Universal Covering and Deck Transformations

May, 2020

Hoang Son Tran
Institute of Mathematics of Hanoi

The Classification of Covering Spaces

The Classification of Covering Spaces

Let X be a connected manifold.

The Classification of Covering Spaces

The Classification of Covering Spaces

Let X be a connected manifold.

Recall

A covering space of X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ with the following property: for each point $x \in X$ there exists an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U.

The Classification of Covering Spaces

The Classification of Covering Spaces

Let X be a connected manifold.

Recall

A covering space of X is a space $\tilde{X}$ together with a map $p : \tilde{X} \to X$ with the following property: for each point $x \in X$ there exists an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U.

From now we only consider the connected covering spaces of X.

Suppose $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map.

Suppose $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a covering map.
Then p induces a group homomorphism:

$$p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$$

by setting $p_*([\tilde{u}]) := [p \circ \tilde{u}]$.

Suppose $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a covering map.
Then p induces a group homomorphism:

$$p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$$

by setting $p_*([\tilde{u}]) := [p \circ \tilde{u}]$.

Theorem

- $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ *is injective*

Suppose $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a covering map.
Then p induces a group homomorphism:

$$p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$$

by setting $p_*([\tilde{u}]) := [p \circ \tilde{u}]$.

Theorem

- $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ *is injective*
- *The subgroup* $p_*(\pi_1(\tilde{X}, \tilde{x_0})) < \pi_1(X, x_0)$ *consists of the homotopy classes of loops in* X *based at* x_0 *whose lifts to* $\tilde{X}$ *starting at* $\tilde{x_0}$ *are loops.*

Suppose $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a covering map.
Then p induces a group homomorphism:

$$p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$$

by setting $p_*([\tilde{u}]) := [p \circ \tilde{u}]$.

Theorem

- $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ *is injective*
- *The subgroup* $p_*(\pi_1(\tilde{X}, \tilde{x_0})) < \pi_1(X, x_0)$ *consists of the homotopy classes of loops in* X *based at* x_0 *whose lifts to* $\tilde{X}$ *starting at* $\tilde{x_0}$ *are loops.*
- *The number of sheets of the covering* p *equals the index of* $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ *in* $\pi_1(X, x_0)$.

The Classification of Covering Spaces

Lifting Criterion

Theorem

Given a covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ *and a continuous map* $f : (Y, y_0) \to (X, x_0)$ *with* Y *path-connected and locally path-connected. Then a lift* $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ *of* f *exists iff*

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x_0}))$$

.

The Classification of Covering Spaces

The Classification of Covering Spaces

The main thrust of the classification will be the Galois correspondence between connected covering spaces of X and subgroups of $\pi_1(X)$.

The Classification of Covering Spaces

The Classification of Covering Spaces

The main thrust of the classification will be the Galois correspondence between connected covering spaces of X and subgroups of $\pi_1(X)$.
We assign to each covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.

The Classification of Covering Spaces

The Classification of Covering Spaces

The main thrust of the classification will be the Galois correspondence between connected covering spaces of X and subgroups of $\pi_1(X)$.
We assign to each covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.

Question

- Is every subgroup of $\pi_1(X, x_0)$ realized as $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ for some covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$?

The Classification of Covering Spaces

The Classification of Covering Spaces

The main thrust of the classification will be the Galois correspondence between connected covering spaces of X and subgroups of $\pi_1(X)$. We assign to each covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.

Question

- Is every subgroup of $\pi_1(X, x_0)$ realized as $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ for some covering space $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$?
- In particular, whether the trivial subgroup is realized or not? Or equivalently, whether X has a simply-connected covering space or not?

Theorem

Suppose X is a connected manifold. Then X has a connected, simply-connected covering space.

Theorem

Suppose X is a connected manifold. Then X has a connected, simply-connected covering space.

Sketch of the proof

Let $\tilde{X} := \{[\gamma] : \gamma$ is a path in X starting at x_0 $\}$.

Theorem

Suppose X is a connected manifold. Then X has a connected, simply-connected covering space.

