Cohomology Groups

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Let X be a topological space, and let $\mathscr{U} = (U_i)_{i \in I}$ be an open covering of X.
Let $\mathcal{F}$ be a sheaf of abelian groups on X.
For each p, define the *qth* cochain group of $\mathcal{F}$ as

$$C^q(\mathscr{U}, \mathcal{F}) = \prod_{(i_0, i_1, \ldots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0, \ldots, i_q})$$

Thus a $q-$cochain is a family

$$f = (f_{i_0, \ldots, i_q}) \text{ such that } f_{i_0, \ldots, i_q} \in \mathcal{F}(U_{i_0, \ldots, i_q})$$

We define the coboundary map $d : C^q \to C^{q+1}$ by setting

$$(df)_{i_0, \ldots, i_{q+1}} = \sum_{k=0}^{q+1} (-1)^k f_{i_0, \ldots, \widehat{i_k}, \ldots, i_{q+1}} \big|_{U_{i_0, \ldots, i_{q+1}}}$$

At the 0 level, d sends $0-$cochain (f_i) to the $1-$cochain (g_{ij}) where

$$g_{ij} = f_i - f_j$$

At the 1 level, d sends $1-$cochain (f_{ij}) to the $2-$cochain (g_{ijk}) where

$$g_{ijk} = f_{jk} - f_{ik} + f_{ij}$$

The elements of $Z^q(\mathscr{U}, \mathcal{F}) := Kerd^q$ are called $q-$cocycles.

The elements of $B^q(\mathscr{U}, \mathcal{F}) := Imd^{q-1}$ are called $q-$coboundaries.

We can check that $d^2 = 0$, so we get a complex $C^{\cdot}(\mathscr{U}, \mathcal{F})$

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \ldots \longrightarrow C^q \longrightarrow \ldots$$

We define the *qth* cohomology group of $\mathcal{F}$, with respect to the covering $\mathscr{U}$, to be

$$H^q(\mathscr{U}, \mathcal{F}) = H^q(C^{\cdot}(\mathscr{U}, \mathcal{F}))$$

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Example

- $H^0(\mathscr{U}, \mathcal{F}) = \mathcal{F}(X)$
  Thus $H^0(\mathscr{U}, \mathcal{F})$ is independent of $\mathscr{U}$.
  This is not true in general.
- $H^1(\{\mathbb{C}^*\}, \mathbb{Z}) = 0$ but $H^1(\{U_1, U_2\}, \mathbb{Z}) = \mathbb{Z}$
  where $U_1 = \mathbb{C}^* \setminus \mathbb{R}_-$ and $U_2 = \mathbb{C}^* \setminus \mathbb{R}_+$.

We would like to associate to every sheaf $\mathcal{F}$ a cohomology group which does not depend on the covering.

Definition

An covering $\mathscr{B} = (V_k)_{k\in K}$ is called finer than $\mathscr{U} = (U_i)_{i\in I}$, denoted $\mathscr{B} < \mathscr{U}$, if there is a mapping $\tau : K \to I$ such that

$$V_k \subset U_{\tau k} \text{ for every } k \in K.$$

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$$V_k \subset U_{\tau k} \text{ for every } k \in K.$$

In this case, τ induces a mapping on $q-$cochains

$$\tau : C^q(\mathscr{U}, \mathcal{F}) \to C^q(\mathscr{B}, \mathcal{F})$$

by the formula

$$(\tau f)_{i_0,\ldots,i_q} = f_{\tau i_0,\ldots,\tau i_q}\big|_{V_{i_0,\ldots,i_q}}$$

Proposition

The mapping τ commutes with d, thus it defines homomorphisms

$$H(\tau) : H^q(\mathscr{U}, \mathcal{F}) \to H^q(\mathscr{B}, \mathcal{F})$$

The mapping $H(\tau)$ is independent of the choice of τ.

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1. $(d(\tau f))_{i_0,\ldots,i_q} = \sum_{k=0}^{q}(-1)^k(\tau f)_{i_0,\ldots,\widehat{i_k},\ldots,i_q}$

   $= \sum_{k=0}^{q}(-1)^k f_{\tau i_0,\ldots,\widehat{\tau i_k},\ldots,\tau i_q} = (df)_{\tau i_0,\ldots,\tau i_q} = (\tau(df))_{i_0,\ldots,i_q}$.

   Thus $d\tau = \tau d$.
2. Suppose that τ and τ' are both refining maps.

   Fix $h = [(f)] \in H^q(\mathscr{U}, \mathcal{F})$.

