Separated Morphisms

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Content

1. Motivation

Usual point of view	**Relative point of view**
X	$X \xrightarrow{f} S \quad S$:base
Properties of objects	Properties of morphisms
$X \xrightarrow{f} Y$ $S \subset X$ $X \times Y$	Stable under base change Local on the base Stable under composition

$$\begin{array}{ccc} X \times_S S' & \longrightarrow & X \\ \downarrow & \square & \downarrow \\ S' & \longrightarrow & S \end{array} \quad \text{base change}$$

$\bullet$: final object

$$\begin{array}{ccc} X \longrightarrow \bullet & \leftrightarrow & X \longrightarrow S \\ \text{"Constant"} & & \text{"Parameter"} \end{array}$$

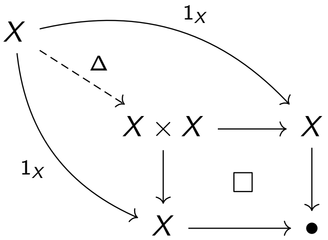
2. Separated morphism

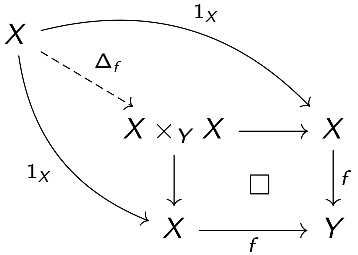
Diagonal map

$$\Delta\colon X \longrightarrow X \times X$$
$$x \longmapsto (x, x)$$

Theorem

X is a Hausdorff space if and only if $\Delta(X)$ is closed subset of $X \times X$





Δ_f is called **diagonal morphism**. (Simply Δ if there is no confusion possible).

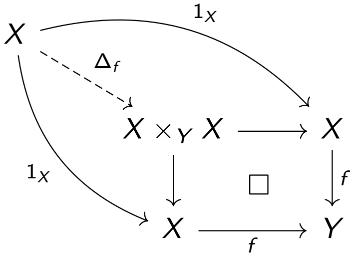
Definition (Separated morphism)

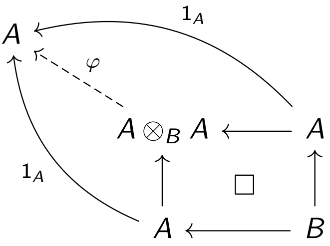
Let $f: X \to Y$ be a morphism of schemes. We say that the morphism f is separated if Δ is closed immersion. We also say X is separated over Y. A scheme X is separated if it is separated over $Spec(\mathbb{Z})$.

Proposition

If $f: X \to Y$ is any morphism of affine schemes then f is separated.

Proof: Let $X = Spec(A)$, $Y = Spec(B)$





$$\varphi \colon A \otimes_B A \longrightarrow A \qquad \text{surjective}$$

$$a_1 \otimes a_2 \longmapsto a_1 a_2$$

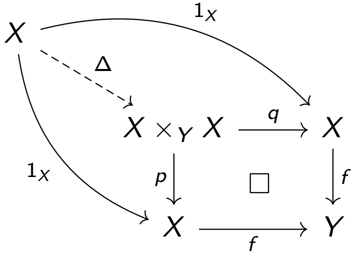
$$\Delta \colon X \to X \times_Y X \text{ closed immersion}$$
$$\Rightarrow f \colon X \to Y \text{ is separated.}$$

Corollary

An arbitrary morphism $f: X \to Y$ is separated if and only if $\Delta(X)$ is closed subset of $X \times_Y X$.