Sketch of the proof

Let $\tilde{X} := \{[\gamma] : \gamma$ is a path in X starting at x_0 $\}$.
Define $p : \tilde{X} \to X$ sending $[\gamma]$ to $\gamma(1)$.

Theorem

Suppose X is a connected manifold. Then X has a connected, simply-connected covering space.

Sketch of the proof

Let $\tilde{X} := \{[\gamma] : \gamma$ is a path in X starting at x_0 $\}$.
Define $p : \tilde{X} \to X$ sending $[\gamma]$ to $\gamma(1)$.
Given an open connected, simply-connected subset $U \subset X$ and a path γ from x_0 to a point $x \in U$, let:
$U_{[\gamma]} := \{[\gamma \cdot \eta] : \eta$ is a path in U with $\eta(0) = \gamma(1)\}$

Theorem

Suppose X is a connected manifold. Then X has a connected, simply-connected covering space.

Sketch of the proof

Let $\tilde{X} := \{[\gamma] : \gamma$ is a path in X starting at x_0 $\}$.
Define $p : \tilde{X} \to X$ sending $[\gamma]$ to $\gamma(1)$.
Given an open connected, simply-connected subset $U \subset X$ and a path γ from x_0 to a point $x \in U$, let:
$U_{[\gamma]} := \{[\gamma \cdot \eta] : \eta$ is a path in U with $\eta(0) = \gamma(1)\}$
The sets $U_{[\gamma]}$ form a basis for a topology on $\tilde{X}$, and make $p|_{U_{[\gamma]} \to U}$ a homeomorphism.

Theorem

Suppose X is a connected manifold. Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \tilde{x_0})) = H$ for a suitably chosen basepoint $\tilde{x_0} \in X_H$.

Theorem

Suppose X is a connected manifold. Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \tilde{x_0})) = H$ for a suitably chosen basepoint $\tilde{x_0} \in X_H$.

Sketch of the proof

In the simply-connected covering space $\tilde{X}$ constructed as above, define:

Theorem

Suppose X is a connected manifold. Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \tilde{x_0})) = H$ for a suitably chosen basepoint $\tilde{x_0} \in X_H$.

Sketch of the proof

In the simply-connected covering space $\tilde{X}$ constructed as above, define: $[\gamma] \sim [\gamma'] \Leftrightarrow \gamma(1) = \gamma'(1)$ and $[\gamma \cdot \gamma'^{-}] \in H$.

Theorem

Suppose X is a connected manifold. Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \tilde{x_0})) = H$ for a suitably chosen basepoint $\tilde{x_0} \in X_H$.

Sketch of the proof

In the simply-connected covering space $\tilde{X}$ constructed as above, define:
$[\gamma] \sim [\gamma'] \Leftrightarrow \gamma(1) = \gamma'(1)$ and $[\gamma \cdot \gamma'^{-}] \in H$.
Let $X_H := X/\sim$. We can check that the natural projection $X_H \to X$ induced by $[\gamma] \mapsto \gamma(1)$ makes X_H to be a covering space of X.

Theorem

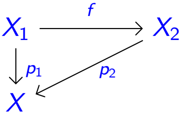
Suppose X is a connected manifold. Then for every subgroup $H \subset \pi_1(X, x_0)$ there is a covering space $p : X_H \to X$ such that $p_*(\pi_1(X_H, \tilde{x_0})) = H$ for a suitably chosen basepoint $\tilde{x_0} \in X_H$.

Sketch of the proof

In the simply-connected covering space $\tilde{X}$ constructed as above, define:
$[\gamma] \sim [\gamma'] \Leftrightarrow \gamma(1) = \gamma'(1)$ and $[\gamma \cdot \gamma'^{-}] \in H$.
Let $X_H := X/\sim$. We can check that the natural projection $X_H \to X$ induced by $[\gamma] \mapsto \gamma(1)$ makes X_H to be a covering space of X.
If we choose for $\tilde{x_0}$ the equivalence classe of the constant path c at x_0, then the image $p_*(\pi_1(X_H, \tilde{x_0}))$ is exactly H.

Definition

An **isomorphism** between covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_1 = p_2 \circ f$.