Then $H(\tau)(h)$ is presented by (τf) and $H(\tau')(h)$ is presented by $(\tau' f)$, where

$$(\tau f)_{i_0,\ldots,i_q} = f_{\tau i_0,\ldots,\tau i_q} \text{ and } (\tau' f)_{i_0,\ldots,i_q} = f_{\tau' i_0,\ldots,\tau' i_q}.$$

Form the $(q-1)-$cochain (b) defined by

$$b_{l_0,\ldots,l_{q-1}} = \sum_{k=0}^{q-1} (-1)^k f_{\tau l_0,\ldots,\tau l_k,\tau' l_k,\ldots,\tau' l_{q-1}}.$$

We see that $d(b) = (\tau' f) - (\tau f)$.

Therefore $H(r)(h) = H(r')(h)$.

We denote $H(\tau)$ by $t_{\mathscr{B}}^{\mathscr{U}}$.

Proposition

The refinement map $t_{\mathscr{B}}^{\mathscr{U}}$ is injective when $q = 1$.

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The refinement map $t^{\mathscr{U}}_{\mathscr{B}}$ is injective when $q = 1$.

Consider a cocycle $f = (f_{ij}) \in Z^1(\mathscr{U}, \mathcal{F})$ s.t $t^{\mathscr{U}}_{\mathscr{B}}([f]) = 0$.

There exists $g = (g_k) \in C^0(\mathscr{B}, \mathcal{F})$ s.t $f_{\tau k, \tau l} = g_k - g_l$ on $V_k \cap V_l$.

Then $g_k - g_l = f_{\tau k, \tau l} = f_{i, \tau l} - f_{i, \tau k}$ on $U_i \cap V_k \cap V_l$
and thus $f_{i,\tau k} + g_k = f_{i, \tau l} + g_l$.

Applying sheaf axioms to the family $(U_i \cap V_k)_{k \in K}$,
one obtains $h_i \in \mathcal{F}(U_i)$ s.t $h_i = f_{i, \tau k} + g_k$.

Thus $f_{ij} = f_{t, \tau k} + f_{\tau k, j} = h_i - h_j$ on $U_i \cap U_j \cap V_k$ for any $k \in K$.

Then on $U_i \cap U_j$, $f_{ij} = h_i - h_j$.

Therefore, $[f] = [d(h)] = 0 \in H^1(\mathscr{U}, \mathcal{F})$.

The relation $<$ between coverings of X is directed.

It is clear that $t_{\mathscr{U}}^{\mathscr{U}} = id$ and $t_{\mathscr{V}}^{\mathscr{B}} \circ t_{\mathscr{B}}^{\mathscr{U}} = t_{\mathscr{U}}^{\mathscr{V}}$ if $\mathscr{V} < \mathscr{B} < \mathscr{U}$.

Thus the collection $\{H^q(\mathscr{U}, \mathcal{F})\}_{\mathscr{U}}$ is a directed system.

Definition

The *qth* cohomology groups of X with coefficient $\mathcal{F}$ is defined by

$$H^q(X, \mathcal{F}) := \varinjlim H^q(\mathscr{U}, \mathcal{F})$$

From the definition, an element of $H^q(X, \mathcal{F})$ is a class $[\alpha, \mathscr{U}]$ where $\mathscr{U}$ is a covering of X and $\alpha \in H^q(\mathscr{U}, \mathcal{F})$.

Two classes $[\alpha, \mathscr{U}]$ and $[\beta, \mathscr{V}]$ are the same if there exists a covering $\mathscr{B}$ with $\mathscr{B} < \mathscr{U}$ and $\mathscr{B} < \mathscr{V}$ such that

$$t^{\mathscr{U}}_{\mathscr{B}}(\alpha) = t^{\mathscr{V}}_{\mathscr{B}}(\beta).$$

Corollary

$$H^1(X, \mathcal{F}) = 0$$

if and only if

$$H^1(\mathscr{U}, \mathcal{F}) = 0 \text{ for all open covering } \mathscr{U}.$$

1. $H^0(X, \mathcal{F}) = \mathcal{F}(X)$
2. If X is a compact Riemann surface, $H^0(X, \mathscr{O}) = \mathbb{C}$
3. $H^0(\mathbb{P}^1, \mathscr{M}) = \mathbb{C}(x)$
4. For any $q \geq 1$, $H^q(X, \mathcal{F}_p) = 0$ where X is a space and $\mathcal{F}_p$ is a skyscraper sheaf at $p \in X$.

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Consider $[\alpha, \mathscr{U}] \in H^q(X, \mathcal{F}_p)$.