Proof: Suppose $X \xrightarrow{(\Delta,\Delta^\sharp)} X \times_Y X$

1. Topo



$\Rightarrow p|_{\Delta(X)}.\Delta = 1_X, \Delta.p|_{\Delta(X)} = 1_{X \times_Y X}.$

2. Sheaf $(\Delta^\sharp)_x \colon \mathcal{O}_{X\times_Y X,\Delta(x)} \to \mathcal{O}_{X,x}$ is surjective, $\forall x \in X$

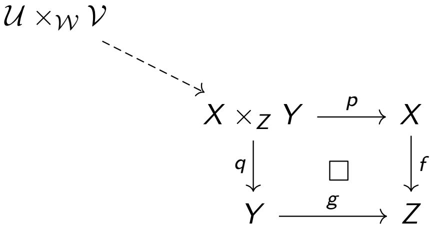
$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times_Y X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{U} & \xrightarrow{\Delta|_{\mathcal{U}}} & \mathcal{U}' \end{array}$$

$$\mathcal{O}_{\mathcal{U}',\Delta(x)} \longrightarrow \mathcal{O}_{\mathcal{U},x}$$

Lemma

Let $f : X \to Z, g : Y \to Z$ be morphisms of schemes. Let $X \times_Z Y, p, q$ be fiber product. Suppose $\mathcal{U} \subset X, \mathcal{V} \subset Y, \mathcal{W} \subset Z$ are open subschemes such that $f(\mathcal{U}) \subset \mathcal{W}, g(\mathcal{V}) \subset \mathcal{W}$. Then $\mathcal{U} \times_{\mathcal{W}} \mathcal{V} = p^{-1}(\mathcal{U}) \cap q^{-1}(\mathcal{V})$.

In particular, if $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are all affine then $\mathcal{U} \times_{\mathcal{W}} \mathcal{V}$ is affine open subschemes of $X \times_Z Y$.



Lemma

Let $f : X \to S, g : Y \to S$ *be morphisms of schemes. Let* $S = \bigcup_I \mathcal{W}_i$ *be any affine open covering of* S*. For each* $i \in I$*, let* $f^{-1}(\mathcal{W}_i) = \bigcup_J \mathcal{U}_j$ *be an affine open covering of* $f^{-1}(\mathcal{W}_i)$ *and let* $g^{-1}(\mathcal{W}_i) = \bigcup_K \mathcal{V}_k$*. Then*

$$X \times_S Y = \bigcup_I \bigcup_{J \times K} (\mathcal{U}_j \times_{\mathcal{W}_i} \mathcal{V}_k).$$

is an affine open covering of $X \times_S Y$

$(\Delta^\sharp)_x \colon \mathcal{O}_{X\times_Y X,\Delta(x)} \to \mathcal{O}_{X,x}$ is surjective, $\forall x \in X$

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times_Y X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{U} & \xrightarrow{\Delta|_{\mathcal{U}}} & \mathcal{U}' \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{U} & \xrightarrow{f|_{\mathcal{U}}} & \mathcal{V} \end{array}$$

$\forall x \in X$, Let $f(x) \in \mathcal{V}$ be an affine open subscheme of Y.
$\exists x \in \mathcal{U}$ be an affine open subscheme of X such that $f(\mathcal{U}) \subset \mathcal{V}$.
Choose $\mathcal{U}' = \mathcal{U} \times_{\mathcal{V}} \mathcal{U} = p^{-1}(\mathcal{U}) \cap q^{-1}(\mathcal{U})$ is affine open subscheme of $X \times_Y X$.
$\Rightarrow \Delta^{-1}(\mathcal{U} \times_{\mathcal{V}} \mathcal{U}) = \Delta^{-1}(p^{-1}(\mathcal{U}) \cap q^{-1}(\mathcal{U})) = \mathcal{U}$.

$$\mathcal{U} \xrightarrow{\Delta|_{\mathcal{U}}} \mathcal{U} \times_{\mathcal{V}} \mathcal{U}$$

Since $\mathcal{U}, \mathcal{V}$ are affine, $\mathcal{U} \xrightarrow{f|_{\mathcal{U}}} \mathcal{V}$ is separated morphism i.e $\Delta|_{\mathcal{U}}$ is closed immersion.
$\Rightarrow \ \mathcal{O}_{\mathcal{U}',\Delta(x)} \longrightarrow \mathcal{O}_{\mathcal{U},x}$ is surjective. $\square$