Theorem

Two connected covering spaces $p_1 : (\tilde{X}_1, \tilde{x_1}) \to (X, x_0)$ *and* $p_2 : (\tilde{X}_2, \tilde{x_2}) \to (X, x_0)$ *are isomorphic if and only if*

$$p_{1*}(\pi_1(\tilde{X}_1, \tilde{x_1})) = p_{2*}(\pi_1(\tilde{X}_2, x_2))$$

.

Theorem

Let X be a connected manifold.
Then there is a bijection between the set of basepoint-preserving isomorphism classes of connected covering spaces $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ to the covering space $(\tilde{X}, \tilde{x_0})$.

The Classification of Covering Spaces

The Universal Covering

Suppose X is a connected manifold, $\tilde{X}$ is the simply-connected, connected covering space constructed as above.

The Classification of Covering Spaces

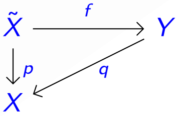
The Universal Covering

Suppose X is a connected manifold, $\tilde{X}$ is the simply-connected, connected covering space constructed as above.
Let $q : Y \to X$ be any connected covering space.

The Classification of Covering Spaces

The Universal Covering

Suppose X is a connected manifold, $\tilde{X}$ is the simply-connected, connected covering space constructed as above.
Let $q : Y \to X$ be any connected covering space.



The Classification of Covering Spaces

The Universal Covering

Suppose X is a connected manifold, $\tilde{X}$ is the simply-connected, connected covering space constructed as above.
Let $q : Y \to X$ be any connected covering space.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & Y \\ \downarrow p & \swarrow q & \\ X & & \end{array}$$

Then $f : \tilde{X} \to Y$ is a covering space of Y.
$\tilde{X}$ is called a **universal covering space** of X, it is unique up to isomorphism.

Examples

- $\exp : \mathbb{C} \to \mathbb{C}^*$.
- $\exp : \mathbb{H} \to \mathbb{D}^*$.
- $f : \mathbb{D} \to \mathbb{D}$ given by $f(z) = z^k, k \in \mathbb{N}^*$.
- $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ for $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ is a lattice in $\mathbb{C}$.

Deck Transformations and Group Action

Deck Transformations

Let $p : \tilde{X} \to X$ be a covering space.

Deck Transformations and Group Action

Deck Transformations

Let $p : \tilde{X} \to X$ be a covering space.

Definition

- A deck transformation is an isomorphism $f : \tilde{X} \to \tilde{X}$.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ f\ } & \tilde{X} \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle p} & \\ X & & \end{array}$$

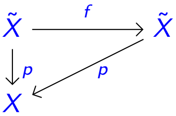
Deck Transformations and Group Action

Deck Transformations

Let $p : \tilde{X} \to X$ be a covering space.

Definition

- A deck transformation is an isomorphism $f : \tilde{X} \to \tilde{X}$.



- These deck transformations form a group $G(\tilde{X})$ under composition.

Remark

By the uniqueness of lifting, a deck transformation is completely determined by where it sends a single point. In particular, only the identity deck transformation can fix a point of $\tilde{X}$.

Remark

By the uniqueness of lifting, a deck transformation is completely determined by where it sends a single point. In particular, only the identity deck transformation can fix a point of $\tilde{X}$.

Definition

A covering $p : \tilde{X} \to X$ is called **normal** (or **Galois**) if for each $x \in X$ and each pair $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

Theorem

Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ *be a connected covering space of a connected manifold* X*, and let* H *be the subgroup* $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$*. Then:*

Theorem

Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ *be a connected covering space of a connected manifold* X*, and let* H *be the subgroup* $p_*(\pi_1(\tilde{X}, \tilde{x_0})) \subset \pi_1(X, x_0)$*. Then:*

- *This covering is normal iff* H *is a normal subgroup of* $\pi_1(X, x_0)$*.*

Theorem

Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a connected covering space of a connected manifold X, and let H be the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0})) \subset \pi_1(X, x_0)$. Then:

- *This covering is normal iff H is a normal subgroup of $\pi_1(X, x_0)$.*
- *$G(\tilde{X}) \simeq N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.*

Theorem

Let $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ be a connected covering space of a connected manifold X, and let H be the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0})) \subset \pi_1(X, x_0)$. Then:

- *This covering is normal iff H is a normal subgroup of $\pi_1(X, x_0)$.*
- *$G(\tilde{X}) \simeq N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.*

Consequence

If $\tilde{X}$ is a normal covering then $G(\tilde{X}) \simeq \pi_1(X, x_0)/H$. Hence for the universal cover $\tilde{X} \to X$ we have $G(\tilde{X}) \simeq \pi_1(X)$.

