N_1 \sqsupseteq \ldots \sqsupseteq N_t = C(M)$ of subsemimodules of M. An *absorbing quasiseries* for M is an absorbing series for $A(M)$. Any chain obtained from a given absorbing series by inserting further terms is a *refinement* of that series. If new subsemimodules are actually inserted, such a refinement is *proper*. Two absorbing series $M = N_0 \sqsupseteq N_1 \sqsupseteq \ldots \sqsupseteq N_t = C(M)$ and $M = L_0 \sqsupseteq L_1 \sqsupseteq \ldots \sqsupseteq L_s = C(M)$ for M are *isomophic* if and only if $t = s$ and there is a permutation σ of $\{1, \ldots, t\}$ such that $N_{i-1}//N_i \cong L_{\sigma(i)-1}//L_{\sigma(i)}$ for each $1 \leq i \leq t$.

Finally, we come to Poyatos' extension of the Jordan - Hölder theorem.

4.9 PROPOSITION [**Poyatos, 1973b**]: *Let R be an information algebra and let M be a left R-semimodule. Then any two absorbing quasiseries of M have isomorphic refinements.*

REFERENCES

Samson Abramsky & Steven Vickers:

[1990] Quantales, observational logic, and process semantics, preprint, 1990.

J.C.M. Baeten & J.A. Bergstra:

[1990] Process algebra with a zero object, in J.C.M. Baeten (ed): CONCUR '90, *Theories of Concurrency: Unification and Extension*, Lecture Notes in Computer Science # 458, Springer - Verlag, Berlin, 1990.

J.C.M. Baeten, J.A. Bergstra & J.W. Klop:

[1987] On the consistency of Koomen's fair abstraction rule, Theor. Comp. Sci. 51 (1987), 129 - 176.

J.C.M. Baeten & W.P. Weijland:

[1990] *Process Algebra*, Cambridge University Press, Cambridge, 1990.

Jean Berstel & Christophe Reutenauer:

[1988] *Rational Series and Their Languages*, Spinger - Verlag, Berlin, 1988.

Peter Butkovic:

[1985] Necessary solvability conditions of systems of linear extremal equations, Discrete Appl. Math. 10 (1985), 19-26.

Raymond A. Cuninghame - Green:

[1979] *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems # 166, Spinger - Verlag, Berlin, 1979.

G. Duchamp & J.Y. Thibon:

[1988] Theoremes de transfert pour les polynomes partiellement commutatifs, Theor. Comp. Sci. 57 (1988), 239 - 249.

Samuel Eilenberg:

[1974] *Automata, Languages, and Machines*, vol. A, Academic Press, New York, 1974.

Calvin C. Elgot:

[1976] Matricial theories, J. Algebra 42 (1976), 391 - 422.

Jonathan S. Golan:
[1991] *Theory of Semirings*, Longman, Harlow, 1991.

Martin Goldstern:
[1985] *Vervollstaendigung von Halbringen*, Diplomarbeit, Technische Universitaet Wien, 1985.

Michel Gondran & Michel Minoux:
[1984] *Graphs and Algorithms*, Wiley - Interscience, New York, 19854.

Udo Hebisch & Hanns J. Weinert:
[1990a] Semirings without zero divisors, Math. Pannon. 1 (1990), 73-94.
[1990b] Generalized semigroup semirings which are zero divisor free or multiplicatively left cancellative, preprint, 1990.

Wim H. Hesselink:
[1990] Command algebra, recursion and program transformation, Formal Aspects of Computing 2 (1990), 60 - 104.

Yannis E. Ioannidis & Eugene Wong:
[1990] Toward an algebraic theory of recursion, preprint, 1990.

Kazuo Iwano & Kenneth Steiglitz:
[1990] A semiring on convex polygons and zero - sum cycle problems, SIAM J. Comput. 19 (1990), 883 -901.

Stephen C. Kleene:
[1956] Representation of events in nerve nets and finite automata, in C. E. Shannon & J. McCarthy (eds): *Automata Studies,* Princeton University Press, Princeton, 1956.

Daniel Krob:
[1987] Monoides et semi - anneaux complets, Semigroup Forum 36 (1987), 323-339.

J. Kuntzmann:
[1972] *Theorie des Reseaux (Graphes)*, Dunod, Paris, 1972.

Gerard Lallement:
[1979] *Semigroups and Combinatorial Applications*, Wiley, New York, 1979.

Michael G. Main & David L. Black:
[1989] Semantic models for total correctness and fairness, preprint,1989.

Jean - Paul Mascle:
[1986] Torsion matrix semigroups and recognizable transductions, in L. Kott (ed): *Automata, Languages and Programming*, Lecture Notes in Computer Science # 226, Springer - Verlag, Berlin, 1986.

Sidney S. Mitchell & Porntip Sinutoke:

[1982] The theory of semifields, Kyungpook Math. J. (1982), 325 - 348.

Francisco Poyatos:

[1972] El teorema de Jordan - Hoelder para A - semimodulos, Rev. Mat. Hispano - Americana (4) 32 (1972), 251 - 260.

[1973a] El teorema de Jordan - Hoelder para A - semimodulos (continuacion), Rev. Mat. Hispano - Americana (4) 33 (1973), 36 - 48.

[1973b] El teorema de Jordan - Hoelder para A - semimodulos (conclusion), Rev. Mat. Hispano - Americana (4) 33 (1973), 122-132.

[1974] Sucesiones principales y series de composicion de A - semimodulos, Rev, Mat. Hispano - Americana (4) 34 (1974), 60-66.

[1976] Series principales y series de composicion de semianillos, Rev. Mat. Hispano - Americana (4) 36 (1976), 63 - 76

[1977] Sobre ciertas propiedades de los bi - ideales de un semianillo, Rev. Mat. Hispano - Americana (4) 37 (1977), 107-133.

[1980] The Jordan - Holder theorem for semirings, Rev. Mat. Hispano - Americana (4) 40 (1980), 49-65.

Arto Salomaa & Matti Soittola:

[1978] *Automata - theoretic Aspects of Formal Power Series*, Springer - Verlag, Berlin, 1978.

Imre Simon:

[1988] Recognizable sets with multiplicities in the tropical semiring, in M. P. Chytil et al. (eds): *Mathematical Foundations for Computer Science 1988*, Lecture Notes in Computer Science # 324, Spinger-Verlag, Berlin, 1988.

F. A. Smith:

[1966] A structure theory for a class of lattice ordered semirings, Fund. Math. 59 (1966), 49-64.

Michihiro Takahashi:

[1984a] On semimodules I, Kobe J. Math. 1(1984), 67-97.

[1984b] On semimodules. II, Kobe J. Math. 1(1984), 177-190.

[1985] On semimodules. III, Kobe J. Math. 2(1985), 131-141.

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