Deligne-Lusztig variety

1 Introduction

1.1 Deligne-Lusztig varieties

Let $\mathbb{G}$ be a connected reductive group over an algebraically closed field $\mathbf{k}$ and set $G = \mathbb{G}(\mathbf{k})$. Let $\mathbf{B}$ be a Borel subgroup of G and W be the finite Weyl group. Then we have the Bruhat decomposition:

$$G = \sqcup_{w \in W} \mathbf{B} w \mathbf{B}$$

which play a fundamental role in understanding the structure and representations of G.
Assume that $\mathbf{k} = \bar{\mathbb{F}}_q$ and σ is the Frobenius of $\mathbf{k}$ over $\mathbb{F}_q$. We also assume that $\mathbb{G}$ is defined over $\mathbb{F}_q$ and denote by σ the corresponding Frobenius morphism on G. In the seminal work of Deligne and Lusztig [DL76], they introduced Deligne-Lusztig varieties as follows: for any element $w \in W$, the corresponding Deligne-Lusztig variety X_w is a subvariety of the flag variety G/B defined by

$$X_w = \{gB \in G/B \mid g^{-1}\sigma(g) \in BwB\}.$$

Lang's theorem says that the Lang-Steinberg map $G \to G$ given by $g \mapsto g^{-1}\sigma(g)$ is surjective (see [Ste, Lem. 7.3], so this implies that X_w is non-empty. Additionally, by a transversal argument, Deligne and Lusztig showed that X_w is a locally closed, smooth variety of pure dimension $l(w)$, the length of w. The finite reductive group $\mathbb{G}(\mathbb{F}_q)$ acts naturally on X_w and on the cohomology of X_w. Moreover, $\mathbb{G}(\mathbb{F}_q)$ acts transitively on the set of irreducible components of X_w (see [BR06]). Then the representations on the compact support $l-$adic etale cohomology of $X'_w s$ will contain all irreducible representations of $\mathbb{G}(\mathbb{F}_q)$.

For example, let's look at the character table of $GL_2(\mathbb{F}_p)$ for a prime p. We then see three kinds of irreducible representations (other than the one-dimensional characters which factor through the determinant $det : GL_2(\mathbb{F}_p) \to \mathbb{F}_p^\times$): principal series representations $I(\chi_1, \chi_2)$ which is obtained by inducing a pair of characters χ_1, χ_2 of $\mathbb{F}_p^\times$ regarded as a character of the split torus of diagonal matrices $\left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}$ via the Borel subgroup of upper-triangular matrices $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$; and the Steinberg representations Sp (and its twists Sp_χ by one-dimensional characters $\chi \circ det$) which is obtained from the action of $GL_2(\mathbb{F}_p)$ on $\mathbb{P}^1(\mathbb{F}_p)$; and the cuspidal (or discrete series) representations $\Theta(\chi)$ corresponding to the character χ of the non-split torus isomorphic to $\mathbb{F}_{p^2}^\times$. Among them, the construction of the cuspidal representations is the most mysterious. Then Drinfeld (in his seminal 1974 paper on Langlands correspondence for GL_2) realized that these are obtained from the first $l-$adic cohomology group of the affine curve $(XY^p - X^pY)^{p-1} = 1$, on which the groups $GL_2(\mathbb{F}_p)$ and the non-split torus act, and the actions commute with each other to give the correspondence of the representations of two groups on the cohomology group.

The Deligne-Lusztig theory generalizes the above fact to the vast general setting of the general reductive groups over finite fields.

Besides the crucial role in the representation theory of finite reductive groups ([DL76]). The structure of X_w has also found important applications in number theory (see [RUW13] for instance).

1.2 Affine Deligne Lusztig varieties

Affine Deligne-Lusztig varieties are analogous to classical Deligne-Lusztig varieties for loop groups. We denote $k = \bar{\mathbb{F}}_q((\epsilon))$, and let σ be the Frobenius morphism of k over $\mathbb{F}_q((\epsilon))$. We assume that the reductive group $\mathbb{G}$ is defined over $\mathbb{F}_q((\epsilon))$ and denote by σ the corresponding Frobenius morphism on the loop group $G = \mathbb{G}(k)$. We choose a $\sigma-$stable Iwahori subgroup I of G. The affine flag variety $Fl = G/I$ has a natural scheme structure.
Let S be a maximal $k-$split torus of G defined over $\mathbb{F}_q((\epsilon))$ and let T be its centralizer, a maximal torus of G. The Iwahori-Weyl group associated to S is

$$\tilde{W} = N(k)/T(k)_1,$$

where N is the normalizer of S in G and $T(k)_1$ is the maximal open compact subgroup of $T(k)$. The group $\tilde{W}$ has a natural quasi-Coxeter structure. We have the following generalization of the Bruhat decomposition

$$G = \bigsqcup_{w \in \tilde{W}} IwI,$$

due to [BT72]. The following definition of affine Deligne-Lusztig varieties was introduced by Rapoport in [Ra05].

Definition 1.1. Given $w \in \tilde{W}$ and $b \in G$, the corresponding affine Deligne-Lusztig varieties (in the affine flag variety) is defined as

$$X_w(b) = \{gI \in G/I;\ g^{-1}b\sigma(g) \in IwI\} \subset Fl.$$

Affine Deligne-Lusztig varieties are schemes locally of finite type over $\bar{\mathbb{F}}_q$. They serve as group-theoretic models for Shimura varieties and shtukas. The $\sigma-$centralizer

$$J_b = \{g \in G(k)\,;\, g^{-1}b\sigma(g) = g\}$$

acts naturally on $X_w(b)$. Unlike classical Deligne-Lusztig varieties, which have the nice geometric structure described in Section 1.1, the geometric structures of affine Deligne-Lusztig varieties are very complicated. Here are some phenomena and open questions in this field:

- For many pairs (w, b), $X_w(b)$ are empty.
- Assume that $X_w(b)$ is nonempty, is it equidimensional? How do we determine its dimension?
- In general, the group J_b does not act transitively on the set of irreducible components of $X_w(b)$ (compare to the action of $G(\mathbb{F}_q)$ in the classical case).