Examples

- $\exp : \mathbb{C} \to \mathbb{C}^*$.
  For $n \in \mathbb{Z}$ let $\tau_n : \mathbb{C} \to \mathbb{C}$ be the translation by $2\pi in$.

Examples

- $\exp : \mathbb{C} \to \mathbb{C}^*$.
  For $n \in \mathbb{Z}$ let $\tau_n : \mathbb{C} \to \mathbb{C}$ be the translation by $2\pi in$.
  Then $\exp(\tau_n(z)) = \exp(z)$ for every $z \in \mathbb{C}$ and thus τ_n is a deck transformation.

Examples

- $\exp : \mathbb{C} \to \mathbb{C}^*$.
  For $n \in \mathbb{Z}$ let $\tau_n : \mathbb{C} \to \mathbb{C}$ be the translation by $2\pi in$.
  Then $\exp(\tau_n(z)) = \exp(z)$ for every $z \in \mathbb{C}$ and thus τ_n is a deck transformation.
  Conversely, if σ is any deck transformation then by definition $\exp(\sigma(0)) = \exp(0) = 1$ and thus there exists $n \in \mathbb{Z}$ such that $\sigma(0) = 2\pi in$. Since $\tau_n(0) = 2\pi in$ as well, $\sigma = \tau_n$.

Examples

- $\exp : \mathbb{C} \to \mathbb{C}^*$.
  For $n \in \mathbb{Z}$ let $\tau_n : \mathbb{C} \to \mathbb{C}$ be the translation by $2\pi i n$.
  Then $\exp(\tau_n(z)) = \exp(z)$ for every $z \in \mathbb{C}$ and thus τ_n is a deck transformation.
  Conversely, if σ is any deck transformation then by definition $\exp(\sigma(0)) = \exp(0) = 1$ and thus there exists $n \in \mathbb{Z}$ such that $\sigma(0) = 2\pi i n$. Since $\tau_n(0) = 2\pi i n$ as well, $\sigma = \tau_n$. Thus

$$\pi_1(\mathbb{C}^*) \simeq G(\mathbb{C}) = \{\tau_n : n \in \mathbb{Z}\} \simeq \mathbb{Z}$$

Examples

- $\exp : \mathbb{C} \to \mathbb{C}^*$.
  For $n \in \mathbb{Z}$ let $\tau_n : \mathbb{C} \to \mathbb{C}$ be the translation by $2\pi in$.
  Then $\exp(\tau_n(z)) = \exp(z)$ for every $z \in \mathbb{C}$ and thus τ_n is a deck transformation.
  Conversely, if σ is any deck transformation then by definition $\exp(\sigma(0)) = \exp(0) = 1$ and thus there exists $n \in \mathbb{Z}$ such that $\sigma(0) = 2\pi in$. Since $\tau_n(0) = 2\pi in$ as well, $\sigma = \tau_n$. Thus
  $$\pi_1(\mathbb{C}^*) \simeq G(\mathbb{C}) = \{\tau_n : n \in \mathbb{Z}\} \simeq \mathbb{Z}$$
- $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ is a lattice in $\mathbb{C}$.