The covering $\mathscr{U}$ has a refinement $\mathscr{B} = (V_i)$ such that the point p is contained in only one V_j.

Thus $C^q(\mathscr{B}, \mathcal{F}_p) \cong \mathcal{F}_p(V_j)$ and $Z^q(\mathscr{B}, \mathcal{F}_p) = 0$,

i.e. $H^q(\mathscr{B}, \mathcal{F}_p) = 0$ and $[\alpha, \mathscr{U}] = [t^{\mathscr{U}}_{\mathscr{B}}(\alpha), \mathscr{B}] = 0$.

Theorem

Let X be a Riemann surface. Then $H^1(X, \mathscr{E}) = 0$

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Proposition (partition of unity)

On any paracompact differentiable manifold X, one has partition of unity for any open covering $\mathscr{U} = \{U_i\}$,i.e there is a set of $C^\infty-$ functions $\{\varphi_i\}$ such that

- every point in X has a neighborhood meeting only finitely many of the sets $\mathrm{Supp}(\varphi_i)$
- $\mathrm{Supp}(\varphi_i) \subset U_i$
- $\sum \varphi_i = 1$

Theorem

Let X be a Riemann surface. Then $H^1(X, \mathscr{E}) = 0$

We will show that $H^1(\mathscr{U}, \mathscr{E}) = 0$ for every covering $\mathscr{U} = \{U_i\}$.
Let (f_{ij}) be a $1-$cocycle, i.e $f_{jk} - f_{ik} + f_{ij} = 0$ for all $(i, j, k) \in I^3$.
In particular, it implies $f_{ii} = 0$ and $f_{ij} = -f_{ji}$.
The $C^\infty-$function $\varphi_j f_{ij}$ on U_{ij} may be differentiably extended to all of U_i by zero outside U_{ij}.

Set $g_i = -\sum_j \varphi_j f_{ij} \in \mathscr{E}(U_i)$, then

$$g_j - g_i = -\sum_k \varphi_k f_{jk} + \sum_k \varphi_k f_{ik} = \sum_k \varphi_k (f_{ik} - f_{jk}) = \sum_k \varphi_k f_{ij} = f_{ij}$$

so that $(f_{ij}) = d(g_i)$ is a coboundary.

Similarly, by setting

$$g_{i_0,\dots,i_{q-1}} = (-1)^n \sum_k \varphi_k f_{i_0,\dots,i_{q-1},k}$$

we can show that

Proposition

For any $q \geq 1$ and $\mathcal{F} \in \{\mathscr{E}, \mathscr{E}^1, \mathscr{E}^{(1,0)}, \mathscr{E}^{(0,1)}, \mathscr{E}^2\}$

$$H^q(X, \mathcal{F}) = 0$$

Theorem

Suppose X is a simply connected Riemann surface. Then

1. $H^1(X, \mathbb{C}) = 0$
2. $H^1(X, \mathbb{Z}) = 0$

We will show that $H^1(\mathscr{U}, \mathbb{C}) = 0$ for every covering $\mathscr{U} = \{U_i\}$.
Let (c_{ij}) be a $1-$cocycle. Since $Z^1(\mathscr{U}, \mathbb{C}) \subset Z^1(\mathscr{U}, \mathscr{E})$ and $H^1(X, \mathscr{E}) = 0$, there exists $(f_i) \in C^0(\mathscr{U}, \mathscr{E})$ such that $c_{ij} = f_i - f_j$ on U_{ij}.
Since $dc_{ij} = 0$, it follows that $df_i = df_j$ on U_{ij}. Thus there exists $\omega \in \mathscr{E}^{(1)}(X)$ such that $\omega\big|_{U_i} = df_i$. Thus $d\omega = 0$, then there exists $f \in \mathscr{E}(X)$ such that $df = \omega$.
Set

$$c_i := f_i - f \text{ on } U_i.$$

Therefore $dc_i = df_i - df = \omega - \omega = 0$ on U_i, c_i is locally constant, i.e., $(c_i) \in C^0(\mathscr{U}, \mathbb{C})$.
On U_{ij}, one has

$$c_{ij} = f_i - f_j = (f_i - f) - (f_j - f) = c_i - c_j,$$

and thus (c_{ij}) is a coboundary.