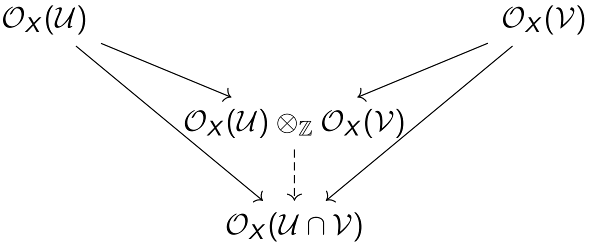
Definition (Separated morphism)

Let $f\colon X \to Y$ be a morphism of schemes. f is separated if Δ is closed immersion.

Corollary

An arbitrary morphism $f\colon X \to Y$ is separated if and only if $\Delta(X)$ is closed subset of $X \times_Y X$.

Let X be a scheme. If $\mathcal{U}$ and $\mathcal{V}$ are open subsets of X, we have a canonical homomorphism



Proposition

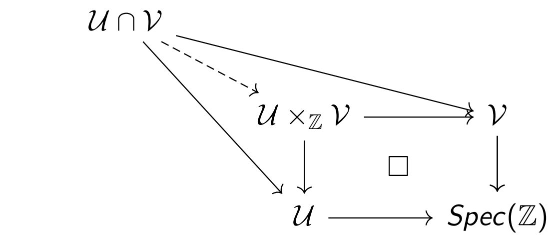
Let X be a scheme. Then the following properties are equivalent:

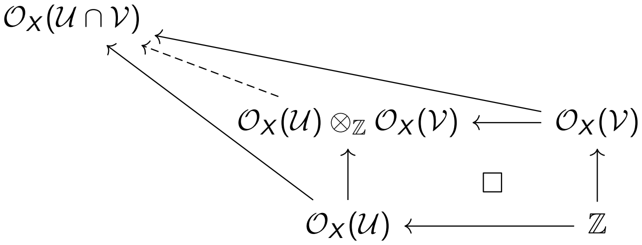
1. *X is separated;*
2. *for every pair of affine open subsets $\mathcal{U}, \mathcal{V}$ of X, $\mathcal{U} \cap \mathcal{V}$ is affine and the canonical homomorphism $\mathcal{O}_X(\mathcal{U}) \otimes_{\mathbb{Z}} \mathcal{O}_X(\mathcal{V}) \to \mathcal{O}_X(\mathcal{U} \cap \mathcal{V})$ is surjective;*
3. *there exists a covering of X by affine open subsets $\mathcal{U}_i$ such that for all i, j, $\mathcal{U}_i \cap \mathcal{U}_j$ is affine and $\mathcal{O}_X(\mathcal{U}) \otimes_{\mathbb{Z}} \mathcal{O}_X(\mathcal{V}) \to \mathcal{O}_X(\mathcal{U} \cap \mathcal{V})$ is surjective.*

Proof: (1) $\Rightarrow$ (2) Since X is separated, $\Delta \colon X \to X \times_{\mathbb{Z}} X$ is closed immersion.
We have $\mathcal{U} \times_{\mathbb{Z}} \mathcal{V} = p^{-1}(\mathcal{U}) \cap q^{-1}(\mathcal{V})$
$\Rightarrow \Delta^{-1}(\mathcal{U} \times_{\mathbb{Z}} \mathcal{V}) = \Delta^{-1}(p^{-1}(\mathcal{U}) \cap q^{-1}(\mathcal{V})) = \mathcal{U} \cap \mathcal{V}$.

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times_{\mathbb{Z}} X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{U} \cap \mathcal{V} & \xrightarrow{\Delta|_{\mathcal{U}\cap\mathcal{V}}} & \mathcal{U} \times_{\mathbb{Z}} \mathcal{V} \end{array}$$

As being a closed immersion is local on the base, $\Delta|_{\mathcal{U}\cap\mathcal{V}}$ is closed immersion. Thus $\mathcal{U}\cap\mathcal{V}$ affine.