Examples

- $\exp : \mathbb{C} \to \mathbb{C}^*$.
  For $n \in \mathbb{Z}$ let $\tau_n : \mathbb{C} \to \mathbb{C}$ be the translation by $2\pi in$.
  Then $\exp(\tau_n(z)) = \exp(z)$ for every $z \in \mathbb{C}$ and thus τ_n is a deck transformation.
  Conversely, if σ is any deck transformation then by definition $\exp(\sigma(0)) = \exp(0) = 1$ and thus there exists $n \in \mathbb{Z}$ such that $\sigma(0) = 2\pi in$. Since $\tau_n(0) = 2\pi in$ as well, $\sigma = \tau_n$. Thus

$$\pi_1(\mathbb{C}^*) \simeq G(\mathbb{C}) = \{\tau_n : n \in \mathbb{Z}\} \simeq \mathbb{Z}$$

- $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ is a lattice in $\mathbb{C}$. For $\gamma \in \Gamma$ denote by $\tau_\gamma : \mathbb{C} \to \mathbb{C}$ the translation by γ. Analogous to the first example, one has:

$$\pi_1(\mathbb{C}/\Gamma) \simeq G(\mathbb{C}) = \{\tau_\gamma : \gamma \in \Gamma\} \simeq \mathbb{Z} \times \mathbb{Z}$$

Deck Transformations and Group Action

Group Actions

Given a group G and a space Y.

Deck Transformations and Group Action

Group Actions

Given a group G and a space Y.

Definition

An **action** of G on Y is a homomorphism $\rho : G \to Hom(Y)$ where $Hom(Y)$ is the group of all homeomorphisms from Y to itself.

Deck Transformations and Group Action

Group Actions

Given a group G and a space Y.

Definition

An **action** of G on Y is a homomorphism $\rho : G \to Hom(Y)$ where $Hom(Y)$ is the group of all homeomorphisms from Y to itself.

Remark

Each $g \in G$ is associated to a homeomorphism $\rho(g) : Y \to Y$, for simplicity we just write as $g : Y \to Y$.

We consider the following condition:

We consider the following condition:

"Each $y \in Y$ has a neighborhood U such that all the images $g(U), g \in G$ are disjoint, i.e, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$." (*)

We consider the following condition:

"Each $y \in Y$ has a neighborhood U such that all the images $g(U), g \in G$ are disjoint, i.e, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$." (*)

Propostion

The action of the deck transformation group $G(\tilde{X})$ on $\tilde{X}$ satisfies (*).

We consider the following condition:

"Each $y \in Y$ has a neighborhood U such that all the images $g(U), g \in G$ are disjoint, i.e, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$." (*)

Propostion

The action of the deck transformation group $G(\tilde{X})$ on $\tilde{X}$ satisfies (*).

To see this, let U be an evenly covered neighborhood of X and $\tilde{U}$ is a sheet of U.

We consider the following condition:

"Each $y \in Y$ has a neighborhood U such that all the images $g(U), g \in G$ are disjoint, i.e, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$." (*)

Propostion

The action of the deck transformation group $G(\tilde{X})$ on $\tilde{X}$ satisfies (*).

To see this, let U be an evenly covered neighborhood of X and $\tilde{U}$ is a sheet of U.
If $g_1(\tilde{U}) \cap g_2(\tilde{U}) \neq \varnothing$ for some $g_1, g_2 \in G(\tilde{X})$, then $g_1(\tilde{x_1}) = g_2(\tilde{x_2})$ for some $\tilde{x_1}, \tilde{x_2} \in \tilde{U}$.

We consider the following condition:

"Each $y \in Y$ has a neighborhood U such that all the images $g(U), g \in G$ are disjoint, i.e, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$." (*)

Propostion

The action of the deck transformation group $G(\tilde{X})$ on $\tilde{X}$ satisfies (*).

To see this, let U be an evenly covered neighborhood of X and $\tilde{U}$ is a sheet of U.
If $g_1(\tilde{U}) \cap g_2(\tilde{U}) \neq \varnothing$ for some $g_1, g_2 \in G(\tilde{X})$, then $g_1(\tilde{x_1}) = g_2(\tilde{x_2})$ for some $\tilde{x_1}, \tilde{x_2} \in \tilde{U}$. But it means that $\tilde{x_1}, \tilde{x_2}$ must lie in the same fiber $p^{-1}(x)$, which intersects $\tilde{U}$ in only one point. Hence $\tilde{x_1} = \tilde{x_2}$ and then $g_1^{-1} g_2$ fixes this point, so $g_1 = g_2$.