- The irreducible components of $X_w(b)$ have a very complicated geometric structure.

Remark 1.2. A major difference between affine Deligne-Lusztig varieties and classical Deligne- Lusztig varieties is that affine Deligne-Lusztig varieties have the second parameter: the element b, or the $\sigma-$conjugacy class $[b]$ in the loop group G; while in the classical case considered in Section 1.1, by Langs theorem there is only one $\sigma-$conjugacy class in $\mathbb{G}(\bar{\mathbb{F}}_q)$ and thus adding a parameter $b \in \mathbb{G}(\bar{\mathbb{F}}_q)$ does not give any new variety.

2 Plan

This small workshop is designed to learn some basic knowledge of Deligne-Lusztig varieties and their affine versions. We will try to go through some basic concepts and the main example will be PGL_2, GL_2, SL_2, and GL_n. The main references are [Bon11] and [Ra05]. Here is the plan:

Talk 1: Structure of $SL_2(\mathbb{F}_q)$. Reference: [Bon11, Chapter 1].

- Transitive action of $G = SL_2(\mathbb{F}_q)$. Structure of centralizers.
- Some subgroups: Borel subgroup, unipotent radical, torus (split).
- Bruhat decomposition
- Conjugacy classes (there are $q+4$ conjugacy classes), Sylow subgroups and their normalizers.

Talk 2: Harish-Chandra theory for $SL_2(\mathbb{F}_q)$. Reference: [Bon11, Chapter 3].

- Recall some group representation terminology: representation, (split) group algebra, irreducible representation, characters,...
- Construction of Harish-Chandra induction and restriction in case of $SL_2(\mathbb{F}_q)$.
- Mackey Formula.
- $(q+5)/2$ irreducible characters which are built out from Harish-Chandra theory. There are still $(q+3)/2$ irreducible characters.

Talk 3: Drinfeld curve and its $l-$adic cohomology. Reference: [Bon11, Chapter 2, Appendix A].

- Definition of Drinfeld curve Y and its basic properties: smoothness, irreducibility.
- Action of G and the non-split torus $T' \cong \mu_{q+1}$ on Y. These two actions commute.
- Some quotients of the above actions. Compute the $l-$adic cohomology of those quotients.
- Prove that for a group H acting on a variety V then $H_c^i(V)^H \cong H_c^i(V/H)$. And if V is irreducible then $H_c^{2dim(V)}(V)$ is one dimensional.

Talk 4: Deligne-Lusztig theory for $SL_2(\mathbb{F}_q)$. Reference: [Bon11, Chapter 4]. In this talk, we will see $(q+3)/2$ remaining irreducible characters of G : cuspidal characters.

- Action of $G \times \mu_{q+1}$ on the Drinfeld curve Y.
- Delinge-Lusztig induction, Harish-Chandra induction is orthogonal to Deligne-Lusztig induction.
- Cuspidality of irreducible components of Deligne-Lusztig induction.
- Mackey formula
- $(q+3)/2$ cuspidal characters.

Talk 5: An overview of Deligne-Lusztig theory for general groups. Reference: [Yo] and [Bon11, Chapter 12]

- The construction of the Deligne-Lusztig varieties, some basic properties: quasi-affine, smooth, pure dimension.
- How Drinfeld curve appears as a Deligne-Lusztig variety of SL_2.
- Deligne-Lusztig induction.
- Main theorems: Mackey formula, disjointness theorem.

Talk 6: Affine Deligne-Lusztig varieties: some definitions and GL_2 example. Reference [Ra05, Section 4]. Note that the referenced paper is for p-adic case whereas you can work over function field case as well.

- Recall some necessary Bruhat-Tits building terminologies (for the case $G = GL_n$ is enough): facet, apartment, parahoric subgroup, Iwahori subgroup.
- Affine Deligne-Lusztig varieties, GL_2 as an illustrated example.
- Kottwitz and Newton map. Prove Proposition 4.4 in [Ra05] about the non-emptiness of affine Deligne-Lusztig varieties in GL_n case.

Talk 7: SL_2, PGL_2 examples and non-emptiness of affine Deligne-Lusztig varieties. Reference: [Reu02, Chapter 2].

- Recall affine building $\mathcal{B}_\infty$, the main apartment A_M, and the main chamber C_M. Let $\rho : \mathcal{B}_\infty \to A_M$ be the retraction of $\mathcal{B}_\infty$ onto A_M relative to C_M.
- Identity $\sigma-$conjugacy class: $b = 1$. (Section 2.1.1)
- Non-Identity $\sigma-$conjugacy class: $b \neq 1$. (Section 2.1.2)
- Relationship between $X_w(1\sigma)$ and $X_w(b\sigma)$. (Section 2.2.4)
- Other non-simply connected groups of semisimple rank 1. (Section 2.2)

Talk 8: Relation to local Shimura variety. Reference [Ra05, Section 3,5,6].

- Admissible and permissible $\mu-$sets. Prove [Ra05, Prop.3.11 and Cor. 3.12].
- Local model of Shimura variety (Rapoport-Zink space), denoted by $\mathcal{M}^{loc}$ in the reference (Section 6).
- Explain in details some cases in [Ra05, Section 6] where local models are related to affine Deligne-Lusztig varieties.

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