$\Rightarrow \mathcal{O}_X(\mathcal{U})\otimes_{\mathbb{Z}}\mathcal{O}_X(\mathcal{V})\to\mathcal{O}_X(\mathcal{U}\cap\mathcal{V})$is surjective.

Proposition

Let X be a scheme. Then the following properties are equivalent:

1. *X is separated;*
2. *for every pair of affine open subsets $\mathcal{U}, \mathcal{V}$ of $X, \mathcal{U} \cap \mathcal{V}$ is affine and the canonical homomorphism $\mathcal{O}_X(\mathcal{U}) \otimes_{\mathbb{Z}} \mathcal{O}_X(\mathcal{V}) \to \mathcal{O}_X(\mathcal{U} \cap \mathcal{V})$ is surjective;*
3. *there exists a covering of X by affine open subsets $\mathcal{U}_i$ such that for all $i, j, \mathcal{U}_i \cap \mathcal{U}_j$ is affine and $\mathcal{O}_X(\mathcal{U}) \otimes_{\mathbb{Z}} \mathcal{O}_X(\mathcal{V}) \to \mathcal{O}_X(\mathcal{U} \cap \mathcal{V})$ is surjective.*

(2) $\Rightarrow$ (3) trivial
(3) $\Rightarrow$ (1) Need: $\Delta \colon X \to X \times_{\mathbb{Z}} X$ is closed immersion.

$$X \times_{\mathbb{Z}} X = \bigcup_{I \times I} (\mathcal{U}_i \times_{\mathbb{Z}} \mathcal{U}_j).$$

$\Rightarrow \Delta|_{\mathcal{U}_i \cap \mathcal{U}_j} \colon \mathcal{U}_i \cap \mathcal{U}_j \to \mathcal{U}_i \times_{\mathbb{Z}} \mathcal{U}_j$ is closed immersion.
As being a closed immersion is local on the base, $\Delta \colon X \to X \times_{\mathbb{Z}} X$ is closed immersion. $\square$

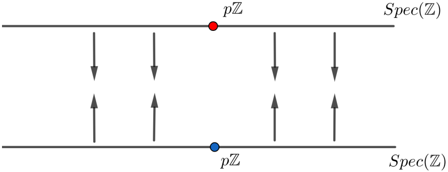
Example (line with a double point)

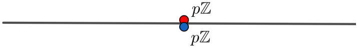
Let p be a prime. Let $X_1 = X_2 = Spec(\mathbb{Z})$ be two affine schemes and $X_{12} = D(p) \subset X_1, X_{21} = D(p) \subset X_2$ be two open subschemes. Then the identity on $X_{12} = X_{21}$ makes it possible to glue the two schemes X_1 and X_2 to a scheme X.

This scheme is not separated. Indeed, condition (ii) is not verified $\mathcal{U} = X_1, \mathcal{V} = X_2 \Rightarrow \mathcal{U} \cap \mathcal{V} = X_1 \cap X_2 = X_{12} = D(p)$ affine. However,

$$\mathcal{O}_X(X_1) \otimes_{\mathbb{Z}} \mathcal{O}_X(X_2) \to \mathcal{O}_X(X_1 \cap X_2) = \mathcal{O}_X(X_{12})$$

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}[1/p]$$





3. Some properties of separated morphism

Proposition

1. *Separated morphisms are local on the base.*
2. *Separated morphisms are stable under base change.*
3. *Separated morphisms are stable under composition.*
4. *Let $f : X \to Y, g : Y \to Z$ be two morphisms such that $g \circ f$ is separated then f is separated.*

Proof:

(1) Suppose $f : X \to Y$ is separated morphism and $Y = \bigcup_I \mathcal{V}_i$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ f^{-1}(\mathcal{V}_i) & \xrightarrow{f|_{f^{-1}(\mathcal{V}_i)}} & \mathcal{V}_i \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\Delta} & X \times_Y X \\ \downarrow & & \downarrow \\ f^{-1}(\mathcal{V}_i) & \xrightarrow{\Delta|_{f^{-1}(\mathcal{V}_i)}} & f^{-1}(\mathcal{V}_i) \times_{\mathcal{V}_i} f^{-1}(\mathcal{V}_i) \end{array}$$

$$X \times_Y X = \bigcup_I (f^{-1}(\mathcal{V}_i) \times_{\mathcal{V}_i} f^{-1}(\mathcal{V}_i))$$

Lemma (Pullback lemma)

Suppose right square is pullback. Left square is pullback if and only if outer rectangle is pullback.

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & \square & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

(2) Let $f : X \to Y$ be a separated morphism

$$\begin{array}{ccc} X' = X \times_Y Z & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ Z & \longrightarrow & Y \end{array}$$

Need: $\Delta_{f'} : X' \to X' \times_Z X'$ is closed immersion.

$$
\begin{array}{ccccc}
X' \times_Z X' & \xrightarrow{q'} & X' & \xrightarrow{h} & X \\
{\scriptstyle p'}\downarrow & \square & {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\
X' & \xrightarrow[f']{} & Z & \xrightarrow[g]{} & Y
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \overset{q' \circ h}{\frown} & & \\
X' \times_Z X' & \overset{\sigma}{\dashrightarrow} & X \times_Y X & \xrightarrow{q} & X \\
{\scriptstyle p'}\downarrow & & {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle f} \\
X' & \xrightarrow[h]{} & X & \xrightarrow[f]{} & Y
\end{array}
$$

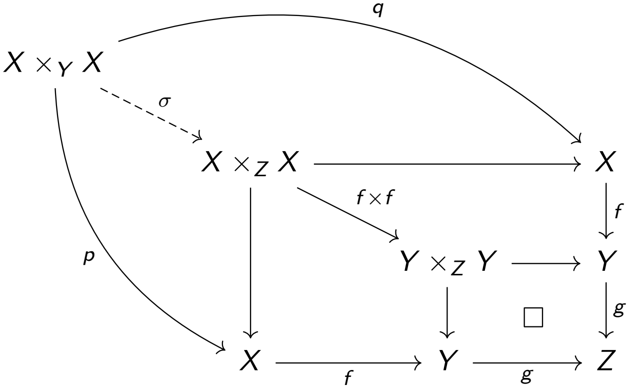
$$
\begin{array}{ccccc}
X' & \xrightarrow{\Delta_{f'}} & X' \times_Z X' & \xrightarrow{p'} & X' \\
{\scriptstyle h}\downarrow & & \vdots{\scriptstyle \sigma} & \square & \downarrow{\scriptstyle h} \\
X & \xrightarrow{\Delta_f} & X \times_Y X & \xrightarrow{p} & X
\end{array}
$$

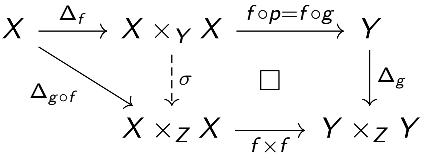
As being a closed immersion is stable under base change, $\Delta_{f'} \colon X' \to X' \times_Z X'$ is closed immersion i.e $f' \colon X \times_Y Z \to Z$ is separated morphism.

Proposition

1. *Separated morphisms are local on the base.*
2. *Separated morphisms are stable under base change.*
3. *Separated morphisms are stable under composition.*
4. *Let $f: X \to Y, g: Y \to Z$ be two morphisms such that $g \circ f$ is separated then f is separated.*

(3) Let $f: X \to Y$ and $g: Y \to Z$ be two separated morphisms of schemes.
Need: $g \circ f: X \to Z$ is separated morphism
i.e $\Delta_{g\circ f}: X \to X \times_Z X$ is closed immersion.