Theorem (Leray)

Suppose $\mathcal{F}$ is a sheaf of abelian group on the topological space X and $\mathscr{U} = \{U_i\}$ is an open covering of X such that $H^1(U_i, \mathcal{F}) = 0$ for every $i \in I$. Then

$$H^1(X, \mathcal{F}) \cong H^1(\mathscr{U}, \mathcal{F}).$$

Example

$H^1(\mathbb{C}^*, \mathbb{Z}) = \mathbb{Z}$

Let $U_1 = \mathbb{C}^* \setminus \mathbb{R}_-$, $U_2 = \mathbb{C}^* \setminus \mathbb{R}_+$ and $\mathscr{U} = \{U_1, U_2\}$. Since U_i is simply connected, $H^1(U_i, \mathbb{Z}) = 0$. Thus $H^1(\mathbb{C}^*, \mathbb{Z}) = H^1(\mathscr{U}, \mathbb{Z})$. Since $\mathbb{Z}(U_i) = \mathbb{Z}$ and $\mathbb{Z}(U_1 \cap U_2) = \mathbb{Z} \times \mathbb{Z}$, we have the complex

$$0 \longrightarrow C^0 = \mathbb{Z}^2 \longrightarrow C^1 = \mathbb{Z}^6 \longrightarrow C^2$$

So $B^1 = \{(0, 0, b_2 - b_1, b_2 - b_1, b_2 - b_1, b_2 - b_1) : b_2, b_1 \in \mathbb{Z}\}$
$= \{(0, 0, a, a, a, a) : a \in \mathbb{Z}\}$
and $Z^1 = \{(0, 0, a_{12}, -a_{12}) : a_{12} \in \mathbb{Z}^2\}$
$= \{(0, 0, a, b, -a, -b) : a, b \in \mathbb{Z}\}$.
Thus $H^1(\mathbb{C}^*, \mathbb{Z}) \cong H^1(\mathscr{U}, \mathbb{Z}) \cong \mathbb{Z}$.

Similarly, one can show that

1. $H^1(\mathbb{C}^*, \mathbb{C}) = \mathbb{C}$
2. $H^1(\mathbb{C} \setminus \{p_1, ..., p_n\}, \mathbb{Z}) = \mathbb{Z}^n$

Similarly, we can define $H^q(X, \mathcal{P})$ where $\mathcal{P}$ is a presheaf on X.
Let $\phi : \mathcal{P} \to \mathcal{G}$ be a morphism of presheaves and $\mathscr{B} < \mathscr{U}$ are coverings of X.
Then ϕ induces a mapping on $q-$cochains

$$\phi : C^q(\mathscr{U}, \mathcal{P}) \to C^q(\mathscr{U}, \mathcal{G})$$

by the formula

$$(\phi f)_{i_0,\dots,i_q} = \phi(f_{i_0,\dots,i_q}).$$

The mapping ϕ commutes with d, thus it defines homomorphisms

$$\phi_* : H^q(\mathscr{U}, \mathcal{P}) \to H^q(\mathscr{U}, \mathcal{G}).$$

One has $\phi_* \circ t^{\mathscr{U}}_{\mathscr{B}} = t^{\mathscr{U}}_{\mathscr{B}} \circ \phi_*$.
By passing to limit, we get homomorphisms

$$\phi_* : H^q(X, \mathcal{P}) \to H^q(X, \mathcal{G}).$$

Proposition

Let X be any topological space.
An exact sequences of presheaves on X

$$0 \to \mathcal{P}' \to \mathcal{P} \to \mathcal{P}'' \to 0$$

induces a long exact sequences

$$0 \to H^0(X, \mathcal{P}') \to H^0(X, \mathcal{P}) \to H^0(X, \mathcal{P}'') \to H^1(X, \mathcal{P}) \to \dots$$
$$\to H^q(X, \mathcal{P}') \to H^q(X, \mathcal{P}) \to H^q(X, \mathcal{P}'') \to H^{q+1}(X, \mathcal{P}') \to \dots$$

For any covering $\mathscr{U}$, the exact sequence of presheaves

$$0 \to \mathcal{P}' \to \mathcal{P} \to \mathcal{P}'' \to 0$$

induces an exact sequence of complexes

$$0 \to C^{\cdot}(\mathscr{U}, \mathcal{P}') \to C^{\cdot}(\mathscr{U}, \mathcal{P}) \to C^{\cdot}(\mathscr{U}, \mathcal{P}'') \to 0$$

which in turn induces the long exact sequences

$$0 \to H^0(\mathscr{U}, \mathcal{P}') \to H^0(\mathscr{U}, \mathcal{P}) \to H^0(\mathscr{U}, \mathcal{P}'') \to H^1(\mathscr{U}, \mathcal{P}) \to \ldots$$
$$\to H^q(\mathscr{U}, \mathcal{P}') \to H^q(\mathscr{U}, \mathcal{P}) \to H^q(\mathscr{U}, \mathcal{P}'') \to H^{q+1}(\mathscr{U}, \mathcal{P}') \to \ldots$$

Since direct limits preserve exactness, we obtain the desired exact sequence.

