Cousin complexes via total fractions

I-Chiau Huang

Cousin complexes appeared in Grothendieck's theory of duality [1]. While looking for some sort of natural minimal injective resolution for a Gorenstein ring, R.Y. Sharp also defined Cousin complexes [2]. Concrete Cousin complexes are rarely seen for explicitly given Noetherian rings except for easy cases. We have a new framework for Cousin complexes so that computations become feasible. Let R be a commutative Noetherian ring. For an R-module M, its Cousin complex

$$\cdots \bigoplus_{\text{height } \mathfrak{p}=r} \mathrm{H}^r_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \to \bigoplus_{\text{height } \mathfrak{q}=r+1} \mathrm{H}^{r+1}_{\mathfrak{q}R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \to \cdots$$

is built up by local cohomology modules. Elements of top local cohomology modules can be described by generalized fractions. A fundamental and natural question is to describe $\mathrm{H}^r_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \to \mathrm{H}^{r+1}_{\mathfrak{q}R_{\mathfrak{q}}}(M_{\mathfrak{q}})$ in the coboundary map in terms of generalized fractions. For a height r prime $\mathfrak{p}$, a system of parameters $f_1, \ldots, f_r$ of $\mathfrak{p}$ and $m \in M$, $f \in R \setminus \mathfrak{p}$, the expected formula

$$\begin{bmatrix} m/f \\ f_1, \ldots, f_r \end{bmatrix} \mapsto \begin{bmatrix} m \\ f_1, \ldots, f_r, f \end{bmatrix}$$

is not even well-defined, *cf.* [3]. To remedy the defect, we introduce new notions: total fractions and faithful representations. Here is an explicit formula occurring in the Cousin complex. Let κ be a field and $R := \kappa[\![X^4, X^3Y, XY^3, Y^4]\!]$. For the maximal ideal $\mathfrak{m}$ and the prime ideal $\mathfrak{p}$ generated by X^4 up to radical, the R-linear map $\mathrm{H}^1_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \mapsto \mathrm{H}^2_{\mathfrak{m}R_{\mathfrak{m}}}(R_{\mathfrak{m}})$ in our approach gives rise to

$$\begin{bmatrix} 1/Y^4 \\ X^3Y \end{bmatrix} \mapsto \begin{bmatrix} XY^3 \\ X^4, Y^8 \end{bmatrix}.$$

It is not clear how other approaches to Cousin complexes provide such a formula.

References

[1] R. Hartshorne, *Residues and Duality*, volume 20 of Lecture Notes in Mathematics, Springer-Verlag, 1966.

[2] R. Y. Sharp, *The Cousin complex for a module over a commutative Noetherian ring*, Math. Z. **112** (1969), 340-356.

[3] F. El Zein, *Complexe dualisant et applications à la classe fondamentale d'un cycle.* Bull. Soc. Math. France Mém., No. 58, 1978.

Institute of Mathematics, Academia Sinica
6F, Astronomy-Mathematics Building, No. 1, Sec. 4, Roosevelt Road
Taipei 10617
Taiwan *E-mail address*: ichuang@math.sinica.edu.tw