We consider the following condition:

"Each $y \in Y$ has a neighborhood U such that all the images $g(U), g \in G$ are disjoint, i.e, $g_1(U) \cap g_2(U) \neq \varnothing$ implies $g_1 = g_2$." (*)

Propostion

The action of the deck transformation group $G(\tilde{X})$ on $\tilde{X}$ satisfies (*).

To see this, let U be an evenly covered neighborhood of X and $\tilde{U}$ is a sheet of U.
If $g_1(\tilde{U}) \cap g_2(\tilde{U}) \neq \varnothing$ for some $g_1, g_2 \in G(\tilde{X})$, then $g_1(\tilde{x_1}) = g_2(\tilde{x_2})$ for some $\tilde{x_1}, \tilde{x_2} \in \tilde{U}$. But it means that $\tilde{x_1}, \tilde{x_2}$ must lie in the same fiber $p^{-1}(x)$, which intersects $\tilde{U}$ in only one point. Hence $\tilde{x_1} = \tilde{x_2}$ and then $g_1^{-1} g_2$ fixes this point, so $g_1 = g_2$.

Remark

(*) is equivalent to "$U \cap g(U) \neq \varnothing \Leftrightarrow g = id$".

Given an action of a group G on a space Y, we can form a space Y/G, the quotient space of Y in which each point y is identified with all its images $g(y)$ as g ranges over G.

Given an action of a group G on a space Y, we can form a space Y/G, the quotient space of Y in which each point y is identified with all its images $g(y)$ as g ranges over G.
The points of Y/G are the **orbits** $Gy := \{g(y) : g \in G\}$, and Y/G is called the **orbit space** of the action.

Given an action of a group G on a space Y, we can form a space Y/G, the quotient space of Y in which each point y is identified with all its images $g(y)$ as g ranges over G.
The points of Y/G are the **orbits** $Gy := \{g(y) : g \in G\}$, and Y/G is called the **orbit space** of the action.
For example, for a normal covering space $\tilde{X} \to X$, the orbit space $\tilde{X}/G(\tilde{X})$ is just X.

Theorem

If an action of a group G on a space Y satisfies (*), then:

Theorem

If an action of a group G on a space Y satisfies (*), then:

- *The quotient map $p : Y \to Y/G, p(y) = Gy$, is a normal covering space.*

Theorem

If an action of a group G on a space Y satisfies (*), then:

- *The quotient map $p : Y \to Y/G, p(y) = Gy$, is a normal covering space.*
- *If Y is path-connected then G is the group of deck transformations of this covering space $Y \to Y/G$.*

Theorem

If an action of a group G on a space Y satisfies (*), then:

- *The quotient map $p : Y \to Y/G, p(y) = Gy$, is a normal covering space.*
- *If Y is path-connected then G is the group of deck transformations of this covering space $Y \to Y/G$.*
- *If Y is path-connected and locally path-connected then $G \simeq \pi_1(Y/G)/p_*(\pi_1(Y))$.*

Definition

- An action is called **properly discontinuous** if: every point $x \in X$ has a neighborhood U such that $U \cap g(U) \neq \varnothing$ for only finitely many g.

Definition

- An action is called **properly discontinuous** if: every point $x \in X$ has a neighborhood U such that $U \cap g(U) \neq \varnothing$ for only finitely many g.
- An action without fixed points is called a **free** action.

Definition

- An action is called **properly discontinuous** if: every point $x \in X$ has a neighborhood U such that $U \cap g(U) \neq \varnothing$ for only finitely many g.
- An action without fixed points is called a **free** action.

Remark

For actions on Hausdorff spaces, freeness plus properly discontinuity implies condition (*).

Theorem

Uniformization Any simply-connected Riemann surface is biholomorphic either to:

- *the unit disc* $\mathbb{D}$
- *the complex plane* $\mathbb{C}$
- *the Riemann sphere* $\mathbb{P}^1$

Theorem

Uniformization Any simply-connected Riemann surface is biholomorphic either to:

- *the unit disc* $\mathbb{D}$
- *the complex plane* $\mathbb{C}$
- *the Riemann sphere* $\mathbb{P}^1$

Consequence

Every Riemann surface is the quotient of one of the above by a discrete group of holomorphic maps acting freely and properly discontinuous.