Proposition

Let X be a paracompact space, $\mathcal{P}$ a presheaf whose associated sheaf is the zero sheaf, then $H^q(X, \mathcal{P}) = 0$ for all q.

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Let $[s, \mathscr{U}] \in H^q(X, \mathcal{P})$ where $\mathscr{U}$ is a locally finite cover of X.
For $x \in X$, let B_x be an open neighborhood of x intersecting only a finite number of elements $U_1, \ldots, U_l$ of $\mathscr{U}$.
One can assume that each U_i contains x.
Thus there is an open neighborhood W_x of x contained in $B_x \cap U_1 \cap \ldots \cap U_l$ s.t

$$\alpha_{i_0,\ldots,i_q}\big|_{W_x} = 0 \text{ for every } \{i_o, \ldots, i_q\}, \text{ where } s = [(\alpha_{i_o,\ldots,i_q})].$$

Let $\mathscr{W} = (W_x)_{x \in X}$.
Thus $\mathscr{W}$ is a refinement of $\mathscr{U}$ and $t^{\mathscr{U}}_{\mathscr{W}}(s) = 0$, i.e. $[s, \mathscr{U}] = 0$.

Proposition

Let X be a paracompact space,let $\mathcal{P}$ be a presheaf.
Then the natural morphism

$$H^q(X, \mathcal{P}) \to H^q(X, \widehat{\mathcal{P}})$$

is an isomorphism.

One has an exact sequences of presheaves

$$0 \to \mathcal{Q}_1 \to \mathcal{P} \to \widehat{\mathcal{P}} \to \mathcal{Q}_2 \to 0$$

Splitting this into short exact sequences

$$0 \to \mathcal{Q}_1 \to \mathcal{P} \to \mathcal{G} \to 0 \text{ and } 0 \to \mathcal{G} \to \widehat{\mathcal{P}} \to \mathcal{Q}_2 \to 0,$$

where $\mathcal{G} = \mathcal{P}/\mathcal{Q}_1$.

Since $\widehat{\mathcal{Q}_1} = \widehat{\mathcal{Q}_2} = 0$, one has

$$H^q(X, \mathcal{Q}_1) = H^{q+1}(X, \mathcal{Q}_1) = H^q(X, \mathcal{Q}_2) = H^{q+1}(X, \mathcal{Q}_2) = 0.$$

From the long exact sequences of cohomology groups, we have

$$H^q(X, \mathcal{P}) \cong H^q(X, \mathcal{G}) \cong H^q(X, \widehat{\mathcal{P}}).$$

From these results, we obtain

Proposition

Let X be a paracompact space.
An exact sequences of sheaves on X

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

induces a long exact sequences

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \to H^1(X, \mathcal{F}) \to \dots$$
$$\to H^q(X, \mathcal{F}') \to H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}'') \to H^{q+1}(X, \mathcal{F}') \to \dots$$

Using an exact sequence of presheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{P} \to 0$$

where $\mathcal{P} = \mathcal{F}/\mathcal{F}'$.

Proposition

If $\mathcal{F}$ is an injective sheaf on a topological space X, then

$$H^i(X, \mathcal{F}) = 0$$

for all $i > 0$.

Lemma 2.4+Proposition 2.5, Algebraic Geometry - R.Hartshorne.

Proposition

Let X be a paracompact space.
For any sheaf $\mathcal{F}$ on X, we have isomorphisms

$$H^q(X, \mathcal{F}) \cong H^q_{sheaf}(X, \mathcal{F}) \text{ for all } q \geq 0$$

between *Čech* cohomology and *sheaf* cohomology.

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between *Čech* cohomology and *sheaf* cohomology.

For $q = 0$, $H^0(X, \mathcal{F}) = \mathcal{F}(X) = H^0_{sheaf}(X, \mathcal{F})$.

For the general case, embed $\mathcal{F}$ in an injective sheaf $\mathcal{G}$ and let $\mathcal{Q}$ be the quotient sheaf

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$$

One gets a long exact sequence of *Čech* cohomology groups.

Since $\mathcal{G}$ is injective, one has an exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{Q}(X) \to H^1(X, \mathcal{F}) \to 0$$

and isomorphisms

$$H^q(X, \mathcal{Q}) \cong H^{q+1}(X, \mathcal{F})$$

for each $q \geq 1$.

Comparing with the long exact sequence of *sheaf* cohomology and using induction, we obtain the desired isomorphisms.

Thank You For Listening!

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