$\Rightarrow g \circ f : X \to Z$ is separated morphism.

Proposition

1. *Separated morphisms are local on the base.*
2. *Separated morphisms are stable under base change.*
3. *Separated morphisms are stable under composition.*
4. *Let $f : X \to Y, g : Y \to Z$ be two morphisms such that $g \circ f$ is separated then f is separated.*

(4)

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Corollary

An arbitrary morphism $f : X \to Y$ is separated if and only if $\Delta(X)$ is closed subset of $X \times_Y X$.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_f} & X \times_Y X & \xrightarrow{f \circ p = f \circ g} & Y \\ & \searrow^{\Delta_{g \circ f}} & \downarrow \sigma & \square & \downarrow \Delta_g \\ & & X \times_Z X & \xrightarrow[f \times f]{} & Y \times_Z Y \end{array}$$

Need: $\Delta_f(X)$ is closed subset $X \times_Y X$.

$$\Delta_f(X) = \sigma^{-1}(\Delta_{g \circ f}(X))$$

- $\Delta_f(X) \subseteq \sigma^{-1}(\Delta_{g \circ f}(X))$

$$\sigma \circ \Delta_f(X) \subseteq \Delta_{g \circ f}(X)$$
$$\Rightarrow \Delta_f(X) \subseteq \sigma^{-1}(\Delta_{g \circ f}(X))$$

$$\begin{array}{ccccc} x \in X & \xrightarrow{\Delta_f} & a, b \in X \times_Y X & \xrightarrow{f \circ p = f \circ g} & Y \\ & \searrow^{\Delta_{g\circ f}} & \downarrow \sigma & & \downarrow \Delta_g \\ & & \sigma(a) = \sigma(b) \in X \times_Z X & \xrightarrow{f \times f} & Y \times_Z Y \end{array}$$

- $\Delta_f(X) \supseteq \sigma^{-1}(\Delta_{g\circ f}(X))$
  Suppose $a \in \sigma^{-1}(\Delta_{g\circ f}(X))$.
  $\Rightarrow \sigma(a) \in \Delta_{g\circ f}(X). \exists x \in X : \sigma(a) = \Delta_{g\circ f}(x) = \sigma(\Delta_f(x))$
  Let $b = \Delta_f(x)$. We have $\sigma(a) = \sigma(b)$
  Need: $a = b = \Delta_f(x)$
  Take affine open subsets $\mathcal{U} \subseteq X, \mathcal{V} \subseteq Y, \mathcal{W} \subseteq Z$ containing $x, f(x), g(f(x))$, repectively, such that $\mathcal{U} \subseteq f^{-1}(\mathcal{V})$ and $\mathcal{V} \subseteq g^{-1}(\mathcal{W})$.

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ i \uparrow & & \uparrow i & & i \uparrow \\ \mathcal{U} & \xrightarrow{f|_{\mathcal{U}}} & \mathcal{V} & \xrightarrow{g|_{\mathcal{V}}} & \mathcal{W} \end{array}$$

$$\begin{array}{ccccc} x \in X & \xrightarrow{\Delta_f} & a, b \in X \times_Y X & \xrightarrow{f\circ p = f \circ g} & Y \\ & \searrow^{\Delta_{g\circ f}} & \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle \Delta_g} \\ & & \sigma(a) = \sigma(b) \in X \times_Z X & \xrightarrow{f \times f} & Y \times_Z Y \end{array}$$