Theorem

- *If a Riemann surface is covered by* $\mathbb{P}^1$ *then it is biholomorphic to* $\mathbb{P}^1$.

Theorem

- *If a Riemann surface is covered by* $\mathbb{P}^1$ *then it is biholomorphic to* $\mathbb{P}^1$.
- *If it is covered by* $\mathbb{C}$, *it is biholomorphic to* $\mathbb{C}, \mathbb{C}^*$ *or a torus*

Theorem

- *If a Riemann surface is covered by* $\mathbb{P}^1$ *then it is biholomorphic to* $\mathbb{P}^1$*.*
- *If it is covered by* $\mathbb{C}$*, it is biholomorphic to* $\mathbb{C}, \mathbb{C}^*$ *or a torus*
- *The other Riemann surfaces are covered by the unit disc.*

Theorem

- *If a Riemann surface is covered by $\mathbb{P}^1$ then it is biholomorphic to $\mathbb{P}^1$.*
- *If it is covered by $\mathbb{C}$, it is biholomorphic to $\mathbb{C}, \mathbb{C}^*$ or a torus*
- *The other Riemann surfaces are covered by the unit disc.*

Proof.

Since $Aut(\mathbb{P}^1) = \{\frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0\}$, then every automorphism of $\mathbb{P}^1$ admits a fixed point. This gives us the first assertion.

Theorem

- *If a Riemann surface is covered by $\mathbb{P}^1$ then it is biholomorphic to $\mathbb{P}^1$.*
- *If it is covered by $\mathbb{C}$, it is biholomorphic to $\mathbb{C}, \mathbb{C}^*$ or a torus*
- *The other Riemann surfaces are covered by the unit disc.*

Proof.

Since $Aut(\mathbb{P}^1) = \{\frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0\}$, then every automorphism of $\mathbb{P}^1$ admits a fixed point. This gives us the first assertion. The subgroup of $Aut(\mathbb{C})$ acting properly discontinuous and freely on $\mathbb{C}$ are conjugated either to $\mathbb{Z}$ or $\mathbb{Z} + \tau\mathbb{Z}$, this gives us the second assertion.

Theorem

- *If a Riemann surface is covered by $\mathbb{P}^1$ then it is biholomorphic to $\mathbb{P}^1$.*
- *If it is covered by $\mathbb{C}$, it is biholomorphic to $\mathbb{C}, \mathbb{C}^*$ or a torus*
- *The other Riemann surfaces are covered by the unit disc.*

Proof.

Since $Aut(\mathbb{P}^1) = \{\frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0\}$, then every automorphism of $\mathbb{P}^1$ admits a fixed point. This gives us the first assertion. The subgroup of $Aut(\mathbb{C})$ acting properly discontinuous and freely on $\mathbb{C}$ are conjugated either to $\mathbb{Z}$ or $\mathbb{Z} + \tau\mathbb{Z}$, this gives us the second assertion. The last one is deduced from Riemann uniformization theorem. □

Consequence

There exists an unbranched holomorphic covering map $\phi : \mathbb{D} \to \mathbb{C} \setminus \{0, 1\}$.

Uniformization Theorem

Picard's Little Theorem

Theorem (Picard's Little Theorem)

A non-constant entire function f omits at most one complex value $a \in \mathbb{C}$.

Uniformization Theorem

Picard's Little Theorem

Theorem (Picard's Little Theorem)

A non-constant entire function f omits at most one complex value $a \in \mathbb{C}$.

Proof.

Suppose f omits two different complex values. Without loss of generality, we can assume that the omitted values are 0 and 1.

$$\begin{array}{ccc} & & \mathbb{D} \\ & \overset{F}{\nearrow} & \downarrow \phi \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \setminus \{0,1\} \end{array}$$

There exists a holomorphic lifting F of f, i.e, $f = \phi \circ F$. But F is constant (Liouville Theorem) $\Rightarrow$ contradiction. □

THANK YOU FOR LISTENING