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle i}\Big\uparrow & & \Big\uparrow{\scriptstyle i} & & {\scriptstyle i}\Big\uparrow \\ \mathcal{U} & \xrightarrow{f|_{\mathcal{U}}} & \mathcal{V} & \xrightarrow{g|_{\mathcal{V}}} & \mathcal{W} \end{array}$$

$$\begin{array}{ccccc} x \in \mathcal{U} & \xrightarrow{\Delta_f|_{\mathcal{U}}} & a, b \in \mathcal{U} \times_{\mathcal{V}} \mathcal{U} & \longrightarrow & \mathcal{V} \\ & \searrow^{\Delta_{g\circ f}|_{\mathcal{U}}} & \Big\downarrow{\scriptstyle \sigma|_{\mathcal{U}\times_{\mathcal{V}}\mathcal{U}}} & & \Big\downarrow{\scriptstyle \Delta_g|_{\mathcal{V}}} \\ & & \sigma(a) = \sigma(b) \in \mathcal{U} \times_{\mathcal{W}} \mathcal{U} & \longrightarrow & \mathcal{V} \times_{\mathcal{W}} \mathcal{V} \end{array}$$

$$\sigma^{-1}(\mathcal{U} \times_{\mathcal{W}} \mathcal{U}) = \mathcal{U} \times_{\mathcal{V}} \mathcal{U}$$

As being a closed immersion is stable under base change,
$\sigma|_{\mathcal{U}\times_{\mathcal{V}}\mathcal{U}} : \mathcal{U} \times_{\mathcal{V}} \mathcal{U} \to \mathcal{U} \times_{\mathcal{W}} \mathcal{U}$ is closed immersion.
$\Rightarrow a = b \in \Delta_f(X)$ and $\Delta_f(X) = \sigma^{-1}(\Delta_{g\circ f}(X))$.
Therefore, $f : X \to Y$ is separated morphism.

4. Quasi-separated morphism

Proposition

A morphism $f : X \to Y$ of schemes is called quasi-compact, if the following equivalent conditions are satisfied:

1. *For every quasi-compact open subset $\mathcal{V} \subset Y$ the pre-image $f^{-1}(\mathcal{V})$ is quasi-compact.*
2. *There exists a covering $Y = \bigcup_I \mathcal{V}_i$ by affine open subsets $\mathcal{V}_i$ such that for every i, the inverse image $f^{-1}(\mathcal{V}_i)$ is quasi-compact.*

Proof:

$(1) \Rightarrow (2)$ trivial

$(2) \Rightarrow (1)$

- Finite union of quasi-compact sets is quasi-compact.
- Let A be a ring. Then $D(f)$ is quasi-compact $, \forall f \in A$.

Definition

A morphism $f : X \to Y$ is called quasi-separated, if the diagonal morphism $\Delta : X \to X \times_Y X$ is quasi-compact.

$\Rightarrow$ every separated morphism is quasi-separated since closed immersions are quasi-compact.

Proposition

Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

1. *The morphism f is quasi-separated.*
2. *For every affine open $\mathcal{V} \subset Y$, and all affine open subsets $\mathcal{U}_1, \mathcal{U}_2 \subset f^{-1}(\mathcal{V})$, the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is quasi-compact.*

Proof: $(1) \Rightarrow (2)$

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times_Y X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{U}_1 \cap \mathcal{U}_2 & \xrightarrow{\Delta|_{\mathcal{U}_1 \cap \mathcal{U}_2}} & \mathcal{U}_1 \times_{\mathcal{V}} \mathcal{U}_2 \end{array}$$

Proposition

Let $f : X \to Y$ be a morphism of schemes. The following are equivalent:

1. *The morphism f is quasi-separated.*
2. *For every affine open $\mathcal{V} \subset Y$, and all affine open subsets $\mathcal{U}_1, \mathcal{U}_2 \subset f^{-1}(\mathcal{V})$, the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ is quasi-compact.*

(2) $\Rightarrow$ (1) Need: $\Delta : X \to X \times_Y X$ is quasi-compact.
Let $Y = \bigcup_I \mathcal{V}_i$ be any affine open covering of S. For each $i \in I$, let $f^{-1}(\mathcal{V}_i) = \bigcup_J \mathcal{U}_j$ be an affine open covering of $f^{-1}(\mathcal{V}_i)$. Then

$$X \times_Y X = \bigcup_I \bigcup_{J \times J} (\mathcal{U}_i \times_{\mathcal{V}_i} \mathcal{U}_j).$$

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times_Y X \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{U}_j \cap \mathcal{U}_k & \xrightarrow{\Delta|_{\mathcal{U}_j \cap \mathcal{U}_k}} & \mathcal{U}_j \times_{\mathcal{V}_i} \mathcal{U}_k \end{array}$$

Corollary

Let X be a (locally) noetherian scheme. Then all morphisms $f : X \to Y$ are quasi-separated.

Proof: *Fact :* Any open subset of the spectrum of a Noetherian ring is quasi-compact.

Indeed, Let $\mathcal{U}$ be an open subset of $X = Spec(R)$, where R is Noetherian.

$$\exists I \in Id(R) : \mathcal{U} = X - V(I)$$

As R is Noetherian, $\exists f_1, \ldots, f_n \in R : I = (f_1, \ldots, f_n)$

$$\mathcal{U} = X - V(f_1, ..., f_n) = X - \bigcup_{i=1}^{n} V(f_i) = \bigcup_{1}^{n} D(f_i) \text{ quasi - compact.}$$

As $\mathcal{U}_1, \mathcal{U}_2$ are affine, $\exists R_1, \ldots, R_m : \mathcal{W} = \mathcal{U}_1 \cap \mathcal{U}_2 \subseteq \bigcup_{i=1}^{m} Spec(R_i)$

$$\mathcal{W} = \bigcup_{i=1}^{m} (\mathcal{W} \cap Spec(R_i)) \text{ quasi - compact.}$$

5. Valuative criterion of separatedness

Denote $\Delta = \{z \in \mathbb{C} |\, |z| < 1\}$ adn $\Delta^* = \{z \in \mathbb{C} - 0 |\, |z| < 1\}$
Suppose X is Hasdorff.

$$\begin{array}{ccc} \Delta^* & \xrightarrow{u} & X \\ {\scriptstyle i}\downarrow & \overset{\overline{u}}{\nearrow} & \\ \Delta & & \end{array}$$

$$\begin{array}{ccc} \xi = \Delta^* & \xrightarrow{u} & X \\ {\scriptstyle i}\downarrow & \overset{\overline{u}}{\nearrow} & \downarrow \\ Spec\ \mathbb{C}((x)) = \Delta & \longrightarrow & \bullet \end{array}$$

$$\begin{array}{ccc} Spec\ K & \xrightarrow{u} & X \\ {\scriptstyle i}\downarrow & \overset{\overline{u}}{\nearrow} & \downarrow{\scriptstyle f} \\ Spec\ A & \xrightarrow[v]{} & Y \end{array}$$

Theorem (Valuative criterion for separated - general version)

Let $f : X \to Y$ be a morphism of schemes. Then the following assertions are equivalent.

1. *f is separated.*
2. *f is quasi-separated and for all diagrams below where A is a valuation ring and $K = Frac(A)$ there exists at most one lift of u.*

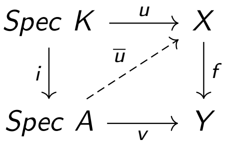
$$\begin{array}{ccc} Spec\ K & \xrightarrow{u} & X \\ {\scriptstyle i}\downarrow & \overset{\bar{u}}{\nearrow} & \downarrow{\scriptstyle f} \\ Spec\ A & \xrightarrow[v]{} & Y \end{array}$$

For many applications the following noetherian version is useful.

Theorem (Valuative criterion for separated - Noetherian version)

Let $f : X \to Y$ be a morphism of schemes and suppose that X is Noetherian. Then the following assertions are equivalent.

1. *f is separated.*
2. *For all diagrams below where A is a valuation ring and $K = Frac(A)$ there exists at most one lift of u.*



THANK YOU FOR YOUR LISTENING!